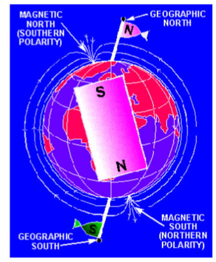
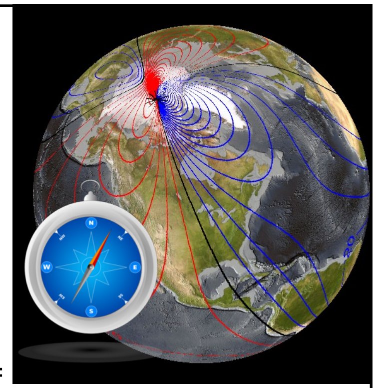
# MAGNETISM

INEL 4151
Dr. Sandra Cruz-Pol
Electrical and Computer Engineering Dept.
UPRM ch 7

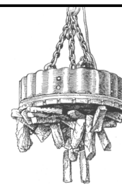


μαγνήτης λίθος= magnetis lithos

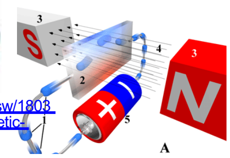
## Magnissia , Grecia

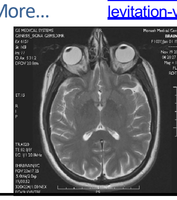

## Applications



Motors
Transformers
MRI
More…

http://videos.howstuffworks.com/hsw/18034-electricity-and-magnetism-magnetic-levitation-video.htm





$$\vec{B} = \mu \vec{H}$$

$H$= magnetic field intensity [A/m]

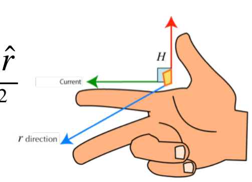
$B$= magnetic field (or flux) density [Teslas]

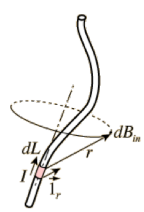
In free space the permeability is:
$\mu_o = 4\pi \times 10^{-7}$ H/m

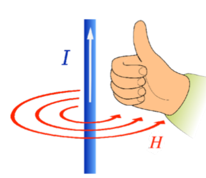
## Magnetic Field
Biot-Savart Law

- States that:

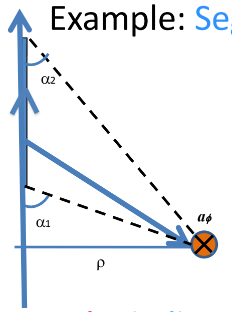
$$d\vec{H} = \frac{Id\vec{l} \times \hat{r}}{4\pi R^2}$$



## Example: Segment of current




$$d\vec{H} = \frac{Id\vec{l} \times \hat{r}}{4\pi R^2}$$

$$\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\hat{a}_\phi$$

For an infinite line filament with current I ($\alpha_1$=180° and $\alpha_2$=0°):

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho \qquad \vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \qquad \hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho$$

## PE. 7.1 Find *H* at (0,0,5)

$$\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\hat{a}_\phi \qquad d\vec{H} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$

Due to 10A current in:

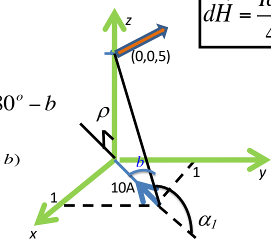
where $\alpha_2$=90° and $\alpha_1 = 180^o - b$

$$\cos b = \frac{\sqrt{1^2+1^2}}{\sqrt{5^2+\sqrt{2}^2}} = -\cos(180^o - b)$$

$$\hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho$$

$$= \left(\frac{-\hat{a}_x - \hat{a}_y}{\sqrt{2}}\right) \times \hat{a}_z$$

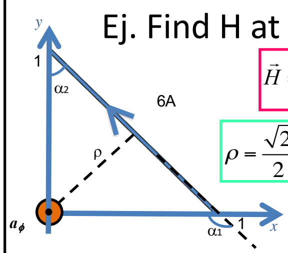
$$= \frac{\hat{a}_y - \hat{a}_x}{\sqrt{2}}$$




