Twiddle Factor Elimination in Multidimensional FFT's

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Abstract

A new approach for the computation of multidimensional FFT without twiddle factors is presented. The procedure uses change of basis in the sample and transform domains to eliminate twiddle factors.

1. Introduction

The multidimensional discrete Fourier transform (MDFT) could be formulated for samples defined on any periodic lattice. The indexes of the sample are view as vectors in the lattice. This work presents a technique for twiddle factor elimination in the multidimensional FFT, by means of a change of basis for the domain of both, the sample and the transform.

2. Definitions

Let N be a nonsingular d×d integer matrix. The set $L_N=\{\mathbf{m}: \mathbf{m} = N\mathbf{r}, \mathbf{r} \in Z^d\}$ is said to be the lattice generated by N. Let $\mathbf{n}, \mathbf{m} \in L_I$, then vector $\mathbf{n}$ is said to be congruent to $\mathbf{m}$ modulo N if $\mathbf{n}-\mathbf{m} \in L_N$. This relation defines a set of equivalence classes denoted $L_{I/N}$ Now consider a sample x ($\mathbf{j}$) periodic on the lattice L_N i.e., for all $\mathbf{j}, \mathbf{m} \in Z^d$, x ($\mathbf{j}$) = x ($\mathbf{j}$ + N$\mathbf{m}$). Then the MDFT is defined to be

$$\hat{x}(\mathbf{k}) = \sum_{\mathbf{j}\in L_{I/N}} x(\mathbf{j}) \exp\left(-2pi\mathbf{k}^T N^{-1}\mathbf{j}\right), \quad (1)$$

$$\mathbf{k} \in L_{I/N^T}$$

where $L_{I/N}$ and L_{I/N^T} represent one period of $x(\mathbf{j})$ and $\hat{x}(\mathbf{k})$, respectively.

Consider a periodic sample on the lattice generated by the matrix,

$$N = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}.$$

Its transform is

$$\hat{x}(\mathbf{k}) = \sum_{\mathbf{j}} x(\mathbf{j}) \exp\left(-2pi\mathbf{k}^T \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}^{-1} \mathbf{j}\right)$$

$$= \sum_{j_2} \exp\left(\tfrac{-2pi}{8} k_1 j_2\right) \times$$

$$\left[\sum_{j_1} x(j_1, j_2) \exp\left(\tfrac{-2pi}{4} k_1 j_1\right)\right] \exp\left(\tfrac{-2pi}{4} k_2 j_2\right)$$

where

$$\exp\left(\tfrac{-2pi}{8} k_1 j_2\right)$$

is the twiddle factor.

3. A New Approach

Let $x_M(\bar{\mathbf{j}}) = x_M\left(\begin{bmatrix} \mathbf{j} \\ \mathbf{j}_1 \end{bmatrix}\right)$ periodic on the

lattice generated by matrix

$$M = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}.$$

The transform is

$$x_M(\bar{\mathbf{k}}) = \sum_{\bar{\mathbf{j}}} x(\bar{\mathbf{j}})\exp\left(-2pi\bar{\mathbf{k}}^t M^{-1}\bar{\mathbf{j}}\right), \quad (2)$$

$$\bar{\mathbf{k}} = [\mathbf{k}, \mathbf{k}_1] \in Z^{2d}.$$

Suppose that there exist integer matrices A and B such that

$$AP+BQ=I.$$

Let

$$R = \begin{bmatrix} -I & P \\ A & I-AP \end{bmatrix} \text{and } C = \begin{bmatrix} -B & I \\ I-QB & Q \end{bmatrix}$$

then

$$RMC = D = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}.$$

Matrices R and C are unimodular.
Since RMC=D, then

$$M=R^{-1}DC^{-1}$$

and

$$M^{-1}=CD^{-1}R. \quad (3)$$

By substituting (3) in (2) one gets

$$\hat{x}_M(\bar{\mathbf{k}}) = \sum_{\bar{\mathbf{j}}} x_M(\bar{\mathbf{j}})\exp\left(-2pi\bar{\mathbf{k}}^t CD^{-1}R\bar{\mathbf{j}}\right).$$

Let $\bar{\mathbf{m}} = C^t\bar{\mathbf{k}}$ and $\bar{\mathbf{n}} = R\bar{\mathbf{j}}$, then
Let C_{ij} and R_{ij} blocks of block-matrices

$$\begin{aligned}
\hat{x}(C^{-t}\bar{\mathbf{m}}) &= \sum_{\bar{\mathbf{n}}\in L_{I/D}} x_M(R^{-1}\bar{\mathbf{n}})\exp\left(-2pi\bar{\mathbf{m}}^t D^{-1}\bar{\mathbf{n}}\right) \\
&= \sum_{\bar{\mathbf{n}}} x_M(R^{-1}\bar{\mathbf{n}})\exp\left(-2pi\left[\mathbf{m}^t, \mathbf{m}_1^{\ t}\right]\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{n} \\ \mathbf{n}_1 \end{bmatrix}\right) \\
&= \sum_{\bar{\mathbf{n}}} x_M(R^{-1}\bar{\mathbf{n}})\exp\left(-2pi\left[\mathbf{m}^t P^{-1}\mathbf{n}\right]\right)\times \\
&\qquad \exp\left(-2pi\left[\mathbf{m}_1^{\ t} Q^{-1}\mathbf{n}_1\right]\right).
\end{aligned}$$

