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RADAR AMBIGUITY FUNCTIONS AND GROUP THEORY*

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Abstract. P. M. Woodward in the early 1950's introduced a mapping from a radar signal f to a function of two variables $W(f)$, called the ambiguity function, that plays a central role in the radar design problem. We may think of W as a nonlinear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$. The description of the range of W has been an open problem. This paper provides, in terms of special functions in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$ a fairly complete description of $W(L^2(\mathbb{R}))$. We show also that $W(L^2(\mathbb{R}))$ is a closed subset of $L^2(\mathbb{R}^2)$ and if $W(f)+W(g)=W(h), f, g, h\in L^2(\mathbb{R})$ then $f=\lambda g$, λ a constant.

1. Introduction. Because radar computations are not familiar to the general mathematical community, we have begun this introduction with a brief simplified version of how ambiguity functions are used in radar computations. We will follow this with the familiar listing of what we consider our important new results.

Let $X_1,\cdots,X_N$ be N objects or targets and assume the radar is at the origin. Let $r_j(t)$, $j=1,\cdots,N$, denote the range (distance from the origin) of X_j and $v_j(t)$ denotes the velocity of X_j at time t. The problem is to transmit an electromagnetic wave or pulse for $-T<t<T$ and from the echo determine the quantities $r_j(0)$ and $v_j(0)$, $j=1,\cdots,N$. Let $s(t)$ denote the pulse, where $s(t)$ is real valued, and let $e(t)$ denote the echo.

We will now briefly outline how information is extracted from $e(t)$. The computational process depends on a "representation" of $s(t)$ and some simplifying assumptions. The first step is to pass from the pulse to a complex valued function (representation) called the waveform of the pulse. If $g(t)\in L^2(\mathbb{R})$ we will use $\hat{g}(f)$ to denote the Fourier transform of g and call the variable f, frequency. Because $s(t)$ is real valued we have

$$\hat{s}(-f)=\hat{s}^*(f)$$

where we will (following electrical engineering notation) use * to denote the complex conjugate. Hence $s(t)$ is completely determined by its positive spectrum. Define

$$\Psi_s(t)=\int_0^\infty \hat{s}(f)e^{2\pi ift}\,df.$$

Then

$$\Psi_s(t)=s(t)+i\sigma(t)$$

where σ is the Hilbert transform of s. Explicitly, using principal part integrals,

$$\sigma(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}\,d\tau,$$

$$s(t)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma(\tau)}{t-\tau}\,d\tau.$$

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Using $\|f\|$, $f \in L^2(\mathbb{R})$, to denote the norm of f, we have

$$\|\Psi_s(t)\|^2 = 2\|s(t)\|^2.$$

It is customary to call $\|f\|^2$ the "energy" of the signal f. For the rest of the motivational discussion we will assume $\|\Psi_s(t)\|^2 = 1$, $t_0 = \int_{-\infty}^{\infty} t|\Psi_s(t)|^2 dt < \infty$ and

$$f_0 = \int_{-\infty}^{\infty} f|\hat{\Psi}_s(t)|^2 df < \infty.$$

It is usual to call t_0 the epoch and f_0 the carrier frequency.

DEFINITION. The waveform $u_s(t)$ of the pulse $s(t)$ is defined by

$$u_s(t) = \Psi_s(t+t_0)e^{-2\pi i f_0(t+t_0)}.$$

It follows that $s(t) = \mathrm{Re}\{\Psi_s(t)\} = \mathrm{Re}\{u_s(t-t_0)e^{2\pi i f_0 t}\}$, where $\mathrm{Re}\{\cdot\}$ denotes the real part of the function in the bracket, and $\|u_s(t)\|^2 = 1$. The function $u_s(t)$ is "slowly varying" in the sense that its spectrum is centered about the 0-frequency.

We would like the echo $e(t)$ to be "as much like" $s(t)$ as possible. If we have one target and the physical assumptions listed later are satisfied then

$$e(t) = \mathrm{Re}\left\{e^{-2\pi i f_0 x_0}u_s(t-t_0-x_0)e^{2\pi i(f_0-y_0)t}\right\} = \mathrm{Re}\{\Psi_e(t)\}$$

where $x_0 = (2/c)r_1(0)$, $y_0 = (2f_0/c)v_1(0)$ and c is the velocity of light. Hence for one target

x_0 = time delay of the echo,
y_0 = doppler or frequency shift of echo

completely determine $r_1(0)$ and $v_1(0)$. One estimates x_0, y_0 by the following method originally suggested by P. M. Woodward [W] and motivated by probabilistic considerations. Consider

$$\Psi_{xy}(t) = e^{-2\pi i f_0 x}u_s(t-t_0-x)e^{-2\pi i y t}e^{2\pi i f_0 t}$$

and form

$$I(x,y) = \left|\int_{-\infty}^{\infty} \Psi_e(t)\Psi_{xy}^*(t)\,dt\right|^2,$$

because $\|u_s(t)\|^2 = I$, $I(x_0,y_0) = 1$ and $I(x,y) \leqq 1$ for all x,y. Thus if we plot $I(x,y)$ by light intensity on a screen the brightest point should be (x_0,y_0) and so we can determine $r_1(0)$ and $v_1(0)$ or the range and velocity of the target. It is crucial for us to observe that

$$I(x,y) = |A_u(x_0-x, y_0-y)|^2$$

where

$$A_u(x,y) = \int_{-\infty}^{\infty} u\left(t-\frac{x}{2}\right)u^*\left(t+\frac{x}{2}\right)e^{-2\pi i y t}dt.$$

We will now list our physical assumptions and then state the results for several targets.

Physical assumptions.

1. Radar cross sections of targets are independent of frequency.
2. All targets are in the far field of the radar.
3. Multiple reflecting waves among the targets are negligible.
4. The functions $r_j(t), j=1,\cdots,N$ are approximately linear for $-T<t<T$.
5. The velocity of the targets is small compared to the speed of electromagnetic propagation.

Then from several targets we have approximately $I(x,y)=M_1^2|A_u(x_1-x,y_1-y)|^2+\cdots+M_N^2|A_u(x_N-x,y_N-y)|^2$ where the M_j depend on the range and the radar cross sections of the targets.

Actually $I(x,y)$ does not determine the number of targets, their range or velocity uniquely and, of course, $I(x,y)$ depends on the form of $A_u(x,y)$. Because of this $A_u(x,y)$ is called the ambiguity function of radar.

Woodward concludes his fundamental book [W] published in 1953 with the following paragraph. (We have changed notation, but nothing else, to fit with our conventions.)

> The reader may feel some disappointment, not unshared by the writer, that the basic question of what to transmit (choice of s) remains unanswered. One might have hoped that practical requirements of range and velocity resolution in any particular problem could be sketched in an x-y diagram and the waveform $u(t)$ then calculated to satisfy the requirements. It seems that this is not possible because the form of $|A_u(x,y)|^2$ cannot be arbitrarily chosen. The precise nature of the restrictions which must be placed in $|A_u(x,y)|^2$ has not been fully investigated.

Calvin H. Wilcox [W1] in 1960 took up the detailed study of ambiguity functions and called the problem posed above by Woodward the "synthesis problem of radar design." Wilcox used only Abelian harmonic analysis in his work. However, it turns out that there is a great deal to be gained by using the representation theory of the Heisenberg group and considering ambiguity functions as special functions on the Heisenberg group. This is not surprising because of the radar uncertainty principle and the deep relation between the Heisenberg group and the Heisenberg uncertainty principle (see [Wg] and [Wyl]). The desire to use the non-Abelian results forces us to operate in a slightly more general setting then Wilcox and so we will have to give slightly different treatments of many of his results.

We will now introduce notation that we will follow for the rest of this paper. It is intentionally slightly different from that used up to now.

If $\langle f,g\rangle$ denotes the usual inner product of functions $f,g\in L^2(\mathbb{R})$, defined by

$$\langle f,g\rangle=\int_{\mathbb{R}} f(t)g^*(t)\,dt,$$

then we can write

$$\mathscr{A}(f)(u,v)=\int_{\mathbb{R}} f\left(t-\frac{u}{2}\right)f^*\left(t+\frac{u}{2}\right)e^{-2\pi ivt}\,dt \tag{1}$$

as

$$\mathscr{A}(f)(u,v)=\left\langle f\left(t-\frac{u}{2}\right)e^{-\pi ivt},f\left(t+\frac{u}{2}\right)e^{\pi ivt}\right\rangle. \tag{2}$$

For $F(u,v)$ and $G(u,v)$ we define

$$\langle F,G\rangle_2=\int_{\mathbb{R}}\int_{\mathbb{R}} F(u,v)G^*(u,v)\,du\,dv$$

and $\|F\|_2^2=\langle F,F\rangle_2$. We will use $L^2(\mathbb{R}^2)$ to denote the above Hilbert space of square summable functions on $\mathbb{R}^2$.

We can now state two results from the paper.

THEOREM A. *The set of ambiguity functions* $\mathscr{A}(f)$, $f\in L^2(\mathbb{R})$, *is a closed subset of* $L^2(\mathbb{R}^2)$.

THEOREM B. *For* f, $g\in L^2(\mathbb{R})$ *let* $\mathscr{A}(f)$ *and* $\mathscr{A}(g)$ *be the corresponding ambiguity functions. Then* $\mathscr{A}(f)+\mathscr{A}(g)$ *is an ambiguity function if and only if* $f=\lambda g$, λ *a constant.*

The last part of this paper is devoted to ways of describing all ambiguity functions. In order to state some of these results we will need the following definition.