$$\vec{H} = \frac{10}{4\pi(5)}\left(0 - \frac{-\sqrt{2}}{\sqrt{27}}\right)\left(\frac{-\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)$$

$$\vec{H} = 30.6(-\hat{a}_x + \hat{a}_y)\frac{mA}{m}$$

## Ej. Find H at the origin for:




$$\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\hat{a}_\phi$$

$$\rho = \frac{\sqrt{2}}{2} \qquad \alpha_2 = 45^o \quad \alpha_1 = 135^o$$

$$\hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho$$

$$= \left(\frac{-\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right) \times \left(\frac{-\hat{a}_x - \hat{a}_y}{\sqrt{2}}\right) = \hat{a}_z$$

$$\vec{H} = \frac{I\sqrt{2}}{4\pi}\left(\cos 45^o - \cos 135^o\right)\hat{a}_z \qquad \vec{H} = 0.95\hat{a}_z \quad A/m$$

## Circular loop of *I*

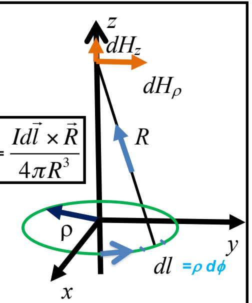
Defined by $x^2 + y^2 = 9, z = 0$

- Apply Biot-Savart:

$$d\vec{l} \times \vec{R} = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix}$$

$$= \rho h d\phi \hat{a}_\rho + \rho^2 d\phi \hat{a}_z$$

$$d\vec{H} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$




- Only $z$-component of $H$ survives due to symmetry:

$$\vec{H} = \int_0^{2\pi} \frac{I\rho^2 d\phi \hat{a}_z}{4\pi\left[\rho^2 + h^2\right]^{3/2}} = \frac{I\rho^2 \hat{a}_z}{4\pi\left[\rho^2 + h^2\right]^{3/2}} \int_0^{2\pi} d\phi = \frac{I\rho^2 \hat{a}_z}{2\left[\rho^2 + h^2\right]^{3/2}}$$

## Ampere's Law

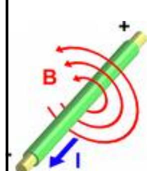
- Simpler
- Analogous to Gauss Law for Coulomb's
- For symmetrical current distributions

Recall Gauss Law:

$$Q_{enc} = \int_v \rho_v \, dv = \oint_S \vec{D} \cdot d\vec{S}$$

## Ampere's Law

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = \int \vec{J} \cdot d\vec{S}$$




$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = I$$

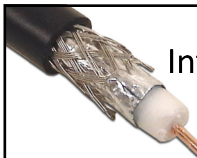
$$d\vec{l} = \rho d\phi \hat{a}_\phi$$

$$\oint H_\phi \cdot \rho d\phi = H_\phi 2\pi\rho$$

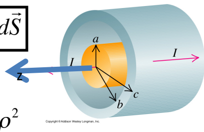
We define an Amperian path where H is constant.

$$\vec{H} = \frac{I}{2\pi\rho} a_\phi$$

## Infinitely long coaxial cable



$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = \int \vec{J}\cdot d\vec{S}$$



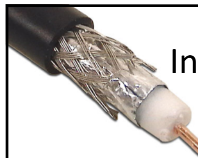

Four cases:
1) For ρ<a

$$I_{enc} = \int \frac{I}{\pi a^2}\cdot \rho\, d\phi\, d\rho\, \hat{a}_z = \frac{I\rho^2}{a^2}$$

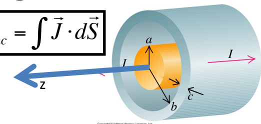
$$\oint H_\phi \cdot \rho d\phi = H_\phi 2\pi\rho$$

$$H_\phi = \frac{I\rho}{2\pi a^2}$$

## Infinitely long coaxial cable



$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = \int \vec{J}\cdot d\vec{S}$$




