Fast Fourier Transform via Projections on Subspaces of a Multiresolution Analysis

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Abstract
Fast Fourier Transform (FFT) algorithms have been investigated since Cooley and Tukey published the first one in 1965. Most of them emphasize on the efficiency but not on the accuracy. In image analysis multiresolution representations are very effective for analyzing the information content of images. In the case of a function in $L^2(R)$, these representations are given by its projections on the subspaces of a multiresolution analysis (MRA). And they approximate the function at a given resolution. In this work these multiresolution approximations are used to compute the discrete Fourier transform of some functions. By choosing the MRA associated with spaces of polynomial splines, and selecting the proper parameters, it is possible to achieve any prescribed accuracy. This method yields estimates of the computation errors in the FFT which are not available via any other approach.

1. INTRODUCTION

The Fast Fourier Transform (FFT) algorithm requires sampling on an equally spaced grid. In this work we study a simple approach, due to Beylkin, for evaluation of the Fourier transform of some functions based on projecting such functions on a subspace of a multiresolution analysis. In this way we obtain the algorithm in [1], which consists of three steps. The first step is the aforesaid projection. The second step is the same as in all algorithms of this type and involves the usual FFT. The third step is a correction step which involves multiplying values at each frequency by a pre-computed factor. In the construction presented in this work, it is chosen the MRA associated with spaces of polynomial splines. This allows us to use properties of the Battle- Lemarié scaling function, while computing projections only with B-splines. These are probably the simplest functions with small supports that are most efficient for both software and hardware implementation.

2. NOTATION

The **Fourier transform** of $f \in L^2(R)$ is defined by

(1.1) $\hat{f}(x) = \int_{-\infty}^{\infty} f(x) e^{-2pixx} dx$.

The **convolution** $f * g$ of two functions $f, g \in L^1(R)$ is defined by

(1.2) $f * g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy$.

The **translation** $T_a f$, where $a \in R$, of a function f, is defined by

(1.3) $T_a f(x) = f(x-a)$.

3. MULTIRESOLUTION ANALYSIS

A function f is said to generate a **multiresolution analysis** (MRA) of $L^2(R)$ if it generates a nested sequence of closed subspaces V_j where

(2.1) $$V_j = clos_{L^2(R)} span\{f_{kj} : k \in Z\}$$

and

(2.2) $$f_{kj}(x) = 2^{-\frac{j}{2}} f\left(2^{-j} x - k\right)$$

such that

1o $\ldots \subset V_1 \subset V_0 \subset V_{-1} \subset \ldots$

2o $clos_{L^2(R)}\left(\bigcup_{j\in Z} V_j\right) = L^2(R)$

3o $\bigcap_{j\in Z} V_j = \{0\}$

4o $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j-1}$

5o $f(x) \in V_0 \Leftrightarrow f(x-n) \in V_0$ for all $n \in Z$

6o $\{f_{k0} : k \in Z\} = \{T_k f : k \in Z\}$ is an orthonormal basis of V_0.

In this case the sequence $\{V_j\}_{j\in Z}$ is said to be a MRA with scaling function f.

Because of 4o and 6o, for each $j \in Z$, $\{f_{kj} : k \in Z\}$ is an orthonormal basis for V_j. Furthermore, the orthonormality of the $T_k f$ in 6o can be relaxed. Indeed, it is sufficient that the set $\{f_{k0} : k \in Z\}$ constitutes a bounded unconditional basis (or Riesz basis) of V_0. If this is the case, the scaling function j of the MRA is defined via its Fourier transform by

(2.3) $$\hat{j}(x) = \frac{\hat{f}(x)}{\sqrt{a(x)}}$$

where

(2.4) $$a(x) = \sum_{l\in Z} \left|\hat{f}(x+l)\right|^2 .$$

In such a way, $\{T_k j : k \in Z\}$ is an orthonormal basis for V_0. It is also possible to start the construction of an MRA from an appropriate choice for the scaling function f. This is possible if

(2.5) $$f(x) = \sum_{k\in Z} c_k f(2x-k)$$ where

$$\sum_{k\in Z} |c_k|^2 < \infty ,$$

(2.6) $\{f(x-k) : k \in Z\}$ is a bounded unconditional basis (or Riesz basis) of V_0 as defined by (2.1) and (2.2), and

(2.7) $$\int_{-\infty}^{\infty} f(x)dx \neq 0 .$$

In this case, if $\{f(x-k) : k \in Z\}$ is not an orthonormal set, the scaling function is the function j defined by (2.3) and (2.4).