DEFINITION. Let $f\in L^2(\mathbb{R})$ and define

$$f_{ab}=e^{2\pi ibt}f(t+a),\qquad a,b\in\mathbb{Z},$$

and let $\mathscr{F}$ denote the set $\{f_{ab}|a,b\in\mathbb{Z}\}$. We will say that f generates an L^2-basis of $L^2(\mathbb{R})$ if $L^2(\mathbb{R})$ is the closure of linear combinations of elements of $\mathscr{F}$, but no proper subset of $\mathscr{F}$ has this property.

Theorem 6 of §4, due to R. Sacksteder, gives necessary and sufficient conditions for $f\in L^2(\mathbb{R})$ to generate an L^2-basis.

THEOREM C. *Let* $f\in L^2(\mathbb{R})$ *generate an* L^2*-basis. The set of ambiguity functions is the closure in* $L^2(\mathbb{R}^2)$ *of the set of functions*

$$\sum_{a,b,c,d\in\mathbb{Z}}\alpha(a,b)\alpha^*(c,d)K(a,b,c,d)A(f)(u+c-a,v+d-b),$$

where

$$K(a,b,c,d)=(-1)^{(a+c)(b+d)}e^{-\pi i[(b+d)u-(a+c)v]}$$

and $\alpha(a,b)$ *is a function on* $\mathbb{Z}\times\mathbb{Z}$ *taking a finite number of nonzero values.*

The importance of Theorem C can perhaps best be illuminated by the following special case.

Let

$$r(t)=\begin{cases}1, & |t|<\dfrac{1}{2},\\[2mm] 0, & |t|\geqq\dfrac{1}{2},\end{cases}$$

and let $r_{ab}=e^{2\pi ibt}r(t+a)$, $a,b\in\mathbb{Z}$. then the set $\{r_{ab}|a,b\in\mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ and

$$A(r)=\begin{cases}\dfrac{\sin\big(\pi v(1-|u|)\big)}{\pi v}, & |u|\leqq 1,\\ 0, & \text{otherwise.}\end{cases}$$

THEOREM D. *Let* Ψ *be the set of complex valued functions* α *on* $\mathbb{Z}\times\mathbb{Z}$ *such that*

$$\sum|\alpha(a,b)|^2<\infty.$$

Then the set of all ambiguity functions is given by

$$F(\alpha)(u,v)=\sum_{a,b,c,d\in\mathbb{Z}}\alpha(a,b)\alpha^*(c,d)K(a,b,c,d)A(r)(u+c-a,v+d-b)$$

where $K(a,b,c,d)$ *is defined in Theorem* C *and* $\alpha\in\Psi$. *Further, if* $f=\sum_{a,b\in\mathbb{Z}}\alpha(a,b)r_{ab}$, *then* $\mathscr{A}(f)(u,v)=F(\alpha)(u,v)$.

Thus we can describe the set of all ambiguity functions in terms of well-known functions.

Several theorems will be given different proofs. The techniques used in §2 will probably be accessible to most readers, as they require only the most basic results from Abelian harmonic analysis. Orthonormal bases play an important role, especially in the proofs of Theorems A and B, and essentially, translate the problem under consideration, into a problem of infinite matrices satisfying certain conditions (see the discussion following Theorem 2.4). The inversion formula, given in Lemma 2.2, is the main tool in earlier parts of the section and could be applied to prove these results, as well, Theorem C is about a special kind of orthonormal bases.

In §3, ideas arising from unitary representation theory of the Heisenberg group are applied to the study of ambiguity functions. The definitions and results stated at this point can be discussed within the framework of locally compact groups, but we will not do so.

Equally powerful and related ideas can be introduced from the theory of Hilbert–Schmidt operators. Certain of these ideas have been previously considered in [W1] and [S] and applied to the problem of synthesizing ambiguity functions which best approximate, in the L^2-norm, a given function in $L^2(\mathbb{R}^2)$. This theory will play no direct part in this work.

An interesting aspect of ambiguity theory is that it finds itself within the scope of several mathematical disciplines. However, it should be emphasized, that radar theory and more generally image processing create special classes of problems not usually encountered in these general mathematical theories.

2. Ambiguity functions. The elementary properties of ambiguity functions will be established in this section using methods of Abelian harmonic analysis. Our main reference will be [K].

Consider $f,\, g\in L^1(\mathbb{R})$ and define

$$\mathscr{A}(f,g)(u,v)=\int f\left(t-\frac{u}{2}\right)g^*\left(t+\frac{u}{2}\right)e^{-2\pi ivt}\,dt.$$

We call $\mathscr{A}(f,g)$ the cross-ambiguity function of f with g. The ambiguity function $\mathscr{A}(f)$ of f is given by

$$\mathscr{A}(f)\equiv\mathscr{A}(f,f).$$

A closely related expression $\mathscr{B}(f,g)$ is sometimes also called the cross-ambiguity function of f with g and for some purposes is easier to work with. Set

$$\mathscr{B}(f,g)\equiv e^{\pi iuv}\circ\mathscr{A}(f,g).$$

A simple change of variables argument shows that we can write

$$\mathscr{B}(f,g)(u,v)=\int f(t)g^*(t+u)e^{-2\pi ivt}\,dt.$$

In this paper we will study $\mathscr{B}(f,g)$, but to avoid confusion we will only call $\mathscr{A}(f,g)$ the cross-ambiguity function of f with g.

There are two obvious ways to consider $\mathscr{B}(f,g)$. The first begins by setting

$$h(u,t)=f(t)g^*(t+u)$$

and viewing $h(u,t)$ as a family of functions in t, parameterized by u. In general, if $F(x,y)$ is any function of two variables x and y, for any fixed $x\in\mathbb{R}$, we set

$$F_x(y)=F(x,y)$$

and consider F_x as a function of y. Using this notation, we can write

$$\mathscr{B}(f,g)_u(v)=\hat{h}_u(v).$$

The behavior of $h(u,t)$ determines to a large extent the behavior of $\mathscr{B}(f,g)$. The following elementary result provides the necessary information upon which a great deal of ambiguity function theory rests.

LEMMA 2.1. *For $f,g\in L^2(\mathbb{R})$, the function $h(u,t)=f(t)g^*(t+u)$ is in $L^2(\mathbb{R}^2)$ and*

$$\|h\|_2^2=\|f\|^2\|g\|^2.$$

Proof. By Fubini's theorem and the positivity of $|h(u,t)|$

$$\iint |h(u,t)|^2\,du\,dt=\int\left[\int |h(u,t)|^2\,du\right]dt$$

$$=\int |f(t)|^2\left[\int |g^*(t+u)|^2\,du\right]dt.$$

But

$$\int |g^*(t+u)|^2\,du=\|g\|^2\quad \forall t.$$

Hence

$$\|h\|_2^2=\|g\|^2\int |f(t)|^2\,dt=\|g\|^2\|f\|^2.$$

LEMMA 2.1′. *Let $f_1, f_2, g_1, g_2\in L^2(\mathbb{R})$ and let*

$$h_1(u,t)=f_1(t)g_1^*(t+u),$$
$$h_2(u,t)=f_2(t)g_2^*(t+u).$$

Then $\langle h_1,h_2\rangle_2=\langle f_1,f_2\rangle\langle g_2,g_1\rangle$.

Proof. Formally

$$\iint h_1(u,t)h_2^*(u,t)\,du\,dt=\int\left[\int h_1(u,t)h_2^*(u,t)\,du\right]dt$$

$$=\int f_1(t)f_2^*(t)\left[\int g_1^*(t+u)g_2(t+u)\,du\right]dt.$$

But $\int g_1^*(t+u)g_2(t+u)\,du=\langle g_2,g_2\rangle$ all t. And so Lemma 2.1′ follows.

To make this rigorous, note that if

$$\iint |h_1(u,t)h_2^*(u,t)|\,du\,dt<\infty$$

then we may replace the double integral with the iterated integral in any order. But, if h_1; $h_2\in L^2(\mathbb{R}^2)$, so are $|h_1|$, $|h_2|$, and we know that the dot product $\langle |h_1|,|h_2|\rangle_2<\infty$. Thus our formal manipulations are legitimate.

It follows, also by Fubini's theorem, that for almost every $u\in\mathbb{R}$, the function $h_u\in L^2(\mathbb{R})$. Since h_u is the product of two $L^2(\mathbb{R})$ functions, it is in $L^2(\mathbb{R})$ by the Schwarz inequality. The formula

$$\mathscr{B}(f,g)_u(v)=\hat{h}_u(v)$$

implies that $\mathscr{B}(f,g)_u$ is the Fourier transform of $h_u\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. By standard Abelian harmonic analysis (see [K]) we have the following corollary.

COROLLARY. *Let $C(\mathbb{R})$ denote the continuous functions on $\mathbb{R}$. For almost every $u\in\mathbb{R}$,*

$$\mathscr{B}(f,g)_u(v)\in C(\mathbb{R})\cap L^2(\mathbb{R})$$

and

$$\lim_{|v|\to\infty}\mathscr{B}(f,g)_u(v)=0.$$

THEOREM 2.1. *$\mathscr{B}(f,g)\in L^2(\mathbb{R}^2)$, whenever f, $g\in L^2(\mathbb{R})$. Moreover,*

$$\|\mathscr{B}(f,g)\|_2^2=\|f\|^2\|g\|^2.$$

Proof. By Fubini's theorem,

$$\|\mathscr{B}(f,g)\|_2^2=\int\left[\int |\hat{h}_u(v)|^2\,dv\right]du$$

which by the Plancherel theorem becomes

$$\int\left[\int |h_u(t)|^2\,du\right]du=\|f\|^2\|g\|^2.$$

THEOREM 2.1′. *Let $f_1, f_2, g_1, g_2\in L^2(\mathbb{R})$. Then*

$$\langle\mathscr{B}(f_1,g_1),\mathscr{B}(f_2,g_2)\rangle_2=\langle f_1,f_2\rangle\langle g_2,g_1\rangle.$$

Proof. Since $\mathscr{B}(f_\alpha,g_\alpha)$, $\alpha=1,2$ are in $L^2(\mathbb{R}^2)$ so are $|\mathscr{B}(f_\alpha,g_\alpha)|$ and so we may apply Fubini's theorem and write

$$\iint\mathscr{B}(f_1,g_1)\mathscr{B}^*(f_2,g_2)\,du\,dv=\int\left[\int h_1u(v)h_2^*u(v)\,dv\right]du$$
$$=\langle f_1,f_2\rangle\langle g_2,g_1\rangle.$$

The following "inversion" formulas provide important tools for the further study of $\mathscr{B}(f,g)$.

