A Computational Kronecker-core Array Algebra SAR Raw Data Generation Modeling System

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Abstract

This work presents a new methodology for synthetic aperture radar (SAR) raw data generation modeling system based on fast and efficient computations of cross-ambiguity functions, acting as point target response functions, and multidimensional cyclic correlations of these functions with selected target reflectivity density functions. Computational Kronecker-core array (CKA) algebra, a branch of finite dimensional multilinear algebra, is being utilized as a tool for the analysis, design, implementation and modification of multidimensional fast Fourier transform (FFT) based algorithms for the efficient computation of the cross-ambiguity functions and the multidimensional cyclic correlation operations. A MATLAB® environment was created for the implementation of the complete SAR raw data generation modeling system. A new DSP board based modeling system is currently being implemented for hardware in the loop simulations at the SAR data receiving facility of the University of Puerto Rico.

1 Introduction

This work deals with the fundamental issue of the fast and efficient treatment of microwave remote sensed data in order to extract information important to a surveillance user. Great advances in active sensor technology, communications, and signal processing technology are demanding new computational theories, methods, and techniques to improve our rapid awareness of our physical sensory reality. For the particular case of SAR systems, this implies fast and efficient means for image formation and rendering from raw data[1]. The identification of enhanced raw data generation techniques will certainly contribute to improve at SAR image formation processes. The work presented here concentrates on the formulation of a system environment for the algebraic modeling of SAR raw data generation. Special attention is given to the algebraic modeling of point surface response data using point estimates of discrete cross-ambiguity surface computations. The enhancements and understandings of point surface response functions center on the efficient computation of cross-ambiguity functions. We present the cross-ambiguity function computations under a Weyl-Heisenberg systems setting, following the work of R. Tolimieri[2]. We also present the indirect calculation of multidimensional cyclic correlation operations using new variants of multidimensional FFT algorithms. The advantage of these new FFT variants over conventional formulations is that additive group theoretic properties of multidimensional input-output indexing sets are used for their mathematical formulations, reducing their computational complexity and improving their implementation performance. Also, these FFT variants do not restrict the lengths of the multidimensional data sets to powers of two. There exist many formulations of fast algorithms for computing the multidimensional discrete Fourier transform (DFT). Computational Kronecker-core array (CKA) algebra, a branch of finite dimensional multilinear algebra, is used as a language to identify similarities and differences among various FFT algorithm variants as well as for the creation of new variants. Each multidimensional DFT computation is expressed in matrix form. The multidimensional DFT matrix, in turn, is decomposed into a set of factors, called functional primitives, which are individually identified with underlying software computational constructs. It is in this identification process where the language of CKA algebra becomes instrumental. For a given hard-

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ware computational structure and multidimensional DFT matrix, there are many FFT algorithm variants which can map to this target machine. The language of CKA algebra aids in this mapping effort by identifying the more computational efficient FFT variants and thus reducing the computational effort.

2 Kronecker Signal Algebra

We introduce in this section some mathematical concepts which are useful in describing the work. First, the concept of tensor or Kronecker product of two matrices and then some basic ideas of Kronecker array signal (KAS) algebra, a branch of finite dimensional multilinear algebra. Let A and B be any two matrices. The Kronecker product of A and B is given by $A \otimes B = [a_{k\ell} \cdot B]_{k,\ell \in Z/R}, Z/R = \{0, 1, \cdots, R-1\}$. Here, we have assumed A to be a square matrix of order R. If B is also a square matrix of order, say, S, then the order of $A \otimes B$ is $R \cdot S = N$. Let A and C be $R \times R$ matrices, and B and D be $S \times S$ matrices. Next, form the Kronecker products $A \otimes B$ and $C \otimes D$. Through direct matrix multiplication we can show that $(A \otimes B)(C \otimes D) = AC \otimes BD$. If we denote I_R, I_S as identity matrices of order R and S, respectively, we have $(A \otimes B) = (A \otimes I_S)(I_R \otimes B)$. From this expression we can see that the action of computing with the matrix $(A \otimes B)$ can be performed in two stages: An action for the computation of $(I_R \otimes B)$, followed by an action for the computation of $(A \otimes I_S)$. Let the N-point discrete Fourier transform (DFT) of a one-dimensional discrete, complex, array signal $x[n]$, of length N, be defined by $(\widehat{x})[k] = \sum_{n \in Z/N} x[n] \omega_N^{kn}; \quad k \in Z/N$, where $w_N = e^{-j\frac{2\pi}{N}}$, and $j = \sqrt{-1}$. Written in matrix form, we have $(\widehat{x}) = F_N \cdot x$, $F_N = \left[\omega_N^{kn}\right]_{k,n \in Z/N}$. We call F_N a matrix representation of the DFT operator. In the same manner, the two-dimensional discrete Fourier transform of an $N_1 \times N_2$ discrete complex array signal $x[n_1, n_2]$ is defined by