Four cases:
2) For a<ρ<b

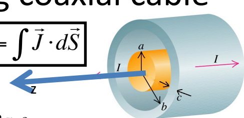
$$I_{enc} = \int_0^a \int_0^{2\pi} \frac{I}{\pi a^2}\hat{a}_z \cdot \rho\, d\phi\, d\rho\, \hat{a}_z = I$$

$$\oint H_\phi \cdot \rho d\phi = H_\phi 2\pi\rho$$

$$H_\phi = \frac{I}{2\pi\rho}$$

## Infinitely long coaxial cable

$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = \int \vec{J}\cdot d\vec{S}$$




Four cases:
3) For b<ρ<b+c

$$I_{enc} = \int \vec{J}\cdot \rho\, d\phi\, d\rho\, \hat{a}_z = I - \int_0^{2\pi}\int_b^{\rho} \frac{I\rho\, d\phi\, d\rho}{\pi\left((b+c)^2 - b^2\right)}$$

$$I_{enc} = I - \frac{I\left(\rho^2 - b^2\right)}{\left(2bc + c^2\right)}$$

$$\oint H_\phi \cdot \rho d\phi = H_\phi 2\pi\rho$$

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{c^2 + 2bc}\right]$$

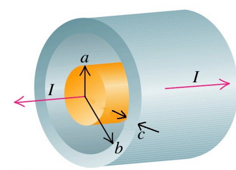
## Infinitely long coaxial cable

$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = \int \vec{J}\cdot d\vec{S}$$

Four cases:
4) For ρ>b+c

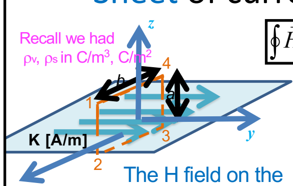
$$I_{enc} = I - I = 0$$

$$0 = H_\phi 2\pi\rho$$




$$H_\phi = 0$$

## Sheet of current distribution




$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = K_y b$$

Cross section is a Line!

The H field is given by:

$$\vec{H} = \begin{cases} H_o \hat{a}_x & z > 0 \\ -H_o \hat{a}_x & z < 0 \end{cases}$$

The H field on the *Amperian* path is given by:

$$\oint \vec{H}\cdot d\vec{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right)\vec{H}\cdot d\vec{l}$$

$$= 0(-a) + (-H_o)(-b) + 0(a) + (H_o)(b)$$

$$= 2H_o b$$

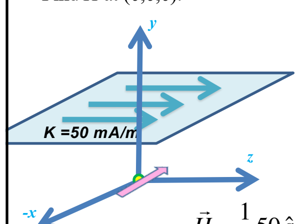
*Despejando por H* :

$$\vec{H} = \begin{cases} \frac{1}{2}\vec{K}_y \hat{a}_x & z > 0 \\ -\frac{1}{2}\vec{K}_y \hat{a}_x & z < 0 \end{cases}$$

In General: $\vec{H} = \frac{1}{2}\vec{K}\times\hat{a}_n$

## PE. 7.5 Sheet of current

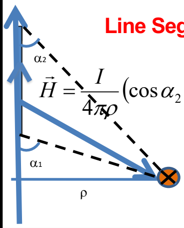
Plane $y=1$ carries a current $K=50\ \mathbf{a}_z$ mA/m.
Find $H$ at (0,0,0).




$$\vec{H} = \frac{1}{2}\vec{K}\times\hat{a}_n$$

$$\vec{H} = \frac{1}{2}50\hat{a}_z \times\left(-\hat{a}_y\right) = 25\hat{a}_x mA/m$$

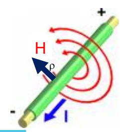
### Line Segment



$$\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\hat{a}_\phi$$

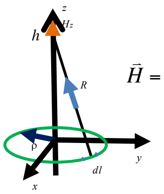
$$\hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho$$

### Infinite Line



$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

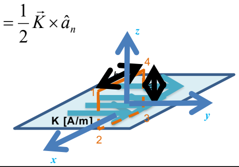
### Loop



$$\vec{H} = \frac{I\rho^2\hat{a}_z}{2\left[\rho^2 + h^2\right]^{3/2}}$$

### Infinite Plane

$$\vec{H} = \frac{1}{2}\vec{K} \times \hat{a}_n$$




---

## Magnetic Flux Density, B

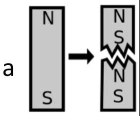
- The magnetic flux is defined as:

$$\Psi = \int_S \vec{B} \cdot d\vec{S} \quad \text{[Wb]}$$

Webers = Teslas * m²

which flows through a surface S.