LEMMA 2.2. *For any* $f, g \in L^2(\mathbb{R})$.

$$f(t)g^*(t+u) = \int \mathscr{B}(f,g)(u,v)e^{2\pi i t v}dv,$$
$$f(t)\hat{g}(x)^* e^{-2\pi i x t} = \check{\mathscr{B}}(f,g)(x,t),$$

for almost every $(u,t) \in \mathbb{R}^2$ *and almost every* $(x,t) \in \mathbb{R}^2$, *where* $\check{\mathscr{B}}$ *denotes the inverse Fourier transform in* $L^2(\mathbb{R}^2)$.

Proof. Since for almost every $u, \mathscr{B}(f,g)_u \in L^2(\mathbb{R})$ we can apply the inverse Fourier transform to $\mathscr{B}(f,g)_u$, in the sense given by the Plancherel theorem. Thus, for almost every u, the first formula holds, for almost every t. By Fubini's theorem, applied to the characteristic function of the set of $(u,t) \in \mathbb{R}^2$ for which the first formula *does not* hold, we get that it holds except on a set in $\mathbb{R}^2$ of measure zero.

To prove the second formula, take the inverse Fourier transform of the first formula with respect to the u variable.

Let $\mathscr{B}(f) = \mathscr{B}(f,f)$. Then

$$\mathscr{B}(f)(-u,-v) = \int f(t)f^*(t-u)e^{2\pi i v t}dt.$$

Let $s = t-u$. Then

$$(**) \qquad \mathscr{B}(f)(-u,-v) = \int f(s+u)f^*(s)e^{2\pi i v(s+u)}ds = e^{2\pi i v u}\mathscr{B}(f)^*(u,v).$$

Now consider the change of variables

$$\tau = t+u, \qquad t = t.$$

and let $H(t,\tau) = f(t)f^*(\tau)$. Then

$$H(t,\tau) = \int \mathscr{B}(f)(\tau - t, v)e^{2\pi i t v}dv \quad \text{and} \quad H^*(t,\tau) = \int \mathscr{B}(f)^*(\tau - t, v)e^{-2\pi i t v}dv.$$

Using formula (**) we have

$$H^*(t,\tau) = \int \mathscr{B}(f)(t-\tau, -v)e^{-2\pi i t v}e^{-2\pi i(\tau - t)v}dv = H(\tau, t).$$

Consider the mapping U: $\mathscr{B}(f)(u,v) \to H(t,\tau)$. This has the property that it is 1 to 1 and norm preserving. Further the $H(t,\tau)$ are easily seen to satisfy the functional equations

1. $H^*(t,\tau) = H(\tau,t)$,
2. $H(t,t) \geq 0$,
3. $H(\xi,\xi)H(t,\tau) = H(t,\xi)H(\xi,\tau)$.

THEOREM 2.2. *Let* $F(t,\tau) \in L^2(\mathbb{R}^2)$ *and satisfy equations* 1,2 *and* 3 *above. Then there exists a* $\mathscr{B}(f)$ *such that* $U(\mathscr{B}(f)) = F(t,\tau)$.

Proof. Equations 1 and 3 combine to yield

$$F(t,t)F(\xi,\xi) = |F(t,\xi)|^2.$$

By hypothesis, $F(t,\xi)\in L^2(\mathbb{R}^2)$ and so

$$\iint F(t,t)F(\xi,\xi)\,dt\,d\xi=\iint |F(t,\xi)|^2\,dt\,d\xi$$
$$=\|F(t,\xi)\|_2^2<\infty.$$

Since $F(t,t)\geqq 0$, we may apply Fubini's theorem to conclude that

$$\left(\int F(t,t)\,dt\right)^2=\|F(t,\xi)\|_2^2\geqq 0.$$

The only interesting case is when $\|F(t,\xi)\|_2>0$ and so there exists ξ_0 such that $F(\xi_0,\xi_0)>0$ and $F(t,\xi_0)\in L^2(\mathbb{R})$.

Define $f(t)=F(t,\xi_0)/(F(\xi_0,\xi_0))^{1/2}$. Then

$$f(t)f^*(\tau)=F(t,\tau).$$

It is clear that $U(\mathscr{B}(f))=F(t,\tau)$ and so we have proven our theorem.

Consider the mapping

$$\mathscr{B}\colon L^2(\mathbb{R})\times L^2(\mathbb{R})\to L^2(\mathbb{R}^2).$$

It is clearly bilinear.

THEOREM 2.3. *$\mathscr{B}$ is continuous and the image of $\mathscr{B}$ spans a dense subspace of $L^2(\mathbb{R}^2)$.*

Proof. If $f_n\to f$ and $g_n\to g$ in $L^2(\mathbb{R})$ then, by the continuity of the Fourier transform $\hat{g}_n\to\hat{g}$ in $L^2(\mathbb{R})$ and

$$f_n(y)\hat{g}_n(x)^*e^{-2\pi ixy}\to f(y)\hat{g}(x)^*e^{-2\pi ixy}$$

in $L^2(\mathbb{R}^2)$. Lemma 2 implies

$$\check{\mathscr{B}}(f_n,g_n)\to\check{\mathscr{B}}(f,g)$$

in $L^2(\mathbb{R}^2)$ and hence, $\mathscr{B}$ is continuous where $\check{}$ denotes the inverse Fourier transform in $L^2(\mathbb{R}^2)$.

Suppose $F\in L^2(\mathbb{R}^2)$ is orthogonal to the span of the image of $\mathscr{B}$. Then, $\check{F}(x,y)e^{2\pi ixy}$ is orthogonal to the span of the space of all products $f(y)\hat{g}^*(x)$ which is known to be dense in $L^2(\mathbb{R}^2)$. It follows $F=0$ and the theorem is proved.

COROLLARY. *The collection of functions $\mathscr{B}(f)$, $f\in L^2(\mathbb{R})$, spans a dense subspace of $L^2(\mathbb{R}^2)$.*

Proof. Suppose $F\in L^2(\mathbb{R})$ is orthogonal to every $\mathscr{B}(f)$, $f\in L^2(\mathbb{R})$. Then, since

$$\mathscr{B}(f+g)=\mathscr{B}(f)+\mathscr{B}(f,g)+\mathscr{B}(g,f)+\mathscr{B}(g),$$

we have F orthogonal to $\mathscr{B}(f,g)+\mathscr{B}(g,f)$. Also, since

$$\mathscr{B}(f+ig)=\mathscr{B}(f)+i\mathscr{B}(g,f)-i\mathscr{B}(f,g)+\mathscr{B}(g)$$

we have F orthogonal to $\mathscr{B}(g,f)-\mathscr{B}(f,g)$. Thus, F is orthogonal to $\mathscr{B}(f,g)$, f, $g\in L^2(\mathbb{R})$, and by the theorem is zero almost everywhere.

The function $\mathscr{B}(f,g)$ can also be viewed as a cross-correlation. For fixed $v\in\mathbb{R}$, set

$$G_v(t)=g(t)e^{2\pi ivt}.$$

Form the cross-correlation $f \circ G_v$ defined by

$$f \circ G_v(u) = \int f(t) G_v^*(u+t)\, dt.$$

and upon writing out the integral, observe that

$$\mathcal{B}(f,g)(u,v) = e^{2\pi i u v} f \circ G_v(u).$$

LEMMA 2.3. $\mathcal{B}(f,g)(u,v) = e^{2\pi i u v}\mathcal{B}(\hat{f},\hat{g})(-v,u)$.

Proof. Since

$$\mathcal{B}(f,g)(u,v) = e^{2\pi i u v}\langle f(t), G_v(u+t)\rangle$$

it follows that

$$\mathcal{B}(f,g)(u,v) = e^{2\pi i u v}\langle \hat{f}, G_v(u+t)\rangle$$

But

$$\int g(u+t) e^{2\pi i v t} e^{-2\pi i t x}\, dt = e^{-2\pi i u v} e^{2\pi i u x}\hat{g}(x-v)$$

which proves the lemma.

COROLLARY. $\hat{f}(x)\hat{g}(x-v)^* = \int \mathcal{B}(f,g)(u,v) e^{2\pi i u(x-v)}\, du$.

We will now show $\mathcal{B}(f,g)$ is continuous. The first step is the next lemma.

LEMMA 2.4. *If $f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R})$ then*

$$\mathcal{B}(f_n, g_n) \to \mathcal{B}(f,g)$$

uniformly over $\mathbb{R}^2$.

