Capacitance and Resistance

Sandra Cruz-Pol, Ph. D.
INEL 4151
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ECE UPRM
Mayagüez, PR

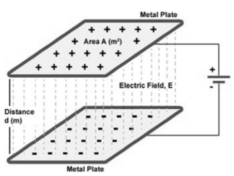
Resistance

- If the cross section of a conductor is not uniform we need to integrate:

$$R = \frac{V}{I} = \frac{\oint_l \vec{E} \cdot d\vec{l}}{\oint_S \sigma \vec{E} \cdot d\vec{S}} \qquad R = \frac{V}{I} = \frac{V}{\oint_S \sigma \vec{E} \cdot \overrightarrow{dS}}$$

1. Solve Laplace eq. to find V
2. Then find E from its differential
3. And substitute in the above equation

Capacitance



- Is defined as the ratio of the charge on one of the plates to the potential difference between the plates:

$$C = \frac{Q}{V} = \frac{\varepsilon \oint_S \vec{E} \cdot d\vec{S}}{\oint_l \vec{E} \cdot d\vec{l}} \quad [Farads]$$

1. Assume Q and find V (Gauss or Coulomb)
2. Assume V and find Q (Laplace)
3. And substitute E in the equation.

To find E, we will use:

from Gauss's Law

$$\nabla \cdot D = \nabla \cdot \varepsilon E = \rho_v$$
$$E = -\nabla V$$

From this we can get:

- Poisson's equation: $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
- Laplace's equation: (if charge-free) $\nabla^2 V = 0$

Relaxation Time

- Recall that: $R = \frac{V}{I} = \frac{V}{\oint_S \sigma \vec{E} \cdot \overrightarrow{dS}}$ $\quad C = \frac{Q}{V} = \frac{\varepsilon \oint_S \vec{E} \cdot d\vec{S}}{\oint_l \vec{E} \cdot d\vec{l}} \quad [Farads]$
- Multiplying, we obtain the Relaxation Time:

$$RC = \frac{\varepsilon}{\sigma}$$

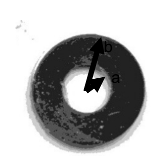
- Solving for R, we obtain it in terms of C:

$$R = \frac{\varepsilon}{\sigma C}$$

P.E. 6.8 find Resistance of disk of radius *b* and central hole of radius *a*.

Laplace's equation:
In cylindrical coordinates (if charge-free)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$



P.E. 6.8 find Resistance of disk of radius *b* and central hole of radius *a*.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0$$

$$V=A\ln\rho+B$$

$$BC:\ \left\rangle \begin{matrix} V(\rho=a)=0 \\ V(\rho=b)=V_o \end{matrix}\right.$$

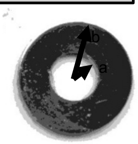
$$V=\frac{V_o}{\ln(b/a)}\ln\frac{\rho}{a}$$

$$E=-\nabla V=-\frac{dV}{d\rho}\hat{\rho}=\frac{V_o}{\rho\ln(b/a)}\hat{\rho}$$

$$d\vec{S}=-\rho\, dz\, d\phi\,\hat{\rho}$$

$$I=\oint_S\sigma\vec{E}\cdot d\vec{S}=\frac{2\pi t V_o\sigma}{\ln(b/a)}$$

$$R=\frac{V_o}{I}=\frac{V_o}{\oint_S\sigma\vec{E}\cdot d\vec{S}}=\frac{\ln(b/a)}{2\pi_o t\sigma}$$



Capacitance

1. Parallel plate
2. Coaxial
3. Spherical

$$C=\frac{Q}{V}=\frac{Q}{\oint_0^l\vec{E}\cdot\vec{dl}}$$

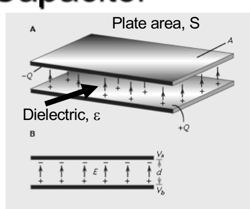
Parallel plate Capacitor

- Charge Q and $-Q$

$$\rho_s=\frac{Q}{S}$$

$$\vec{D}_n=\rho_s\hat{a}_x$$

$$\vec{E}=\frac{\rho_s}{\varepsilon}\hat{a}_x$$

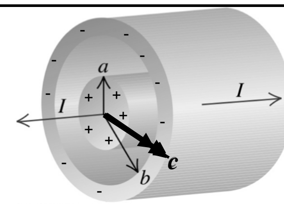


$$V=-\int_d^0\vec{E}\cdot d\vec{l}=-\int_d^0\frac{Q}{\varepsilon S}dx=\frac{Qd}{\varepsilon S}$$

$$C=\frac{Q}{V}=\frac{\varepsilon S}{d}$$

Coaxial Capacitor

- Charge $+Q$ & $-Q$

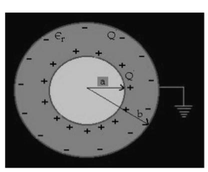


$$Q=\varepsilon\oint\vec{E}\cdot d\vec{S}=\varepsilon E_\rho 2\pi\rho L$$

$$V=-\int\vec{E}\cdot d\vec{l}=-\int_b^a\frac{Q}{2\pi\varepsilon\rho L}\hat{\rho}\cdot d\rho\hat{\rho}=\frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}$$

$$C=\frac{Q}{V}=\frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

Spherical Capacitor



- Charge +Q & -Q

$$Q=\varepsilon\oint\vec{E}\cdot dS=\varepsilon E_r 4\pi r^2$$

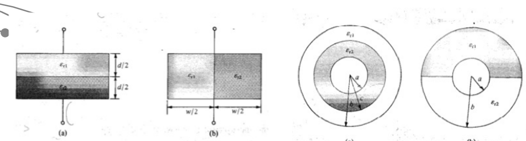
$$V=-\int\vec{E}\cdot d\vec{l}=-\int_b^a\frac{Q}{4\pi\varepsilon r^2}\hat{r}\cdot dr\hat{r}=\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$C=\frac{Q}{V}=\frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}$$

Capacitors connection

- Series $\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}$
- Parallel $C=C_1+C_2$

Examples:

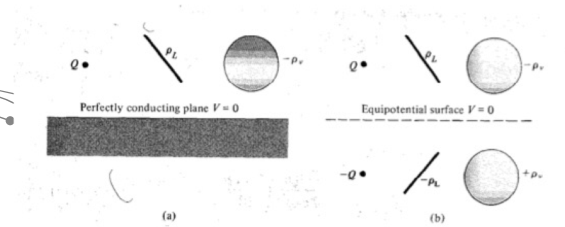


In summary :

	C	R	
Parallel Plate	$\frac{\varepsilon S}{d}$	$\frac{\sigma d}{S}$	
Coaxial	$2\pi\varepsilon L / \ln\frac{b}{a}$	$2\pi\varepsilon L / \ln\frac{b}{a}$	
Spherical	$\frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}$		

Method of Images

- (or mirror charges) is a problem-solving tool in electrostatics.



Method of Images

- Use superposition
- Valid only for top region (where Q is)

$$V(\rho,\varphi,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{\rho^2+(z-a)^2}} + \frac{-q}{\sqrt{\rho^2+(z+a)^2}}\right)$$