- The total flux thru a closed surface in a magnetic field is: $\oint_S \vec{B} \cdot d\vec{S} = 0$




$$\oint_S \vec{B} \cdot d\vec{S} = \int_v \nabla \cdot \vec{B}\, dv = 0$$

$$\nabla \cdot \vec{B} = 0$$

Recall $\nabla \cdot \vec{D} = \rho_v$

Monopole doesn't exist.

---

## Maxwell's Equations for Static Fields

| Differential form | Integral Form | |
|---|---|---|
| $\nabla \cdot \vec{D} = \rho_v$ | $\oint_s \vec{D} \cdot d\vec{S} = \int_v \rho_v dv$ | Gauss's Law for *E* field. |
| $\nabla \cdot \vec{B} = 0$ | $\oint_s \vec{B} \cdot d\vec{S} = 0$ | Gauss's Law for *H* field. Nonexistence of monopole |
| $\nabla \times \vec{E} = 0$ | $\oint_L \vec{E} \cdot d\vec{l} = 0$ | Faraday's Law; E field is conserved. |
| $\nabla \times \vec{H} = \vec{J}$ | $\oint_L \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{S}$ | Ampere's Law |

---

## Magnetic *Scalar and Vector* Potentials, $V_m$ & A

$$\nabla \times \vec{H} = \vec{J}$$

When *J*=0, the curl of *H* is =0, then recalling the vector identity: $\nabla \cdot \left(\nabla \times \vec{H}\right) = 0 = \nabla \cdot (\nabla V)$

- We can define a Magnetic Scalar Potential as:

$$\vec{H} = -\nabla V_m \quad \text{if } \vec{J} = 0$$

- The magnetic Vector Potential A is defined:

$$\vec{B} = \nabla \times \vec{A}$$

---

## The magnetic vector potential, *A*, is defined from:

$$\vec{B} = \nabla \times \vec{A}$$

where $\vec{B} = \mu_o \vec{H} = \int_L \frac{\mu_o I d\vec{l} \times \hat{a}_R}{4\pi R^2}$

It can be shown that:

$$\vec{A} = \int_L \frac{\mu_o I d\vec{l}}{4\pi R}$$

(we used this): $-\nabla\left(\frac{1}{R}\right) = \frac{1}{R^2}$

The magnetic vector potential A is used in antenna theory.

Substituting into equation for Magnetic Flux:

$$\Psi = \int_S \vec{B} \cdot d\vec{S} = \int_S (\nabla \times A) \cdot d\vec{S} = \oint_L \vec{A} \cdot d\vec{l}$$

This is another way of finding magnetic flux. $\Psi = \oint_L \vec{A} \cdot d\vec{l}$

---

## P.E. 7.7 A current distribution causes a magnetic vector potential of:

$$\vec{A} = x^2 y\hat{x} + y^2 x\hat{y} - 4xyz\hat{z}$$

Find :

① *B* at (-1,2,5)

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & y^2 x & -4xyz \end{vmatrix}$$

Answer: $\vec{B} = 20\hat{x} + 40\hat{y} + 3\hat{z}$ [T]

② Flux thru surface *z=1, 0≤ x ≤1, -1≤y ≤4*

$$\Psi = \int_S \vec{B} \cdot d\vec{S} = \oint_S \vec{A} \cdot d\vec{l}$$

$\Psi = 20$ [Wb]

*Answer :* $\Psi = \int_0^1 x^2(-1)dx + \int_{-1}^4 y^2(1)dy + \int_1^0 x^2(4)dx + 0$