Proof. (R. Sacksteder independently suggested this proof to one of the authors.) Set $\mathcal{B}_n(u,v) = \mathcal{B}(f,g)(u,v) - \mathcal{B}(f_n,g_n)(u,v)$. Then

$$\mathcal{B}_n(u,v) = \int \left((f(t) - f_n(t)) g^*(t+u) + f_n(t)(g^*(t+u) - g_n^*(t+u)) \right) e^{-2\pi i v t}\, dt$$

and by the Schwarz inequality,

$$|\mathcal{B}_n(u,v)| \leq \|f - f_n\|\,\|g\| + \|f_n\|\,\|g - g_n\|.$$

The lemma follows.

THEOREM 2.4. *$\mathcal{B}(f,g)$ is a continuous bounded function which achieves its maximum $\langle f,g\rangle$ at the origin.*

Proof. The Schwarz inequality proves everything except for the continuity. By the preceding theorem it is sufficient to prove continuity for f and g taken from a dense subspace of $L^2(\mathbb{R})$. The set of functions

$$\{ e^{-\pi(t+r)^2} : r \in \mathbb{R} \}$$

spans a dense subspace of $L^2(\mathbb{R})$. Taking f and g from this span it is easy to see that we are done once we show

$$\mathcal{B}\left(e^{-\pi t^2}, e^{-\pi(t+r)^2} \right)$$

is continuous, for all r. But,

$$\mathscr{B}\left(e^{-\pi t^2}, e^{-\pi(t+r)^2}\right) = \frac{\sqrt{2}}{2} e^{-\pi/2(u+r)^2} e^{-\pi/2v^2} e^{\pi i(r+u)v}$$

which is clearly continuous.

Let $\mathscr{I}_i$ be an orthonormal basis of $L^2(\mathbb{R})$. Then the set of functions $\psi_{ik} = \mathscr{B}(\mathscr{I}_i, \mathscr{I}_k)$, $i,k \in \mathbb{Z}$ is orthonormal by Lemma 1′ and complete by Theorem 3. Now $f(t) \in L^2(\mathbb{R})$ can be written as

$$f(t) = \sum_{-\infty}^{\infty} a_i \mathscr{I}_i, \qquad \sum_{-\infty}^{\infty} |a_i|^2 = \|f\|^2 < \infty.$$

Similarly, $F(u,v) \in L^2(\mathbb{R}^2)$ can be written as

$$F(u,v) = \sum_{m,n \in \mathbb{Z}} c_{mn} \psi_{mn}, \qquad \sum_{m,n \in \mathbb{Z}} |c_{mn}|^2 < \infty.$$

Now consider $\mathscr{B}(f) \in L^2(\mathbb{R}^2)$. Then if

$$c_{mn} = \int\int \mathscr{B}(f) \bar{\psi}_{mn} \, du \, dv,$$
$$\mathscr{B}(f) = \sum c_{mn} \psi_{mn}.$$

By Lemma 2.1′

$$c_{mn} = \langle f, \mathscr{A}_m \rangle \langle f, \mathscr{I}_n \rangle^* = a_m a_n^*.$$

Conversely, if $c_{mn} = a_m a_n^*$ then $f = \sum a_m \mathscr{I}_m \in L^2(\mathbb{R})$ and $H(t,\tau) = f(t) f(\tau)^*$ satisfies the hypothesis of Theorem 2.2. Hence $F(u,v) \in L^2(\mathbb{R}^2)$ is an ambiguity function if and only if $c_{mn} = a_m \bar{a}_n$.

COROLLARY. *Let $F(u,v) \in L^2(\mathbb{R}^2)$ and $F = \sum_{m,n \in Z} c_{mn} \psi_{mn}$. Then $F(u,v)$ is an ambiguity function if and only if $c_{kk} c_{mn} = c_{mk} c_{kn}$, $c_{mn} = c^*_{mm}$ and $c_{kk} \geqq 0$ all $m, n, k \in \mathbb{Z}$.*

THEOREM A. *The set of ambiguity functions is a closed subset of $L^2(\mathbb{R}^2)$.*

Proof. Let $F(u,v)$ be the limit of sequence of ambiguity functions $\mathscr{B}(f_i)$, $i = 1, \cdots, n \cdots$. Let

$$c_{mn}(i) = \langle \mathscr{B}(f_i), \psi_{mn} \rangle_2,$$
$$c_{mn} = \langle F, \psi_{mn} \rangle_2.$$

Hence for each i, $c_{mn}(i)$ satisfy the conditions in the above corollary. Since $\lim_{i \to \infty} \mathscr{B}(f_i) = F$, we have for all $m, n \lim_{i \to \infty} c_{mn}(i) = c_{mn}$. Hence the c_{mn} satisfy the equations of the above corollary and F is an ambiguity function.

THEOREM B. *For f, $g \in L^2(\mathbb{R})$ let $\mathscr{B}(f)$ and $\mathscr{B}(g)$ be the corresponding ambiguity functions. Then $\mathscr{B}(f) + \mathscr{B}(g)$ is an ambiguity function if and only if $f = \lambda g$, λ a constant.*

Now consider $\mathscr{B}(f)$ and $\mathscr{B}(cf)$ where c is a constant. Then by direct computation

$$\mathscr{B}(f) + \mathscr{B}(cf) = \mathscr{B}\left(\sqrt{1 + |c|^2 f}\right).$$

Now let f, $g \in L^2(\mathbb{R})$ and consider $\mathscr{B}(f) + \mathscr{B}(g)$. If

$$\mathscr{B}(f) = \sum_{a,b,c,d \in \mathbb{Z}} \alpha_1(a,b,c,d) F_{abcd}, \quad \mathscr{B}(g) = \sum_{a,b,c,d \in \mathbb{Z}} \alpha_2(a,b,c,d) F_{abcd}.$$

Then

$$\mathscr{B}(f)+\mathscr{B}(g)=\sum_{a,b,c,d\in\mathbb{Z}}[\alpha_1(a,b,c,d)+\alpha_2(a,b,c,d)]F_{abcd}.$$

Now assume that $\mathscr{B}(f)+\mathscr{B}(g)$ is an ambiguity function. Then the corollary to Theorem 2.4 implies that

$$\begin{aligned}\alpha_1(r,s,r,s)\alpha_2(a,b,c,d)+\alpha_1(a,b,c,d)\alpha_2(r,s,r,s)\\=\alpha_1(r,s,c,d)\alpha_2(a,b,r,s)+\alpha_1(a,b,r,s)\alpha_2(r,s,c,d).\end{aligned}\tag{2.1}$$

Because $\mathscr{B}(f)$ and $\mathscr{B}(g)$ are ambiguity functions, we know that

$$\begin{aligned}\alpha_1(a,b,c,d)&=\alpha_1(a,b)\alpha_1^*(c,d),\\ \alpha_2(a,b,c,d)&=\alpha_2(a,b)\alpha_2^*(c,d).\end{aligned}$$

Hence we can rewrite (2.1) as

$$\begin{aligned}\alpha_1(r,s)\alpha_1^*(r,s)\alpha_2(a,b)\alpha_2^*(c,d)+\alpha_1(a,b)\alpha_1^*(c,d)\alpha_2(r,s)\alpha_2^*(r,s)\\=\alpha_1(r,s)\alpha_1^*(c,d)\alpha_2(a,b)\alpha_2^*(r,s)+\alpha_1(a,b)\alpha_1^*(r,s)\alpha_2(r,s)\alpha_2^*(c,d).\end{aligned}\tag{2.2}$$

Assume f is not the zero function, then $\alpha_1(r_0,s_0)\neq 0$ for some r_0 and s_0. It is easy to see that there is no loss in generality in assuming that $\alpha_1(0,0)\neq 0$. Then setting $v=s=0$ in (2.2) we obtain

$$[\alpha_1(0,0)\alpha_2(a,b)-\alpha_2(0,0)\alpha_1(a,b)][\alpha_1^*(0,0)\alpha_2^*(c,d)-\alpha_2^*(0,0)\alpha_1^*(c,d)]=0;$$

this implies that

$$\alpha_1(0,0)\alpha_2(a,b)=\alpha_2(0,0)\alpha_1(a,b)$$

or

$$\alpha_2(a,b)=\frac{\alpha_2(0,0)}{\alpha_1(0,0)}\alpha_1(a,b),\quad \text{for all } a,b\in\mathbb{Z}.$$

Thus if $c=\alpha_2(0,0)/\alpha_1(0,0)$ we have

$$g=cf.$$

This proves our assertion.

THEOREM 2.5. *Let $f, g\in L^2(\mathbb{R})$ and assume*

$$\mathscr{B}(f)=\mathscr{B}(g).$$

Then $f=c\circ g$ almost everywhere, where c is a constant and $|c|=1$.

Proof. By Lemma 2.2.

$$f(y)\hat{f}(x)^*=g(y)\hat{g}(x)^*$$

for almost all $(x,y)\in\mathbb{R}^2$. If f does not vanish on a set of positive measure, then for some y_0 we have $f(y_0)\neq 0$ and

$$f(y_0)\hat{f}(x)^*=g(y_0)\hat{g}(x)^*$$

holds for almost every x. Thus, there is a constant $c=g(y_0)^*/f(y_0)^*$ such that

$$f(y)=c\circ g(y),$$

for almost every y. The constant must have modulus one since $\mathscr{B}(f)=\mathscr{B}(g)$.

If f vanishes almost everywhere that $\mathscr{B}(f)=0$ and $g(y)=\hat{g}(x)^*=0$ almost everywhere in $(x,y)\in\mathbb{R}^3$. It is easy to see that $g(y)=0$ almost everywhere.

The same argument proves the following.