4. SPACES OF POLYNOMIAL SPLINES

For each positive integer m, the space S_m of cardinal splines of order m and knot sequence Z is the collection of all functions f such that $f, f', \ldots, f^{(m-1)}$ are continuous everywhere and the restrictions of f to any interval $[k, k+1)$, $k \in Z$, are polynomials of degree at most m. Now we define V_0^m as the smallest closed subspace of $L^2(R)$ that contains $S_m \cap L^2(R)$, that is

(3.1) $$V_0^m = clos_{L^2(R)}\left(S_m \cap L^2(R)\right).$$

The central B-spline of order m, denoted by $b^{(m)}$, is defined recursively by (integral) convolution:

(3.2) $$b^{(m)}(x) = b^{(m-1)} * b^{(0)}(x)$$

where $b^{(0)}$ is the characteristic function of the interval $[-1/2, 1/2)$. The B-splines are symmetric bell-shaped functions; they have a number of attractive properties. First, they are compactly supported. Second, they have a simple analytical form in both the time and the frequency domain. The values of the B-splines may be obtained using recursion over the spline order. If

$$(\bullet) = \frac{\frac{(m+1)}{2} + x}{m} b^{(m-1)}\left(x + \frac{1}{2}\right) + \frac{\frac{(m+1)}{2} - x}{m} b^{(m-1)}\left(x - \frac{1}{2}\right)$$

then

(3.3) $$b^{(m)}(x) = (\bullet) \text{ for } m = 1,2,\ldots$$

The Fourier transform of a central B-spline can be found from equation (3.2) and is given by

(3.4) $$\hat{b}^{(m)}(x)=\left(\frac{\sin px}{px}\right)^{m+1}.$$

With a slightly different definition of cardinal B-splines, the proof of the following proposition can be found in [2].

Proposition 1

(i) $Supp b^{(m)}=[-(m+1)/2,(m+1)/2]$

(ii) The two-scale difference: when m is odd

$b^{(m)}(x)=\sum_{|k|\le\frac{m+1}{2}} c_k^{(m)} b^{(m)}(2x-k)$ where

$$c_k^{(m)}=\frac{1}{2^m}\binom{m+1}{k+\frac{m+1}{2}}.$$

(iii) The set of translations $\{b^{(m)}(x-k): k\in Z\}$ is a bounded unconditional basis (or Riesz basis) of V_0^m.

(iv)

$$a^{(m)}(x):=\sum_{l=-\infty}^{\infty}\left|\hat{b}^{(m)}(x+l)\right|^2=\sum_{l=-m}^{m} b^{(2m+1)}(l)e^{2pilx}.$$

The total positivity of the cardinal splines along with Proposition 1 allow us to construct an MRA with starting point the central B-spline of order m, $b^{(m)}$. According to (2.3) the scaling function - called a Battle-Lemarié scaling function- is defined by

(3.5) $$\hat{j}^{(m)}(x)=\frac{\hat{b}^{(m)}(x)}{\sqrt{a^{(m)}(x)}}.$$

5. MAIN RESULTS

The proof of the two propositions in this section can be found in [1].

Proposition 2

Let $\{V_j^{(m)}\}_{j=-\infty}^{\infty}$ be the MRA generated by the cardinal B-spline of order m, let $f\in L^2(R)$ and f_k its projection on $V_j^{(m)}$, $j<0$, that is

(4.1) $$f_k=\int_{-\infty}^{\infty} f(x)b_{kj}^{(m)}(x)dx.$$

Let $F(x)$ be the Fourier series given by

(4.2) $F(x)=\sum_{k\in Z} f_k e^{-2pixk}$. Then

(4.3)

$$2^{\frac{j}{2}}\frac{F(x)}{\sqrt{a^{(m)}(x)}}=\sum_{l\in Z}\hat{f}\left(2^{-j}(x+l)\right)\hat{j}^{(m)}(x+l).$$

Remarks (4.4)

a) From (4.3) we can write

$$2^{\frac{j}{2}}\frac{F(x)}{\sqrt{a^{(m)}(x)}}=\hat{f}\left(2^{-j}x\right)\hat{j}^{(m)}(x)+l(x)$$

where

$$l(x)=\sum_{l=\pm1,\pm2,\ldots}\hat{f}\left(2^{-j}(x+l)\right)\hat{j}^{(m)}(x+l).$$

b) $\hat{j}^{(m)}$ is a low-pass filter [4].

c) From the precedent remarks it is clear that $2^{\frac{j}{2}}\frac{F(x)}{\sqrt{a^{(m)}(x)}}$ is an approximation of $\hat{f}\left(2^{-j}x\right)$ for values of x near the zero frequency. For example, for $m=23$, $\hat{j}^{(m)}(x)\approx1$ and $\hat{j}^{(m)}(x\pm1)\approx0$ for $|x|\le\frac{1}{4}$.

The values of $\hat{j}^{(m)}(x+l)$ are also very small for $l=\pm2,\pm3,\ldots$ and the same x.