$$(\widehat{x})[k_1, k_2] = \sum_{n_1 \in Z/N_1} \sum_{n_2 \in Z/N_2} x[n_1, n_2] \omega_{N_1}^{k_1 n_1} \omega_{N_2}^{k_2 n_2};$$

$$\text{where } k_1 \in Z/K_1,\ k_2 \in Z/K_2.$$

Also, $\omega_{N_1} = e^{-j\frac{2\pi}{N_1}}$ and $\omega_{N_2} = e^{-j\frac{2\pi}{N_2}}$. Let $F_{N_1 \times N_2}$ denote a matrix representation of the two-dimensional discrete Fourier transform operator acting on an $N_1 \times N_2$ complex signal array $x[n_1, n_2]$. Through direct matrix multiplication we can show that

$$F_{N_1 \otimes N_2} = (F_{N_1} \otimes F_{N_2})$$

$$F_{N_1 \otimes N_2} = (I_{N_1} \otimes F_{N_2})(F_{N_1} \otimes I_{N_2})$$

$$F_{N_1 \otimes N_2} = (F_{N_1} \otimes I_{N_2})(I_{N_1} \otimes F_{N_2})$$

If U_1, U_2, V_1, and V_2 are linear spaces over the complex field $\mathbb{C}$, and $\mathcal{T}_i : U_i \to V_i$, $i = 1, 2$, are linear operators acting over these spaces; then, $(\mathcal{T}_1 \otimes \mathcal{T}_2) : U_1 \otimes U_2 \to V_1 \otimes V_2$, termed the Kronecker product of the transformations $\mathcal{T}_1$ and $\mathcal{T}_2$, is the linear transformation satisfying the following condition:

$$(\mathcal{T}_1 \otimes \mathcal{T}_2) \{u_1 \otimes u_2\} = \mathcal{T}_1 \{u_1\} \otimes \mathcal{T}_2 \{u_2\}$$

for all $u_i \in U_i$, $i = 1, 2$. $T_1 \otimes T_2$ is a Kronecker product of matrices $A \otimes B$; where, A and B are the matrix representations of the operators $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, conditioned on bases selection criteria[3]. We call the elements of the linear spaces by the names of vector array signals, array signals, or, simply, signals. The linear spaces are turned into linear algebras by introducing a vector array signal binary multiplication operation invoked by the circular or cyclic convolution. Thus, we are interested in the linear spaces of the form $V = M_{R,S}(\mathbb{C})$, as well as linear $\mathcal{T}$, such that

$$\mathcal{T} : M_{R,S}(\mathbb{C}) \to M_{R,S}(\mathbb{C}),$$

In general, we can say that if we have $V = M_{R,S}(\mathbb{C})$; then, the linear space $M_{R,S}(\mathbb{C})$ can then be represented as the Kronecker product

$$M_{R,S}(\mathbb{C}) = M_{R,1}(\mathbb{C}) \otimes M_{S,1}(\mathbb{C})$$

in which the Kronecker mapping is the dyad mapping $x \otimes y = xy^T$. The signal algebras manifest themselves when we invoke multidimensional array cyclic convolutions as array binary multiplication operations.