COROLLARY. *For* $f,\ g\in L^2(\mathbb{R})$, *if* $\mathscr{B}(f,g)=0$ *then* $f=0$ *almost everywhere or* $g=0$ *almost everywhere.*

We will return now to the ambiguity functions $\mathscr{A}(f)$ and denote by $\mathscr{R}$ the collection of all ambiguity functions.

Let $SL(2,\mathbb{R})$ denote the group of all 2×2 real matrices of determinant one, acting on $\mathbb{R}^2$ by the rule

$$T(u,v)=(au+bv,cu+dv)$$

where

$$T=\begin{bmatrix} a & b\\ c & d\end{bmatrix}.$$

As a group $SL(2,\mathbb{R})$ is generated by the following matrices:

$$J=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},\quad t(a)=\begin{bmatrix} 1 & 0\\ a & 1\end{bmatrix},\quad a\in\mathbb{R},\quad m(b)=\begin{bmatrix} b & 0\\ 0 & 1/b\end{bmatrix},\quad b>0.$$

To see this for $c>0$, write

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}=t\left(\frac{a}{c}\right)J^{-1}m(c)t\left(\frac{d}{c}\right).$$

We will now state how $SL(2,\mathbb{R})$ acts on $\mathscr{R}$.

THEOREM 2.6. $\mathscr{R}$ *is invariant under the action of* $SL(2,\mathbb{R})$, *and*

1. $\mathscr{A}(f)\circ J=\mathscr{A}(\hat{f})$,
2. $\mathscr{A}(f)\circ t(a)=\mathscr{A}(g)$ *where* $g(t)=e^{\pi i a t^2}f(t)$,
3. $\mathscr{A}(f)\circ m(b)=\mathscr{A}(h)$ *where* $h(t)=f(bt)$.

Proof. Statement 1 follows from Lemma 2.3. The last two statements can easily be proved by direct substitution.

3. Ambiguity functions and the Heisenberg group. In this section, we will work with the ambiguity functions $\mathscr{A}(f)$. A unitary operator on $L^2(\mathbb{R})$ is a linear mapping U of $L^2(\mathbb{R})$ satisfying

$$\langle Uf,Ug\rangle=\langle f,g\rangle,$$

for all $f,\ g\in L^2(\mathbb{R})$. The collection of all unitary operators U on $L^2(\mathbb{R})$ forms a group under composition which will be denoted by $\mathscr{U}$. An implication of the Plancherel theorem is that the Fourier transform, denoted by $\mathscr{F}$, is a unitary operator on $L^2(\mathbb{R})$.

The following two unitary operators on $L^2(\mathbb{R})$ play an important role in Abelian harmonic analysis and hence, the development of the theory of the ambiguity function given in the preceding section.

For $f\in L^2(\mathbb{R})$ and $a\in\mathbb{R}$, set

$$(S(a)f)(t)=f(t+a),\qquad t\in\mathbb{R},$$
$$(M(a)f)(t)=e^{2\pi i a t}f(t),\qquad t\in\mathbb{R},$$

and observe that the mappings $S(a)$ and $M(a)$ are unitary operators of $L^2(\mathbb{R})$.

Consider, now, $S: \mathbb{R} \to \mathscr{U}$ and $M: \mathbb{R} \to \mathscr{U}$ as mappings from $\mathbb{R}$ into $\mathscr{U}$. We set

$$\mathscr{S} = S(\mathbb{R}), \qquad \mathscr{M} = M(\mathbb{R}),$$

and call $\mathscr{S}$ the shift operators and $\mathscr{M}$ the multiplication operators. Both $\mathscr{S}$ and $\mathscr{M}$ are subgroups of $\mathscr{U}$ and in fact, we have the following lemma.

LEMMA 3.1. *S and M are group isomorphisms of* $\mathbb{R}$ *into* $\mathscr{U}$.

Proof. Immediate from the definition.

The reason that non-Abelian group theory enters into the study of ambiguity functions is contained in the next result.

LEMMA 3.2. $M(v)S(u) = e^{-2\pi iuv}S(u)M(v)$.

Proof. For $f \in L^2(\mathbb{R})$,

$$(M(v)S(u)f)(t) = e^{2\pi ivt}(S(u)f)(t) = e^{2\pi ivt}f(t+u),$$
$$(S(u)M(v)f)(t) = (M(v)f)(t+u) = e^{2\pi iv(t+u)}f(t+u),$$

which verifies the truth of the lemma.

Thus, the operators $M(v)$ and $S(u)$ do not commute. This observation is the mathematical basis for the introduction of the Heisenberg group in quantum mechanics and is an expression of the uncertainty principal. We will now define the Heisenberg group and study its implications in ambiguity function theory.

Let I denote the identity operator on $L^2(\mathbb{R})$ and set

$$C(\lambda) = \lambda I, \qquad \lambda \in \mathbb{C}, \quad |\lambda| = 1.$$

Then, $C(\lambda)$ is a unitary operator and the mapping

$$C: \mathbb{C}(1) \to \mathscr{U}(L^2(\mathbb{R})) = \mathscr{U},$$

where $\mathbb{C}(1)$ denotes the multiplicative group of complex numbers of modulus 1, is a group monomorphism. We set $\mathscr{C}$ equal to the range of C.

Clearly, $\mathscr{C}$ is a subgroup of $\mathscr{U}$ and is, in fact, the center of $\mathscr{U}$.

Let

$$\mathscr{H} = \mathscr{C} \circ \mathscr{M} \circ \mathscr{S}$$

denote the set of operators of the form

$$C(\lambda)M(b)S(a), \qquad |\lambda| = 1, \quad a, b \in \mathbb{R}.$$

THEOREM 3.1. $\mathscr{H}$ *is a subgroup of* $\mathscr{U}$.

Proof. By Lemma 3.2, we can write

$$\begin{aligned} &C(\lambda_1)M(b_1)S(a_1)C(\lambda_2)M(b_2)S(a_2) \\ &\quad = C(\lambda_1)C(\lambda_2)C(e^{2\pi ia_1b_2})M(b_1)M(b_2)S(a_1)S(a_2), \end{aligned}$$

which by Lemma 3.1, becomes

$$C(\lambda_1\lambda_2 \circ e^{2\pi ia_1b_2})M(b_1+b_2)S(a_1+a_2).$$

Thus, the product of two operators in $\mathscr{H}$ is again in $\mathscr{H}$.

It follows that

$$I = C(\lambda_1)M(b_1)S(a_1)C(\lambda_2)M(-b_1)S(-a_1)$$

if and only if $\lambda_2 = \lambda_1^{-1}e^{2\pi ia_1b_1}$ and hence, the inverse of an operator in $\mathscr{H}$ is again in $\mathscr{H}$.

An alternate definition of $\mathscr{H}$ can be taken to be the group generated by $\mathscr{M}$ and $\mathscr{S}$.

$\mathscr{H}$ is sometimes called the Heisenberg group, however, we will reserve this term for the abstractly defined group N given as follows.

As a set N consists of all points $\mathbf{x}=(x_1,x_2,x)\in\mathbb{R}^3$. The multiplication rule on N is given by the formula

$$\mathbf{x}\circ\mathbf{y}=\left(x_1+y_1,x_2+y_2,x+y+\tfrac{1}{2}(x_2y_1-x_1y_2)\right).$$

It is easy to verify that N is a group having centext X consisting of all points $(0,0,x)$, $x\in\mathbb{R}$.

For future use, we will single out two especially important automorphisms of N. Let $\mathscr{J}$ denote the mapping of N given by

$$\mathscr{J}(\mathbf{x})=(x_2,-x_1,x).$$

Clearly, $\mathscr{J}$ is an automorphism on N which acts by the identity mapping when restricted to the center.

Define D: $N\to\mathscr{U}$ by setting

$$D(\mathbf{x})=C(e^{2\pi i\lambda(\mathbf{x})})M(x_1)S(x_2)$$

where $\lambda(\mathbf{x})=x+\frac{1}{2}x_1x_2$. Equivalently,

$$(D(\mathbf{x})f)(t)=C(e^{2\pi i\lambda(\mathbf{x})})e^{2pix_1t}f(t+x_2).$$

Using Lemma 3.2, the next result is easily proved.

THEOREM 3.2. *D: $N\to\mathscr{U}$ is a group homomorphism satisfying*

1. $\ker D=\{(0,0,x)\colon x\in\mathbb{Z}\}$,
2. $\operatorname{im} D=\mathscr{H}$.

The group homomorphism D has, by necessity, been built in a non-Abelian fashion from the group homomorphisms S and M. In a sense, examined more closely in the next section, the Fourier transform $\mathscr{F}$ is closely related to these group homomorphisms and hence, to the Heisenberg group N. For a more complete discussion see [A–T]. At this time, the formulas of the next theorem will suffice.

THEOREM 3.3.

$$\mathscr{F}S(x)\mathscr{F}^{-1}=M(x),\quad \mathscr{F}M(x)\mathscr{F}^{-1}=S(-x),\quad \mathscr{F}D(\mathbf{x})\mathscr{F}^{-1}=D(\mathscr{J}\mathbf{x}).$$

Proof. The first two formulas are easily proved by Abelian harmonic analysis methods. The last formula comes from the definition of D and Lemma 3.2.

The ambiguity function $\mathscr{A}(f)$ can be expressed in terms of the group homomorphism D. This is accomplished in the next theorem.