In a certain conducting region,

$$\mathbf{H} = yz(x^2 + y^2)\mathbf{a}_x - y^2xz\mathbf{a}_y + 4x^2y^2\mathbf{a}_z \text{ A/m}$$

(a) Determine $\mathbf{J}$ at $(5, 2, -3)$
(b) Find the current passing through $x = -1, 0 < y, z < 2$
(c) Show that $\nabla \cdot \mathbf{B} = 0$

$$\vec{J} = \nabla \times \vec{H}$$

$$\vec{J} = \left(8x^2y + xy^2\right)\hat{x} + \left(y\left(x^2 + y^2\right) - 8xy^2\right)\hat{y} + \left(-y^2z - zx^2 - 3zy^2\right)\hat{z}$$

$\vec{J}$@(5,2,−3)

$$= \left(8\cdot 25\cdot 2 + 5\cdot 4\right)\hat{x} + \left(2\left(25+4\right) - 8\cdot 5\cdot 4\right)\hat{y} + \left(4\cdot 3 + 3\cdot 25 + 3\cdot 3\cdot 4\right)\hat{z}$$

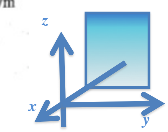
$$= \left(420\right)\hat{x} + \left(58 - 160\right)\hat{y} + \left(12 + 75 + 36\right)\hat{z}$$

$$= 420\hat{x} - 102\hat{y} + 123\hat{z} \quad \text{A/m}^2$$

In a certain conducting region,

$$\mathbf{H} = yz(x^2 + y^2)\mathbf{a}_x - y^2xz\mathbf{a}_y + 4x^2y^2\mathbf{a}_z \text{ A/m}$$

(a) Determine $\mathbf{J}$ at $(5, 2, -3)$
(b) Find the current passing through $x = -1, 0 < y, z < 2$
(c) Show that $\nabla \cdot \mathbf{B} = 0$




0<y<2
0<z<2

$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = \int \vec{J}\cdot d\vec{S}$$

$$\vec{J} = \left(8x^2y + xy^2\right)\hat{x} + \left(y\left(x^2 + y^2\right) - 8xy^2\right)\hat{y} + \left(-y^2z - zx^2 - 3zy^2\right)\hat{z}$$

$$\vec{J}_x = 8x^2y + xy^2\Big|_{x=-1}$$

$$I_{enc} = \int_0^{y=2}\int_0^{z=2}\left(8y - y^2\right)dy\,dz = \int_0^2 dz\left(\frac{8y^2}{2} - \frac{y^3}{3}\right)_0^2 = 2\left(16 - \frac{8}{3}\right)$$

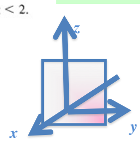
$$= 26.67A$$

**7.28** For a current distribution in free space,

$$\mathbf{A} = (2x^2y + yz)\mathbf{a}_x + (xy^2 - xz^3)\mathbf{a}_y - (6xyz - 2x^2y^2)\mathbf{a}_z \text{ Wb/m}$$

(a) Calculate $\mathbf{B}$.
(b) Find the magnetic flux through a loop described by $x = 1, 0 < y, z < 2$.

$$\vec{B} = \nabla \times \vec{A}$$




$$\Psi = \int \vec{B}\cdot d\vec{S} = \int_0^2\int_0^2 B_x\,dy\,dz$$

$$= \int_0^2\int_0^2\left(-6xz + 4x^2y + 3xz^2\right)\Big|_{x=1} dy\,dz$$

$$= \int_0^2\int_0^2\left(-6z + 4y + 3z^2\right)\Big|_{x=1} dy\,dz = \int_0^2\left(-6zy + 4\frac{y^2}{2} + 3z^2y\right)\Big|_{y=0}^2 dz$$

$$= \int_0^2\left(-12z + 8 + 6z^2\right)dz = \left(-12\frac{z^2}{2} + 8z + 6\frac{z^3}{3}\right)\Big|_{z=0}^2 = -24 + 16 + 6\left(\frac{8}{3}\right) = 8\,Wb$$

Questions?