Let $Z_N = Z/N = \{0, 1, 2, \cdots, N-1\}$. A one-dimensional array signal x, of length N, is said to be periodic, modulo R if R is a divisor of N; that is, $N = R \cdot S$ and $x[a + bR] = x[a], a \in Z_R, b \in Z_S$. A one-dimensional array signal x, of length N, is decimated modulo R if R is a divisor of N; that is, $N = R \cdot S$ and $x[a] = 0, x[a + b \cdot R] = x[a] = 0, a \in Z_R; b \in Z_S$. These observations are very important when considering the additive group theoretic properties (coset decompositions) of the input/output indexing sets of the matrix representation of *unitary* operators $\mathcal{T}$ in linear algebras V. The matrix representation of the operators can be decomposed into a set of factors which we term *Kronecker functional primitives* and are, basically, sparse matrices. This decomposition process usually leads to efficient algorithms for the action of operators. Of particular importance to us is the ubiquitous discrete Fourier transform operator.

2.1 Operators on $L(Z/N)$

The set of all one-dimensional array signals $f : Z/N \rightarrow \mathbb{C}$ forms a linear space which we denote by $L(Z/N)$. The set $L(Z/N)$ is isomorphic to the N-dimensional complex linear space $\mathbb{C}^N$. The set of N, N-point array signals $\{\delta_{\{k\}} : k = 0, 1, \ldots, n-1\}$, where $\delta_{\{k\}}[j] = 1, k = j$ forms a basis for the space $L(Z/N)$ which we call the standard basis. We introduce the shift operator S_N over the space $L(Z/N)$. This operator is the central component in the characterization of shift-invariant, finite impulse response (FIR) operators commonly used for filtering operations. Let the operator S_N over the space $L(Z/N)$ be defined in the following manner. $S_N : L(Z/N) \longrightarrow L(Z/N)$, where $\delta_{\{k\}} \longmapsto S_N\delta_{\{k\}} = \delta_{\{k+1\}}$. Using $\langle f, \delta_{\{k\}}\rangle = f[k] = f_k$ as an orthogonal projection operation, we write $f = \sum_{0\leq k<N}\langle f, \delta_{\{k\}}\rangle\delta_{\{k\}}$. Allowing F_N to operate on f gives $F_N(f) \equiv \widehat{f} = F_N(\sum_{j\in Z/N} f_j\delta_{\{j\}})$. After linearity, $\sum_{j\in Z/N} f_jF_N\delta_{\{j\}} = \sum_{j\in Z/N} f_j\chi_j^*$. To characterize cyclic, finite impulse response (FIR) operators, we start by identifying the vector array signal obtained by letting the FIR operator T_h act on the unit sample array signal δ. Since any N-th order vector array signal f can be written as a linear combination of shifted versions of δ, knowing the response $T_h(\delta)$ will help in determining $T_h(f)$. We call the unit sample response or impulse response of the system T_h the result obtained by applying T_h to the unit sample sequence δ, which sometimes is called the impulse signal. Thus, we have $T_h(\delta) = \sum_{0\leq m<M} h[m]S_M^m\delta_{\{0\}}$ or $\sum_{0\leq m<M} h[m]\delta_{\{m\}} = h$. The unit sample response of an FIR operator T_h is the array signal h. For any given vector signal $f \in L(Z/M)$, we can always write $f = \sum_{k\in Z/M} f[k]S_M^k\delta$ or $\sum_{k\in Z/M} f[k]\delta_{\{k\}}$. Evaluating f at $j \in Z/M$ results in $f[j] = \sum_{k\in Z/M} f[k]\delta_{\{k\}}[j]$ or