THEOREM 3.4. *For $\mathbf{x}\in N$, and $f\in L^2(N)$,*

$$\mathscr{A}(f)(x_2,x_1)=e^{2\pi ix}\langle f,D_{\mathbf{x}}f\rangle.$$

Proof. Since

$$D_{\mathbf{x}}f(t)=e^{2\pi i\lambda(\mathbf{x})}e^{2\pi ix_1t}f(t+x_2),$$

we can write,

$$\langle f, D_{\mathbf{x}} f\rangle = e^{-2\pi i\lambda(\mathbf{x})}\int f(t)f^*(t+x_2)e^{-2\pi i x_1 t}dt,$$

$$= e^{-2\pi i\lambda(\mathbf{x})}\mathscr{B}(f)(x_2,x_1),$$

$$= e^{-2\pi i x}\mathscr{A}(f)(x_2,x_1),$$

which proves the theorem.

The significance of this result is that we can view ambiguity functions as well-known objects in the theory of unitary representations of the Heisenberg group. For the most part, the theory we develop can be generalized to the theory of unitary representations of locally compact groups on Hilbert spaces but we will restrict our analysis to what we need to study ambiguity functions.

A unitary representation of N is a homomorphism U of N into $\mathscr{U}$. Let U be a unitary representation of N and $f\in L^2(\mathbb{R})$. Consider the function on N defined by

$$p(\mathbf{x}) = \langle U_{\mathbf{x}} f, f\rangle, \qquad \mathbf{x}\in N.$$

THEOREM 3.5. *The function p is positive define on N, in the sense that, for any finite number of elements* $g_1,\cdots,g_n$ *in N and complex numbers* $\lambda_1,\cdots,\lambda_n$ *we have,*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} p\left(g_j^{-1}g_k\right)\lambda_j^*\lambda_k \geqq 0.$$

Proof. Let $g=\sum_{k=1}^{n}\lambda_k U_g f$. A direct calculation shows,

$$0 \leqq \langle g,g\rangle = \sum_{k=1}^{n}\sum_{j=1}^{n} p\left(g_j^{-1}g_k\right)\lambda_j^*\lambda_e,$$

which proves the theorem.

We note that, by Theorem 3.4, we have

$$e^{2\pi i x}\mathscr{A}(f)(x_2,x_1)^*$$

is positive definite. This enables us to translate general results about positive definite functions into assertions about ambiguity functions.

The following results are well-known about positive definite functions. Observe the relationship of these results to the corresponding results about $\mathscr{A}(f)$ coming from Theorem 2.4. We list them without proof.

1. $p(\mathbf{0})\geqq 0$,
2. $p(g^{-1})=p(g)^*$, $g\in N$,
3. $|p(g)|\leqq p(\mathbf{0})$, $g\in N$.

The unitary representation U of N is called continuous if, for each $f\in L^2(\mathbb{R})$, the mapping,

$$\mathbf{x}\to U_{\mathbf{x}} f,$$

is continuous from N into the Hilbert space $L^2(\mathbb{R})$. We give N the topology of the underlying Euclidean space. If U is a continuous unitary representation of N and $f\in L^2(\mathbb{R})$, then $p(\mathbf{x}) = \langle U_{\mathbf{x}}, f, f\rangle$ is continuous. Since D can be shown to be continuous, we can prove, by this approach, that $\mathscr{A}(f)$ is continuous.

A deeper result is that D is irreducible, in the sense of the following definition. The unitary representation U is irreducible if, for any closed subspace V of $L^2(\mathbb{R})$ such that

$$U_{\mathbf{x}} f \in V,$$

wherever $f \in V$, we have $V = L^2(\mathbb{R})$.

A proof that D is irreducible can be found in [Wyl]. The first implication of D being irreducible is that, for any $f \in L^2(\mathbb{R})$ which does not vanish on a set of positive measure, the span of the set of functions,

$$\{ D_{\mathbf{x}} f : \mathbf{x} \in N \},$$

is dense in $L^2(\mathbb{R})$. As we see, in the proof of the following result, the uniqueness Theorem 2.5 of §2, the density of this span in $L^2(\mathbb{R})$, can be viewed as the key in the uniqueness theorem.

THEOREM 3.6 *If* $f, g \in L^2(\mathbb{R})$ *and* $\mathscr{A}(f) = \mathscr{A}(g)$ *then*

$$f = \lambda g,$$

for some constant λ, $|\lambda| = 1$.

Proof. From $\langle D_{\mathbf{x}} f, f \rangle = \langle D_{\mathbf{x}} g, g \rangle$, $\mathbf{x} \in N$ it easily follows that $\langle D_{\mathbf{x}} f, D_{\mathbf{y}} f \rangle = \langle D_{\mathbf{x}} g, D_{\mathbf{y}} g \rangle$ for all $\mathbf{x}, \mathbf{y} \in N$.

Note $\|f\|_2 = \|g\|$ implies $f = 0$ almost everywhere if and only if $g = 0$ almost everywhere. We will assume, therefore, that both f and g are nonzero on set of positive measure. From the irreducibility of D it follows that each of the set

$$A = \{ D_{\mathbf{x}} f : \mathbf{x} \in N \}$$

and

$$B = \{ D_{\mathbf{x}} g : \mathbf{x} \in N \}$$

spans a dense subspace of $L^2(\mathbb{R})$.

Define the mapping $U: A \to B$ by setting $U(D_{\mathbf{x}} f) = D_{\mathbf{x}} g$, $\mathbf{x} \in N$. We have to show U is well defined. Suppose $D_{\mathbf{u}} f = D_{\mathbf{v}} f$. Then

$$\langle D_{\mathbf{u}} f, D_{\mathbf{x}} f \rangle = \langle D_{\mathbf{u}} g, D_{\mathbf{x}} g \rangle,$$

for all $\mathbf{x}$, by the remarks above. This implies by the assumption $D_{\mathbf{u}} f = D_{\mathbf{v}} f$ that

$$\langle D_{\mathbf{u}} g, D_{\mathbf{x}} g \rangle = \langle D_{\mathbf{v}} g, D_{\mathbf{x}} g \rangle, R$$

for all $\mathbf{x} \in N$. Since B spans a dense subspace of $L^2(\mathbb{R})$, $D_{\mathbf{u}} g = D_{\mathbf{v}} g$. Thus, U is well defined. The condition $\langle D_{\mathbf{x}} f, D_{\mathbf{v}} f \rangle = \langle D_{\mathbf{x}} g, D_{\mathbf{v}} g \rangle$ immediately implies, along with the previous described property of B, that U extends to a unitary operator of $L^2(\mathbb{R})$.

It is trivial to see that

$$U D_{\mathbf{x}} U^{-1} = D_{\mathbf{x}}, \qquad \mathbf{x} \in N$$

and so $U = \lambda I$, $|\lambda| = 1$ which proves the theorem.

Another consequence of the condition of irreducibility will now be discussed. Consider two positive definite functions, p_1 and p_2, on N. We say that p_2 dominates p_1 if $p_2 - p_1$ is positive definite. A positive definite function p on N is called indecomposable if every positive definite function on N which is dominated by p is a scalar multiple of p.

The following theorem can be found in [A], in a slightly different setting, and will be asserted without proof.

THEOREM 3.7. *If U is an irreducible unitary representation of N and $f \in L^2(\mathbb{R})$ which does not vanish on a set of positive measure, then the corresponding positive definite function, p,*

$$p(\mathbf{x}) = \langle U_{\mathbf{x}} f, f \rangle,$$

is indecomposable.

An immediate implication is that $\langle D_{\mathbf{x}} f, f \rangle$ is indecomposable for every $f \in L^2(\mathbb{R})$ which does not vanish on a set of positive measure.

We will now reprove Theorem B using these ideas from unitary representation theory. For $f \in L^2(\mathbb{R})$, we write,

$$p_f(\mathbf{x}) = \langle D_{\mathbf{x}} f, f \rangle, \qquad \mathbf{x} \in N.$$

Suppose $f, g, h \in L^2(\mathbb{R})$ and

$$\mathscr{A}(h) = \mathscr{A}(f) + \mathscr{A}(g).$$

Then,

$$p_h = p_f + p_g.$$

Since p_h is indecomposable and p_h dominates both p_f and p_g, neglecting the trivial case, we can write,

$$p_f = c p_g,$$

where $c \neq 0$ is constant. From $p_f(\mathbf{0}) \geqq 0$ and $p_g(\mathbf{0}) \geqq 0$, we can infer $c > 0$. Let $g' = \sqrt{c}\, g$. Then,

$$p_f = p_{g'},$$

and

$$\mathscr{A}(f) = \mathscr{A}(g'),$$

from which it follows, by Theorem 3.6, that,

$$f = \lambda g' = \lambda \sqrt{c}\, g,$$

which is the conclusion of Theorem B.

4. Another unitary representation of N. A "piece" of another unitary representation of N will be defined which is unitarily equivalent to the representation D defined in the proceeding section. We will avoid as many technical details as possible. For further details see [A–T].

Let Γ be the subgroup of N generated by $(1,0,0)$ and $(0,1,0)$ and denote by H the space of all functions F on N which satisfies the following conditions:

1. $F(\gamma \mathbf{x}) = F(\mathbf{x})$, $\gamma \in \Gamma$, $\mathbf{x} \in N$,
2. $\|F\|_H^2 = \int_0^1 \int_0^1 \int_0^1 |F(\mathbf{x})|^2 \, d\mathbf{x} < \infty$,
3. $F(\mathbf{x}\mathbf{z}) = e^{2\pi i z} F(\mathbf{x})$, $\mathbf{x} \in N$, $\mathbf{z} \in Z$.

