Regularization Methods for Solution of Linear Systems

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Abstract

The advent of the computer had forced the application of mathematics to all branches of human endeavor. One important property of mathematical problems is the stability of their solutions to small changes in the initial data. Problems that fail to satisfy this stability condition are called ill-posed. If the initial data in such problems are known only approximately and contain a random error, then the above mentioned instability of their solutions leads to non-uniqueness of the classically derived approximate solutions and to serious difficulties in their physical interpretation. In solving such problems, one must first define the concept of an approximate solution that is stable to small changes in the initial data, and to use special methods for deriving this solution. We are working in the development of the regularization method in the construction of approximate solutions of ill-posed problems.

1. INTRODUCTION

The rapidly increasing use of computational technology requires the development of computational algorithms for solving broad classes of problems. But just what do we mean by the "solution" of a problem? What requirements must the algorithms for finding a "solution" satisfy?

Many features of problems encountered in practice are not reflected by the classical conceptions and formulation of problems. Let us illustrate this. Consider the following equation:

$$Az = u \qquad (1)$$

where z is an unknown vector, u is a known vector and $A = \{a_{ij}\}$ is a square matrix with elements a_{ij}.

If the system is nonsingular (Eq. 2), that is, if det $A \neq 0$, it has a unique solution, which we can find by Cramer's rule or by some other method.

$$\begin{aligned} 5z_1 + 7z_2 &= 5 \\ \sqrt{2}z_1 + \sqrt{98}z_2 &= \sqrt{50} \end{aligned} \qquad (2)$$

If the system is singular (Eq 3), det $A = 0$, it will have a solution (not unique) only when the condition for existence of a solution (vanishing of the relevant determinants) is satisfied.

$$\begin{aligned} z_1 + 7z_2 &= 5 \\ \sqrt{2}z_1 + \sqrt{98}z_2 &= \sqrt{50} \end{aligned} \qquad (3)$$

We are dealing with some other system

$\tilde{A}z = v$ such that $\|\tilde{A}-A\| \leq \delta$ and $\| v - u\| \leq \delta$, where the meaning of the norm is usually determined by the nature of the problem. Since we have the approximate system $\tilde{A}z = v$ rather than the exact one, we can speak only of finding an approximate solution.

2. SOLUTION OF ILL-POSED PROBLEMS

We know the difficulties involved in solving ill-posed linear systems. Large changes in the solution may result from small changes in the right hand members of the system.

If a singular system (Eq. 2) has any solution at all, it has infinitely many. We have to consider a whole class of system of equations that are indistinguishable from each other that may include both singular and unsolvable systems. The methods of constructing approximate solutions of this class must be generally applicable. These solutions must be stable under small changes in the initial data. The construction of such methods is based in the idea of regularization method.

Suppose that the system $Az = u$ is singular and that the vector u constituting the right-hand member satisfies the conditions of solvability of the system. The solution of such system is not unique. Let F_A (Fig. 1) denote the set of its solutions. We then pose the problem of finding a normal solution (Fig. 1) of the system. We define the normal solution of the system $Az = u$ for the vector z_1 as the solution z_0 for which

$$\|z_0-z_1\| = \inf_{z\in F_A} \|z-z_1\|, \qquad (4)$$

where z_1 is a fixed element (vector) determined by the formulation of the problem and $\|z\|$, the norm of the vector z, is defined by the formula in Eq. 5.

$$\|z\| = \sqrt{(z_1^2 + z_2^2 + \ldots z_n^2)} \qquad (5)$$

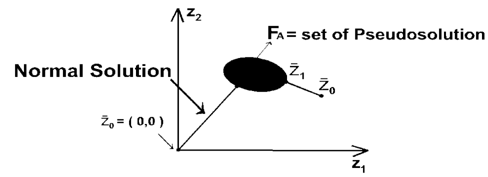

Fig. 1 Geometry of Pseudosolution

We shall assume for simplicity that $z_1 = 0$. The normal solution is unique.

In linear algebra, a vector z minimizing the discrepancy $\|Az-u\|^2$ is called pseudosolution of the system $Az = u$.

Suppose that, instead of the exact singular system Az=u, we have the system with approximate right-hand member

$$Az = v,$$

where $\| v - u\| \leq \delta$ and the vector v may fail to satisfy the solvability condition. It is natural to seek an approximate normal solution of the system $Az = v$ among the vectors z such that $\|Az - v\| \leq \delta$. By the definition of normal solution, this solution will minimize the functional $\Omega[z] = \|z-z_1\|^2$. The problem reduces to minimizing the functional $\|z-z_1\|^2$ on the set of vectors satisfying the inequality $\|Az - v\| \leq \delta$. It reduces to finding the vector z_α minimizing the smoothing functional Eq. 6.

$$M^\alpha[z, v,A] = \|Az - v\|^2 + \alpha\|z - z_1\|^2, \alpha>0 \quad (6)$$

The value of the parameters α is the determined from the condition $\|Az^\alpha- v\| = \delta$, that is, from discrepancy.

If we have the case in which both the right-hand member of the equation and the matrix *A* are inexactly given; that is, let us look at an equation of the form

$$\tilde{A}z = v,$$

where

$$\|\tilde{A}-A\| \leq \delta, \| v -u\| \leq \delta$$

Minimizing the functional form we obtain,

$$M^{\alpha}[z,v,\tilde{A}] = \|\tilde{A}z - v\|^2 + \alpha\|z - z_1\|^2, \alpha>0 \quad (7)$$

based in A. Tikhonov Theorem.

Let us suppose that the vector *u* satisfies the conditions for solvability and the z_0 is the normal solution of the equation $Az = u$.

Applying this method we obtained stability in our nonsingular system. This can be observed in Table 1 when all approximations converge to solution (0.1,0.7).

Table 1. Approximation of Solution for the Linear System (3).

NUMBER OF DIGITS	SOLUTION Z_1	SOLUTION Z_2
1	0	.7
2	.11	.69
3	.098	.7
4	.0994	.7
5	.1	.69999
6	.999896	.700025
7	.1	.6999999
8	.09999999	.7

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