$\sum_{k\in Z/M} f[k]\delta[j-k]$. The indexing set $A = Z/N = Z_N = \{0, 1, \ldots, N-1\}$ forms an abelian group with modulo N addition as the internal binary operation. Its dual is $\widehat{A} = \{{}^N\chi_{\{k\}} : k \in Z/N\}$, with $\chi_{\{k\}} : Z/N \longrightarrow \mathbb{C}$, with $[m] \longmapsto {}^N\chi_{\{k\}}[m] = e^{+2\pi jk\cdot(m)/N}$, $j = \sqrt[2]{-1}$. When no ambiguities arise, we drop the superscript N from the expression ${}^N\chi_{\{k\}}$. The value ${}^N\chi_{\{1\}}[1]$ is usually written as $\omega_N = e^{-2\pi j/N}, j = \sqrt[2]{-1}$. The functions $\chi_{\{k\}}$ are usually termed exponential sequences, characteristic sequences, or, simply, characters. Given an N-point impulse response signal, $h \in L(Z/N)$, and an input vector array signal, $x = \chi_{\{k\}}$, the output y, after acting with T_h, becomes $y = \sum_{j\in Z/N} h[j]S_N^j\chi_{\{k\}}^*$ or $T_h\{\chi_k^*\}$. Another important operator is the cyclic reflection operator, denoted by the symbol R_N. Its action on the linear space $L(Z/N)$ is described by $R_N : L(Z/N) \longrightarrow L(Z/N)$, with $(f) \longmapsto R_Nf = f^{(-)}$. Here,$(R_Nf)[k] = f^{(-)}[k]$ or f_{N-k}, Modulo N, and $k \in Z/N$.

3 Previous Work

New enhanced SAR imaging techniques are needed for improving our fundamental understanding of physical processes pertaining to the environment through remote surveillance. Of particular importance are studies conducted about the earth surface property characteristics for the better understanding of concepts such as monitoring of wetlands, soil moisture content, backscattering from crops, nearshore ocean surface currents, and subsurface imaging in hyperarid regions. To identify these enhanced SAR imaging techniques, a better understanding of efficient SAR raw data generation operations is needed. We follow a basic model for understanding the principles of raw data generation due to R. Blahut [4]. It presents SAR raw data as a correlation operation between the point spread function, or, as we call it, *point surface response function*, of a SAR system and the reflectivity density function depicting a physical object environment. The model also defines the cross-ambiguity function between the SAR system's transmitted and received signals as the point surface response. The work of R. Tolimieri [2] presents mathematical formulations of algorithms for the computation of the finite, discrete, cross-ambiguity function in the context of discrete Weyl-Heisenberg systems. The attributes of a point spread function in a SAR system determine, to a great extent, the better-quality of the image formation process. The books of G. Franceschetti and R. Lanari, [5], and M. Soumekh [6] present many algorithms for SAR image formation operations. Special emphasis is given in these books to the role that the DFT plays in image formation operations. An efficient parallel SAR processor using multidimensional FFT's is presented by G. Franceschetti, et al., in [7]. The tutorial work of Johnson, et. al., presents a methodology for analyzing, designing, modifying, and implementing FFT algorithms on various computing structures using Kronecker products algebra [8]. For the general formulation of multidimensional FFT algorithms using Kronecker products, we followed the work of R. Tolimieri, et. al., as presented in [9]. In [10], we presented a set of FFT algorithms to aid in the image formation operation detail, and in [11] a Java-based environment was presented for the automatic C-language source code generation of FFT algorithms.

4 Point Surface Response

Kronecker array signal (KAS) algebra has been instrumental in the analysis, design and implementation of different classes of algorithms for signal processing computing methods[12]. In this work we concentrated on the design of variants of algorithms for the computation of the finite, discrete, radar cross-ambiguity function, and their software and hardware realizations. The algorithms implementation methodology is an improvement over existing formulations. Enhancements on the methodology concentrate on group theoretic techniques applied to input/output data indexing sets in a KAS algebra and linear operator setting, on modified re-sampling techniques, and on the efficient computation of two-dimensional fast Fourier transforms. The algorithms have been tested in MATLAB and are currently being ported to DSP computing units[13]. Section 2.1, above, presented some of the linear operators used to compute the finite, discrete, radar cross-ambiguity function as a composition of these operators. As it was point out in Section 2, above, multidimensional FFT algorithms can be expressed as Kronecker products of lower dimensional FFT's. We used this approach throughout this work. We proceed to define the basic formulation for the finite, discrete, radar cross-ambiguity used throughout this work. Let f, g be functions on $L(Z/N)$, the linear, complex space of all N-point, one-dimensional, vector array signals. The finite, discrete, radar cross-ambiguity function $A(f,g)[a,b]$ is defined on the cartesian product indexing set $Z/N \times Z/N$ as $A(f,g)[m,k] = \sum_{n \in Z/n} f[n] \cdot g^*[n+m] e^{-j\frac{2\pi}{N} \cdot k \cdot n}$.We use $g_{m,k}[n] = g[n+m] e^{j\frac{2\pi}{N} \cdot k \cdot n}$ to write $A(f, g_{m,k})[p,q] = e^{-j\frac{2\pi}{N} \cdot k \cdot n} A(f,g)[m+p, k+q]$. This expression introduces the study of various additive group theoretic techniques on the input output indexing sets of the computation as well as time-frequency analyses[14].