One can prove that H is a Hilbert space and that for $\mathbf{x} \in N$ and $F \in H$, the function

$$(\mathscr{D}(\mathbf{x})F)(\mathbf{y}) = F(\mathbf{yx}), \qquad \mathbf{y} \in N$$

is again in H. In fact, we can prove the following

THEOREM 4.1. *$\mathscr{D}$ is a unitary representation of N on H.*

We will tie together D and $\mathscr{D}$ by the Weil–Brezin mapping

$$W\colon L^2(\mathbb{R}) \to H$$

defined by setting

$$W(f)(\mathbf{x}) = e^{2\pi i(x + x_1x_2/2)} \sum_{m \in \mathbb{Z}} f(x_2 + m) e^{2\pi i m x_1}.$$

THEOREM 4.2. *W is an isometry from $L^2(\mathbb{R})$ onto H satisfying*

$$W^{-1}\mathscr{D}(\mathbf{x})W = D(\mathbf{x}), \qquad \mathbf{x} \in N.$$

Proof. Complete details of the proof can be found in [A–T]. We will prove the formula. Since

$$(\mathscr{D}(\mathbf{x})W(f))(\mathbf{y}) = W(f)(\mathbf{yx}) = W(f)\left(y_1 + x_1, y_2 + x_2, y + x + \tfrac{1}{2}(y_2x_1 - y_1x_2)\right),$$

it follows that

$$(\mathscr{D}(\mathbf{x})W(f))(\mathbf{y}) = e^{2\pi i(y + x + y_2x_1/2 - y_1x_2)}e^{2\pi i(y_1 + x_1)(y_2 + x_2)} \sum_{m \in \mathbb{Z}} f(y_2 + x_2 + m) e^{2\pi i m(y_1 + x_1)}.$$

Upon expanding the right-hand side we get

$$(\mathscr{D}(\mathbf{x})W(f))(\mathbf{y}) = W(D(\mathbf{x})F)(\mathbf{y}).$$

We say that W is an intertwining operator between $D(\mathscr{J}) = D \circ \mathscr{J}$ and $\mathscr{D}$.

Consider $\mathbf{a} \in \Gamma$. Then, a_1, $a_2 \in \mathbb{Z}$ and $a \equiv \frac{1}{2}a_1a_2 \bmod \mathbb{Z}$. Let $F \in H$. Recall $F(\mathbf{ay}) = F(\mathbf{y})$. It is easy to see that

$$\mathbf{y} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{y}[\mathbf{y}, \mathbf{a}]$$

where $[\mathbf{y}, \mathbf{a}] = \mathbf{y}^{-1}\mathbf{a}^{-1}\mathbf{ya} = (0, 0, y_2a_1 - y_1a_2)$.

THEOREM 4.3. *For $\mathbf{a} \in \Gamma$ and $F \in H$,*

$$\mathscr{D}(\mathbf{a})F(\mathbf{y}) = \left(WD(\mathbf{a})W^{-1}F\right)(\mathbf{y}) = e^{2\pi i(a_2y_1 - a_1y_2)}F(\mathbf{y}).$$

Proof. By definition,

$$(\mathscr{D}(\mathbf{a})F)(\mathbf{y}) = F(\mathbf{y} \cdot \mathbf{a}) = F(\mathbf{a} \cdot \mathbf{y} \cdot [\mathbf{y}, \mathbf{a}]) = e^{2\pi i(a_2y_1 - a_1y_2)}F(\mathbf{y}).$$

COROLLARY. $e^{2\pi i(a_2y_1 - a_1y_2)}W(f)(\mathbf{y}) = W(g)(\mathbf{y})$ *where*

$$g(y) = D(\mathbf{a})f(y).$$

Consider $H_0 = L^2(\mathbf{R}^2/\mathbf{Z}^2)$. For $F, G \in H$,

$$F(\mathbf{x})G^*(\mathbf{x}) = F_0(x_1, x_2)G_0^*(x_1, x_2)$$

where $F_0(x_1, x_2) = F(x_1, x_2, 0)$. Thus, F_0, $G_0 \in H_0$ and

$$\langle F, G \rangle_H = \langle F_0, G_0 \rangle_{H_0}.$$

For a_1, $a_2 \in \mathbb{Z}$ and x_1, $x_2 \in \mathbb{R}$, define

$$\chi_{a_1 a_2}(x_1, x_2) = e^{2\pi i(a_1 x_1 + a_2 x_2)}.$$

Clearly $\chi_{a_1,a_2} \in H_0$. We also have, for $F, G \in H$,

$$\left\langle \chi_{a_1,a_2} F, G \right\rangle_H = \left\langle \chi_{a_1,a_2}, F_0^* G_0 \right\rangle_{H_0}.$$

THEOREM 4.4. *The set of functions*

$$\left\{ \chi_{a_1,a_2} F \colon a_1, a_2 \in \mathbb{Z} \right\}$$

is an orthonormal basis of H if and only if

$$|F(\mathbf{x})| \equiv 1, \quad \textit{almost everywhere}.$$

Proof. Clearly, if $|F(\mathbf{x})| \equiv 1$, almost everywhere, then the set of functions is orthonormal since

$$\left\langle \chi_{a_1,a_2} F, \chi_{b_1,b_2} F \right\rangle_H = \left\langle \chi_{a_1,a_2}, \chi_{b_1,b_2} \right\rangle_{H_0} = 0.$$

Moreover, if $G \in H$ satisfies

$$\left\langle \chi_{a_1,a_2} F, G \right\rangle_H = \left\langle \chi_{a_1,a_2}, F^* G \right\rangle_{H_0} = 0,$$

for all a_1, $a_2 \in \mathbb{Z}$ then by the completeness of χ_{a_1,a_2}, a_1, $a_2 \in \mathbb{Z}$ in H_0, $F^*G \equiv 0$, almost everywhere. Since $|F| \equiv 1$, almost everywhere, $G \equiv 0$, almost everywhere which implies the set $\{\chi_{a_1,a_2} F \colon a_1, a_2 \in \mathbb{Z}\}$ is an orthonormal basis in H.

Conversely, if $\{\chi_{a_1,a_2} F \colon a_1, a_2 \in \mathbb{Z}\}$ is an orthonormal basis in H, then

$$\left\langle \chi_{a_1,a_2} F, F \right\rangle = \left\langle \chi_{a_1,a_2}, |F|^2 \right\rangle = 0,$$

whenever both a_1 and a_2 are not both 0. Thus, $|F|^2$ is constant almost everywhere. But

$$\langle F, F \rangle_H = 1$$

implies $|F| \equiv 1$, almost everywhere.

Theorem 4.2, the corollary to Theorem 4.3, and Theorem 4.4 immediately imply the next result.

THEOREM 4.5. *For* $f \in L^2(\mathbb{R})$, *satisfying*

$$|W(f)(\mathbf{y})| \equiv \left| \sum_{l \in \mathbb{Z}} f(y_2 + l) e^{2\pi i l y_1} \right| \equiv 1,$$

almost everywhere, *the collection of functions*

$$f_{a_1,a_2}(y) = e^{2\pi i a_2 y} f(y + a_1),$$

as a_1, a_2 *run over* $\mathbb{Z}$, *forms an orthonormal basis of* $L^2(\mathbb{R})$.

If $F(\mathbf{x})$ does not satisfy the condition $|F(\mathbf{x})| = 1$, almost everywhere, then the collection of functions

$$W = \{ \chi_{a,b} \cdot F(\mathbf{x}) \colon a, b \in \mathbb{Z} \}$$

will not be an orthonormal basis but could be an L^2-basis of H, in the sense that, the linear span of W is dense in H and no proper subset of W has this property. It will be convenient to discuss the problem of when W determines an L^2-basis of H by considering the analogous problem on $\pi^2=\mathbb{R}^2/\mathbb{Z}^2$.

Consider $F(u,v)\in L^2(\pi^2)$ and set

$$W_0=\{\chi_{a,b}(u,v)\cdot F(u,v)\colon a,b\in\mathbb{Z}\}.$$

Let

$$g(t)=m\{(u,v)\in\pi^2\colon |F(u,v)|\leqq t\},$$

where m denotes Lebesgue measure on π^2.

Observe that $g(t)$ is the distribution function of $|F(u,v)|$, and hence determines a probability measure on $\mathbb{R}$.

THEOREM 4.6. *W_0 is a minimal basis of $L^2(\pi^2)$ if and only if*

1. $g(0)=0$,
2. $\int_{0^+}^{\infty}(1/t^2)\,dg(t)<\infty$. (*This includes* 1.)

Proof. Take $G\in L^2(\pi^2)$ satisfying

$$\langle G,W_0\rangle=0.$$

Then

$$\langle\chi_{a,b}F,G\rangle=\langle\chi_{a,b},F^*G\rangle=0,\qquad a,b\in\mathbb{Z},$$

which by the completeness of the set $\{\chi_{a,b}\colon a,b\in\mathbb{Z}\}$ in $L^2(\pi^2)$ implies $F^*G=0$ almost everywhere. Thus, $g(0)=0$ implies $G=0$ almost everywhere. We have proved that $g(0)=0$ implies W_0 spans a dense subspace of $L^2(\pi^2)$. The converse is trivial, for if $g(0)\neq 0$, let G be the function which is identically one where F vanishes and zero otherwise. Then G is orthogonal to W_0 but is not the zero function in $L^2(\pi^2)$.

We will now show the equivalence of minimality to Theorem 4.6, statement 2. The argument includes the above discussion.