C^{-t} and R^{-1} respectively, and

$$\begin{aligned}
&\hat{x}(C_{11}\mathbf{m} + C_{12}\mathbf{m}_1) = \\
&= \sum_{\mathbf{n}\in L_{I/P}} \left\{ \sum_{\mathbf{n}_1\in L_{I/Q}} x(R_{11}\mathbf{n} + R_{12}\mathbf{n}_1)\exp\left(-2pi\left[\mathbf{m}_1^{\ t} Q\mathbf{n}_1\right]\right)\right\} \\
&\qquad \times\exp\left(-2pi\left[\mathbf{m}^t P^{-1}\mathbf{n}\right]\right)
\end{aligned}$$

which is the transform without twiddle factor. To illustrate the above consider from the example the matrix

$$N = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}.$$

This matrix can be expressed as N=PQ where

$$P = \begin{bmatrix} 1 & 0 \\ 2 & 8 \end{bmatrix} \text{and } Q = \begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix}.$$

Then the blocks of the block-matrices C^{-t} and R^{-1} are

$$R_{11} = \begin{bmatrix} 4 & 0 \\ 2 & -1 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 1 & 0 \\ 2 & 8 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix} \text{and } C_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The transform is illustrated in Figure 1. The factorization N=PQ could be determined by the next theorem.

Theorem: Let D an m×n diagonal matrix and N any m×n matrix. Then there always exist unitary matrices U, m×m matrix and V, n×n matrix, such that N=UDV. Once the factorization N=UDV is determined, set $D=D_1D_2$ then $N=UD_1D_2V$ and $P=UD_1$ and $Q=D_2V$.

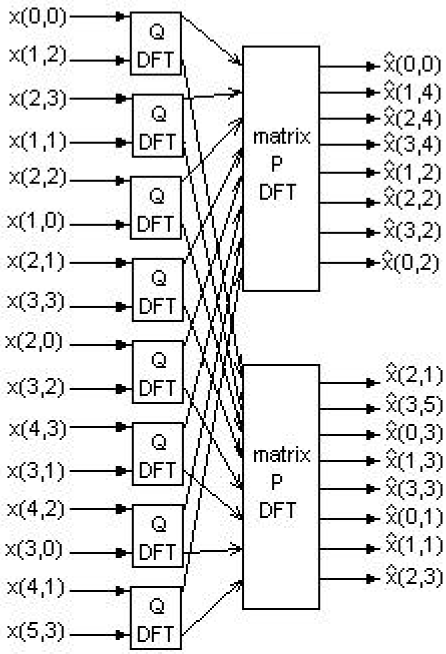

Figure 1

4. Algorithm

The following algorithm describes the procedure presented above.

0. The user provides the matrix N.
1. N is factorize as N=PQ.
   1.1 Matrices U, D and V are computed, such that N=UDV.
   1.2 The user provides matrices D_1 and D_2 such that $D=D_1D_2$.
   1.3 Set $P=UD_1$ and $Q=D_2V$.
2. Compute matrices A and B such that AP+QB=I.
3. Compute matrices R^{-1} and C^{-t}.
4. Output: Matrices P, Q , $R_{11}^{-1}, R_{12}^{-1}, C_{11}^{-t}$ and C_{12}^{-t}.

When either one or both matrices P and Q admit factorizing, the above algorithm is applied recursively.

5. Summary

To obtain the transform without twiddle factors the matrix M is set as

$$M = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}.$$

Then matrices P, Q, R, C are computed such that

$$RMC = D = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}.$$

Therefore $M=R^{-1}DC^{-1}$. Now, if $P=P_1P_2$ and $Q=Q_1Q_2$ the same procedure is applied to matrix D. That is, the same algorithm is applied to

$$\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

The inverse of the computed matrices R_2 and C_2 , return by this application of the algorithm are multiplied by the embedded matrices

$$\begin{bmatrix} R^{-1} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \text{and} \begin{bmatrix} C^{-1} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

to get the new permutation of the sample, for the twiddle factor less MDFFT.

A Matlab toolbox was created to provide a code for computing the factorization N=PQ for any matrix $m\times n$. It also provides a code for the algorithm described above.

Conclusion

Twiddle factor elimination in MDFFT was achieved by using change of basis in the domain of the sample and the transform. These results provide the right parameters for the design of highly efficient MDFFT for data indexed by lattices.

References

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