5 SAR Raw Data Modeling

The contribution presented here on this on going work centers on the development of a theoretical/experimental framework for the efficient modeling of SAR raw data through the efficient computation of the point surface response function and multidimensional cyclic convolution operations. This framework has its foundation on the principles of Kronecker array signal (KAS) algebra to allow a development of a computational signal processing environment where computing methods for radar signal processing are mathematically formulated as sets of algorithms. In this regard, a computational signal processing environment (CSPE) can be thought of as the aggregate of the following components: A set of input signals, a set of output signals, a set of operators, a set of composition rules for these operators, a set of actions rules for the operators to act on input data in order to produce output data, and a user interface. The algorithms are then coded in an identified computing language for a given computing structure. The KAS algebra serves as a language tool to aid in the mapping of the signal processing computing methods to the digital computing machines. We are seeking modalities of computing representations that will serve as interfaces between different computing algebras, and between the KAS algebra and a target digital computing machine language. These modalities of computing representations will assist in automating the process of mapping digital signal processing (DSP) computing methods and models to advanced computing architectures and scalable, reconfigurable computing units, allowing for more low cost, efficient, on board, and real time SAR processing operations in the future. At the present time we establish a linked table of correspondences of relations between Kronecker functional primitives and digital machine computing constructs through computer language coding processes. To this end, new variants of FFT algorithms were constructed tailored to fast computation of the finite cross-ambiguity function. This algorithms are currently being implemented on DSP computing units. It is important to point out that implementing FFT algorithms on advanced computing architectures for optimal results is not a trivial matter. Several reasons can be provided to support this claim. One reason is that, for a specific digital machine or computing hardware structure (CHS), many mathematical variants of a given FFT algorithm (these variants can be viewed as forming a set, this set defined here as an FFT mathematical framework (FMF)), must be coded in the mapping process in order to reach an optimal implementation. An FFT implementation is a realization of a mapping of a mathematical formulation an FFT algorithm through the stages of the generation of the source, object, and execution codes, or their equivalent representations. Also, for a particular algorithm variant, the performance changes from machine to machine, and it is desirable to identify on which machine it performs best. A third important reason is that some observed inherent software and hardware attributes cannot always be expressed as parameters in mathematical formulations or observed through performance evaluation metrics.

6 Summary and Conclusions

The computing approach presented here is based on the successful use of cross- ambiguity functions, placed in a Weyl-Heisenberg computational framework, as point surface response functions for nonlinearly modulated, time-frequency collocated, transmitted signals. These functions were correlated with prescribed target reflectivity density functions to produce desired object domain results. Kronecker-core array signal algebra, a branch of finite dimensional multilinear algebra, was utilized as a mathematical tool-language for formulations of multidimensional fast Fourier transform (FFT) algorithms, prevalent in all cross-ambiguity functions as well as multidimensional correlation computations. An interactive Java-based stand-alone utility was designed and developed to assist in this work through automatic software source code generation of FFT algorithms from Kronecker algebra formulations.

This alternative modality of using Kronecker algebra for mapping multidimensional FFT's to advanced hardware computing structures is showing promising results for allowing to establish identifications between parallel-distributed computing constructs and the mathematical expressions named by us functional primitives. Algorithms were formulated in this work as factored compositions of functional primitives.

This method will, hopefully, contribute to make inferences in estimating computing performance results of certain classes of large-scale multidimensional signal processing algorithms from their mathematical formulations in Kronecker products form, effecting, this way, a potential impact at the essential hardware implementation scales needed when dealing with fundamental understandings of planetary surface energetics and dynamics.

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