Suppose $a_0b_0\in\mathbb{Z}$ and that the closure V in $L^2(\pi^2)$ of the set $F\cdot(\mathbb{C}\cdot\chi_{a_0,b_0})^{\perp}$ is proper in $L^2(\pi^2)$. As is standard $(\mathbb{C}\cdot\chi_{a_0,b_0})^{\perp}$ denotes the orthogonal complement of $\mathbb{C}\cdot\chi_{a_0,b_0}$ in $L^2(\pi^2)$. Choose G_1 orthogonal to V. Then, for every function G_2 orthogonal to χ_{a_0,b_0}, we have

$$\langle G_1,F\cdot G_2\rangle=\langle F^*G_1,G_2\rangle=0.$$

Thus, $F^*G_1=\lambda\cdot\chi_{a_0,b_0}$ for some constant $\lambda\neq0$. This implies $F^{*-1}=\lambda^{-1}\chi_{a_0,b_0}^{-1}\cdot G_1\in L^2(\pi^2)$ and hence $F^{-1}\in L^2(\pi^2)$. The converse is obvious. Thus, we have proved that W_0 is a minimum L^2-basis of $L^2(\pi^2)$ if and only if $F^{-1}\in L^2(\pi^2)$.

We will now show that $F^{-1}\in L^2(\pi^2)$ and only if Theorem 4.6, statement 2 holds. We simply observe that

$$\int_{0^+}^{\infty}\frac{1}{t^2}dg(t)=\int_{\pi^2}|F(u,v)|^{-1}du\,dv,$$

and hence $F^{-1}\in L^2(\pi^2)$ if and only if

$$\int_{0^+}^{\infty}\frac{1}{t^2}dg(t)<\infty.$$

5. Examples of ambiguity functions. In this section, we will build ambiguity functions which include the standard ambiguity functions dealt with in radar theory along with an example coming from Heisenberg group theory. We begin with a few general remarks.

An orthonormal basis of $L^2(\mathbb{R})$ is a set of functions f_n, $n \in \mathbb{Z}$, in $L^2(\mathbb{R})$ such that

$$\langle f_n, f_m \rangle = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}$$

and the closure of the linear span of these functions in $L^2(\mathbb{R})$. More generally, a countable subset M of $L^2(\mathbb{R})$ will be called an L^2-basis of $L^2(\mathbb{R})$ if the closure of its linear span equals $L^2(\mathbb{R})$ and no proper subset of M has this property.

The L^2-basis of $L^2(\mathbb{R})$ we construct will be of the following form. A fixed function $f \in L^2(\mathbb{R})$ will be taken and we define

$$f_{a,b}(t) = (M(b)S(a)f)(t) = e^{2\pi i b t} f(t+a), \qquad a, b \in \mathbb{Z}.$$

We will consider examples where the set of functions

$$\{f_{a,b} \colon a, b \in \mathbb{Z}\}$$

is an L^2-basis and use Theorem 4 of §5 to show we have an orthonormal basis.

In the original manuscript, the authors believed that the Gaussian $g(t) = e^{-\pi t^2}$ leads to a minimal basis. As pointed out by the referee, this is not the case. A proof can be seen by showing that $G = |W(g)|$ does not satisfy condition 2 of Theorem 4.6.

The two L^2-bases we consider will be orthonormal. Consider the rectangular function

$$r(t) = \begin{cases} 1, & |t| < \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}. \end{cases}$$

It is easy to see that $r(t)$ satisfies the hypothesis of Theorem 4.5. Thus, the collection of functions

$$\mathscr{R} = \{r_{a,b} \colon a, b \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$, called the rectangular basis of $L^2(\mathbb{R})$.

The rectangle function r is a standard signal processing function. The next basis we consider is more exotic and comes from Heisenberg group theory, especially Theorem 4.5 and [A–T, pp. 81–82]. Applying the Weil–Brezin mapping W to the Gaussian g gives the Heisenberg group theory analogue of the classical theta function. Explicitly,

$$W(g)(\mathbf{x}) = e^{2\pi i (x + x_1 x_2/2)} \sum_{l \in \mathbb{Z}} e^{-\pi (x_2 + l)^2} e^{2\pi i l x_1}$$

where $g(t) = e^{-\pi t^2}$. Consider

$$F(\mathbf{x}) = \frac{W(g)(\mathbf{x})}{|W(g)(\mathbf{x})|}$$

and observe $F(\mathbf{x})$ satisfies the conditions needed for Theorem 4.5 to assert that the set of functions

$$\{\chi_{a_1, a_2} \cdot F \colon a_1, a_2 \in \mathbb{Z}\}$$

is an orthonormal basis of H. By Theorem 4.5, if

$$t(y) = W^{-1}(F)(y)$$

then the set of functions

$$T=\{t_{a,b}\colon a,b\in\mathbb{Z}\}$$

is an orthonormal basis. The only facts we will need are given in the following lemma.

LEMMA 5.1. *Let $\theta(z)=\sum_{l\in\mathbb{Z}}e^{-\pi l^2}e^{2\pi i l z}$, $z=x+iy$, be the classical theta function and set*

$$t(y)=\int_{x=0}^{1}\frac{\theta(z)}{|\theta(z)|}\,dx,\qquad y\in\mathbb{R}.$$

Then the set of functions

$$T=\{t_{a,b}\colon a,b\in\mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

We call T the theta basis of $L^2(\mathbb{R})$.

We will now state, without proof, how the Fourier transform acts on the three bases considered. First,

$$\hat{t}=t,\qquad \hat{r}(v)=\frac{\sin\pi V}{\pi V}.$$

Since, by Theorem 2.6

$$\hat{f}_{a,b}=(\hat{f})_{+b,-a}$$

we have that T is invariant under the action of the Fourier transform and $\mathcal{R}$ maps onto sinusoidals.

We will now relate the cross-ambiguity function $\mathcal{A}(f_{a,b},f_{c,d})$ to $\mathcal{A}(f)$.

THEOREM 5.1. *Let $f\in L^2(\mathbb{R})$ and $f_{a,b}=M(b)S(a)\cdot f$. Then*

$$\mathcal{A}(f_{a,b},f_{c,d})(u,v)=K\cdot\mathcal{A}(f)(u+c-a,v+d-b)$$

where $K=(-1)^{(a+c)(b+c)}e^{-\pi[(b+d)u-(a+c)v]}$.

Proof. Consider

$$\mathcal{B}(f_{a,b},f_{c,d})(u,v)=\langle f_{a,b},M(v)(u)f_{c,d}\rangle$$

which we can write

$$\langle M(b)S(a)f,M(v)S(u)M(d)S(c)f\rangle=\langle f,S(-a)M(-b)M(v)S(u)M(d)S(c)f\rangle.$$

Using Lemma 3.2, this becomes

$$e^{-2\pi i u c}e^{2\pi i a(v+c-b)}\mathcal{B}(f)(u+c-a,v+d-b).$$

The theorem follows once we observe $\mathcal{B}(f,g)=e^{\pi i u v}\mathcal{A}(f,g)$.

The ambiguity function of r is easy to compute and for convenience we give the answer in the next lemma. The ambiguity function of t does not have a simple form.

LEMMA 5.2.

$$\mathcal{A}(r)(u,v)=\begin{cases}-\sin((u-1)\pi v)/\pi v, & 0<u<1,\\ \sin((u-1)\pi v)/\pi v, & -1\leqq u<0,\end{cases}$$

and vanishes elsewhere.

The corresponding cross-ambiguity functions can be determined by Theorem 5.1.

Suppose $f \in L^2(\mathbb{R})$ and

$$F = \{f_{a,b} : a, b \in \mathbb{Z}\}$$

is an L^2-basis of $L^2(\mathbb{R})$. Then, any $h \in L^2(\mathbb{R})$ can be written as

$$h = \sum_{a,\, b \in \mathbb{Z}} \alpha(a,b) f_{a,b}$$

where

$$\langle h,h\rangle = \sum \alpha(a,b)\alpha^*(c,d)\langle f_{a,b}, f_{c,d}\rangle < \infty.$$

Of course, if F is an orthonormal basis the above condition reduces to

$$\sum_{a,\, b \in \mathbb{Z}} |\alpha(a,b)|^2 < \infty.$$

By Theorem 5.1 and Lemma 5.2, we immediately have the following result.

THEOREM C. *Let $f \in L^2(\mathbb{R})$ generate an L^2 basis. Let Φ_f denote the set of functions α: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ such that*

$$\sum \alpha(a,b)\alpha^*(c,)\langle f_{a,b}, f_{c,d}\rangle < \infty.$$

The set of functions

$$F(\alpha) = \sum_{a,b,c,d} \alpha(a,b)\alpha^*(c,d)K(a,b,c,d)A(f)(u+c-a, v+d-b)$$

where

$$K(a,b,c,d) = (-1)^{(a+c)(b+d)} e^{-\pi i[(b+d)u-(a+c)v]}$$

$\alpha \in \Phi_f$, *is the set of ambiguity functions.*

Theorem D follows easily from Theorem C and the discussion in this section.

We close with an interesting example.

Example. Let $p(t)$ be a periodic function of period 1 and consider

$$f(t) = p(t)e^{-\pi t^2}.$$

Write $p(t) = \sum_{a \in \mathbb{Z}} \alpha(a) e^{2\pi i a t}$.

Then, if

$$\mathscr{B}(p)(u,l) = e^{\pi i l u}\int_0^1 p\left(t - \frac{u}{2}\right)p^*\left(t + \frac{u}{2}\right)e^{2\pi i l t}\,dt$$

we have

$$\mathscr{B}(f)(u,v) = \mathscr{B}(e^{-\pi t^2})(u,v) \sum_{l \in \mathbb{Z}} \mathscr{B}(p)(u,l)e^{-\pi l^2/2}e^{2\pi i l(u+iv)/2}.$$

Thus we can interpret the ambiguity function of f as having a continuous part $\mathscr{B}(e^{-\pi t^2})$ and a "discrete" part which is a theta-like function with coefficients given by the periodic version of the ambiguity function of $p(t)$.

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