Definition

For $a>0$ and $j\in Z$ let us define

(4.5) $$E_\infty=\frac{\sup_{|x|\le a}\left|2^{\frac{j}{2}}\frac{F(x)}{\sqrt{a^{(m)}(x)}}-\hat{f}\left(2^{-j}x\right)\right|}{\sup_{|x|\le a}\left|\hat{f}\left(2^{-j}x\right)\right|}.$$

Proposition 3

(i) If $f\in L^2(R)$ and $\hat{f}$ is bounded, for $o<a<1$ the error E_∞ in approximating the Fourier transform $\hat{f}\left(2^{-j}x\right)$ by a periodic function $2^{\frac{j}{2}}\frac{F(x)}{\sqrt{a^{(m)}(x)}}$ for $|x|<a$, is such

that

(4.6) $E_\infty \le \frac{1}{2\hat{f}^{(m)}(a)-1}\left[1-\hat{f}^{(m)}(a)+q(a)\right]$

where

(4.7)

$$q(a)=\frac{1}{C_{\hat{f}}(0,a)}\sum_{l=\pm1,\pm2,\ldots} C_{\hat{f}}(l,a)\left(\frac{a}{|l|-a}\right)^{m+1}$$

and

$$C_{\hat{f}}(l,a)=\sup_{|x|\le a}\left|\hat{f}\left(2^{-j}(x+l)\right)\right|.$$

(ii) For $e>0$ we may choose m, the order of the central B-spline, and the parameter $a>0$ so that

$$E_\infty \le e \qquad \text{for} \qquad |x|<a .$$

Remark (4.8)

From (4.7), since $\hat{f}$ is bounded, we may choose m without any dependence on j.

6. APPLICATIONS

As an example let us consider a function $f \in L^2(R)$ with support in an interval strictly contained in $[0,1]$; that is $Suppf=[a,b]$ with $0<a<b<1$. Indeed, we can assume this for any function of compact support, without loss of generality. Furthermore, we assume that $\hat{f}$ is bounded. According to Remark (4.8), given $e>0$ and $a=\frac{1}{4}$, we may choose m such that $\hat{f}\left(2^{-j}x\right)=2^{\frac{j}{2}}\frac{F(x)}{\sqrt{a^{(m)}(x)}}$ for $|x|<a$ and any j, with accuracy e. For convenience we only consider splines of odd order. Then, from (4.2) we may write

(5.1) $\hat{f}\left(2^{-j}x\right)=2^{\frac{j}{2}}\frac{1}{\sqrt{a^{(m)}(x)}}\sum_{k\in Z} f_k e^{-2pixk}$ for $|x|<a$ and any j (with accuracy e). Since f is zero outside $[a,b]$, and $0<a<b<1$, we may always choose j in such a way that $f_k=0$ for $k<0$ and $k\ge L$, where $L=2^{-j}$. It is due to the fact that

(5.2)

$$Suppb_{kj}^{(m)}=\left[-2^j\frac{(m+1)}{2}+k2^j, 2^j\frac{(m+1)}{2}+k2^j\right]$$

from part (i) of Proposition 1 and (2.2). Thus, from (4.1)

(5.3) $$f_k=\int_{-2^j\frac{(m+1)}{2}+k2^j}^{2^j\frac{(m+1)}{2}+k2^j} f(x)b_{kj}^{(m)}(x)dx .$$

For such a j the series in (5.1) is a finite sum and we obtain

(5.4) $$\hat{f}(l)=\frac{L^{\frac{1}{2}}}{\sqrt{a^{(m)}\left(l/L\right)}}\frac{1}{L}\sum_{k=0}^{L-1} f_k e^{-2pik\frac{l}{L}},$$

$-\frac{L}{4}\le l\le\frac{L}{4}$, which may be evaluated using the FFT. As a result we have a simple algorithm. The algorithm consists of three steps:

1. computing integrals in (5.3)
2. computing the sum in (5.4) via FFT
3. multiplying by the factor $1/\sqrt{a^{(m)}\left(l/L\right)}$ in (5.4).

REFERENCES

[1] G. Beylkin. *On the Fast Fourier Transform of Functions with Singularities.* PAM Report 195, 1994.

[2] C. Chui. *An Introduction to Wavelets*. Academic Press, 1992.

[3] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Reg. Conf. Ser. Appl. Math. 61, Philadelphia: Soc. Ind. Appl. Math., 1992.

[4] S. Mallat. *A Theory for Multiresolution Signal Decomposition: the Wavelet Representation*. IEEE Trans. PAMI 11, pp. 674-693, 1989.

[5] H. J. Weaver. *Applications of Discrete and Continuous Fourier Analysis*. John Wiley and Sons, 1983.