Maxwell Equations

INEL 4151 ch 9

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Electricity => Magnetism

In 1820 Oersted discovered that a steady current produces a magnetic field while teaching a physics class.

Would magnetism would produce electricity?

Eleven years later, and at the same time, Mike Faraday in London and Joe Henry in New York discovered that a time-varying magnetic field would produce an electric current!

Cruz-Pol, Electromagnetics

V_{emf} = - \int \mathbf{E} \cdot d\mathbf{l} = - \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S}

Len's Law = (-)

If N=1 (1 loop)

The time change

\[ V_{emf} = - \int \mathbf{E} \cdot d\mathbf{l} = - \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} \]

can refer to $B$ or $S$

Would magnetism would produce electricity?

Two cases of

$B$ changes

\[ \int \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \]

Stoke's theorem

\[ \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \]

\[ \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \]

3 cases:

Stationary loop in time-varying $B$ field

\[ \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \]

Moving loop in static $B$ field

\[ \nabla \times \mathbf{E} = \nabla \times \mathbf{u} \times \mathbf{B} \]

Moving loop in time-varying $B$ field

\[ \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{u} \times \mathbf{B} \]
**PE 9.1 If B=**

\[ V_{\text{emf}} = \frac{\int E \cdot dl}{L} = \frac{\int \mathbf{u} \times \mathbf{B} \cdot dl}{L} \]

Figure 9.3 Induced emf due to a moving loop in a static B field

**Example 9.3** Circuit with \(10^{-3}\) m\(^2\) cross-section, radius =10cm, \(l(t)=3 \sin \omega t\) [A], 50 Hz, \(N_1=200\) turns, \(N_2=100\), \(\mu_2=500\mu_0\). find \(V_{\text{emf}}\) on both coils

\(\omega = 2 \pi f\)

**Find reluctance and use Faraday’s Law**

\[ V_{\text{emf}} = -\frac{N_2}{L} \frac{d\Psi}{dt} \]

\[ \Psi = \oint \mathbf{H} \cdot d\mathbf{l} = N_1 I, \mu S \]

\[ V_{\text{emf}} = -100(200)(3\omega \cos \omega) \cdot \frac{500 \mu_0 \cdot 10^{-3}}{2\pi (0.10)} = -6.7 \cos 100 \pi \]

**Some terms**

- \(E\) = electric field intensity [V/m]
- \(D\) = electric field density [C/m\(^2\)]
- \(H\) = magnetic field intensity, [A/m]
- \(B\) = magnetic field density, [Teslas]

**Take a look at a capacitor**

- **dielectric** is an electrical insulator

- d.e., C = open circuits

- What’s the current in an open circuit?
Maxwell’s Eqs.

- the equation of continuity: \( \nabla \cdot J = -\frac{\partial \rho}{\partial t} \)
- Maxwell added the term \( \frac{\partial \mathbf{D}}{\partial t} \) to Ampere’s Law so that it not only works for static conditions but also for time-varying situations.
- This added term is called the displacement current density, \( J_\text{d} \), while \( J=J_\text{c} \) is the conduction current.

For static fields we had: \( \nabla \times \mathbf{H} = \mathbf{J} \)

- But the divergence of the curl of ANY vector \( \mathbf{A} \) is zero: \( \nabla \cdot \nabla \times \mathbf{A} = 0 \)
- The continuity of the current requires: \( \nabla \cdot J = -\frac{\partial \rho}{\partial t} \) This doesn’t work for time-varying fields!

We need to define: \( \nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_\text{d} \)

Now when we take the divergence of curl
\( \nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{J} + \mathbf{J}_\text{d}) \)

Therefore: \( \nabla \cdot J_\text{d} = -\nabla \cdot J \)

Maxwell put them together

If we didn’t have the displacement current, \( J_\text{d} \), then Ampere’s Law:

\[ \int H \cdot dl = \int J \cdot dS = \text{I}_{\text{bc}} = I \]

\[ \int H \cdot dl = \int J \cdot dS = 0 \]

\[ \int H \cdot dl = \int J_\text{d} \cdot dS = \frac{d\mathbf{D}}{dt} = J \]

At low frequencies \( J \approx J_\text{d} \), but at radio frequencies both terms are comparable in magnitude.

For dynamic fields we had:

\[ \nabla \cdot J_\text{d} = -\nabla \cdot J = \frac{\partial \rho}{\partial t} + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \]

Therefore:
\[ \mathbf{J}_\text{d} = \frac{\partial \mathbf{D}}{\partial t} \]

Substituting in curl of \( \mathbf{H} \)
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

This is Maxwell’s equation for Ampere’s Law

Maxwell Equations in General Form

<table>
<thead>
<tr>
<th>Differential form</th>
<th>Integral Form</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla \cdot \mathbf{D} = \rho )</td>
<td>( \int \mathbf{D} \cdot d\mathbf{S} = \int \rho \cdot d\mathbf{v} )</td>
<td>Gauss’s Law for ( \mathbf{E} ) field</td>
</tr>
<tr>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>( \int \mathbf{B} \cdot d\mathbf{S} = 0 )</td>
<td>Gauss’s Law for ( \mathbf{H} ) field. Nonexistence of monopole</td>
</tr>
<tr>
<td>( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} )</td>
<td>( \int \mathbf{E} \cdot d\mathbf{dl} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} )</td>
<td>Faraday’s Law</td>
</tr>
<tr>
<td>( \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} )</td>
<td>( \int \mathbf{H} \cdot d\mathbf{dl} = \int \mathbf{J} \cdot d\mathbf{dl} + \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} )</td>
<td>Ampere’s Circuit Law</td>
</tr>
</tbody>
</table>

Electromagnetics

- This is the principle of meters, hydroelectric generators, and transformers operation.

\[ \int \mathbf{H} \cdot d\mathbf{dl} = \int (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S} \]

* Mention some examples of EM waves

Dr. S. Cruz-Pol, INEL 4152-
Electromagnetics
Phasors & complex #s

Working with harmonic fields is easier, but requires knowledge of phasor, let’s review:

- complex numbers and
- phasors

Maxwell Equations for Harmonic fields

Differential form:

<table>
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<th>Notes</th>
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<tr>
<td>( \nabla \cdot \vec{E} = \rho )</td>
<td>Gauss’s Law for E field.</td>
</tr>
<tr>
<td>( \nabla \cdot \vec{H} = 0 )</td>
<td>Gauss’s Law for H field. No monopole</td>
</tr>
<tr>
<td>( \nabla \times \vec{E} = -j\omega \vec{H} )</td>
<td>Faraday’s Law</td>
</tr>
<tr>
<td>( \nabla \times \vec{H} = \vec{J} + j\omega \vec{E} )</td>
<td>Ampere’s Circuit Law</td>
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* (substituting \( D = \varepsilon \vec{E} \) and \( H = \mu \vec{H} \))

Earth Magnetic Field Declination from 1590 to 1990

News

Magnetic North Pole Shift Affects Tampa Airport

Jan 2011:

The Earth’s magnetic north pole is slowly heading toward Russia, according to scientists, but one of the places being affected by this is Tampa International Airport.

Airport officials closed its main runway this week until Jan. 13 to adjust the taxiway signs accounting for the magnetic pole shift, Tampa Bay Online reports.”