Frames for classification, quantization, and vector-valued codes

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Topics

- Frames
- Quantization methods
- A comparison of Σ-Δ and PCM
- CAZAC codes
- A vector-valued ambiguity function
- Frame potential and Wiener amalgam criteria for classification

Frames

FUNTFs

- A sequence $F = \{E_j\}_{j=1}^N \subseteq \mathbb{C}^d$ is a *frame* for $\mathbb{C}^d$ if

$$\exists A, B > 0 \quad \text{such that} \quad \forall u \in \mathbb{C}^d, \quad A\|u\|^2 \leq \sum_{j=1}^N |\langle u, E_j \rangle|^2 \leq B\|u\|^2.$$

- F is a *tight frame* if $A = B$; and F is a *finite unit-norm tight frame* (FUNTF) if $A = B$ and each $\|E_j\| = 1$.
- Theorem: If $\{E_j\}_{j=0}^{N-1}$ is a FUNTF for $\mathbb{C}^d$, then

$$\forall u \in \mathbb{C}^d, \quad u = \frac{d}{N} \sum_{j=0}^{N-1} \langle u, E_j \rangle E_j.$$

- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.

Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian ”min-max” waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

DFT FUNTFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUNTFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

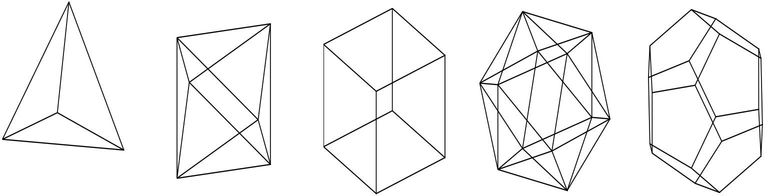
$$N = 8,\, d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}\left(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}}\right)$$

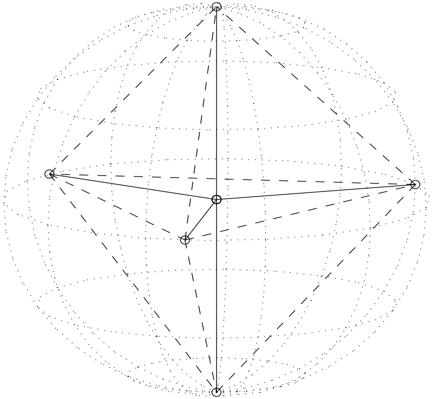
$$m = 1, \ldots, 8.$$

- Sigma-Delta Super Audio CDs - but not all authorities are fans.

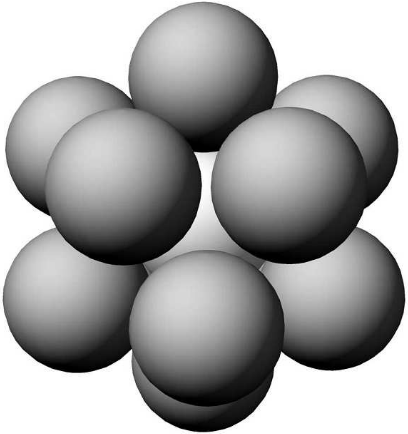
Recent applications of FUNTFs



Recent applications of FUNTFs



Recent applications of FUNTFs



Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = \langle a,b\rangle(a-b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

Characterization of FUNTFs

Theorem

Let $N \leq d$. The minimum value of *TFP*, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUNTFs of N elements for $\mathbb{R}^d$.

Problem

Find FUNTFs analytically, effectively, computationally.

Quantization Methods

A quantization problem

Qualitative Problem Obtain *digital* representations for class X, suitable for storage, transmission, recovery.

Quantitative Problem Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \quad x = \sum x_n e_n, \quad x_n \in \mathbb{K}.$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q: \; X \to \{\sum q_n e_n : \; q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

Methods

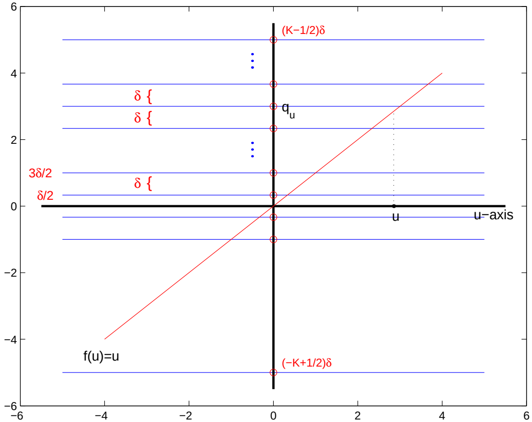
Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given x_n. Reasonable in 16-bit (65,536 levels)digital audio.

Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.

SRQP

Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$



$$Q(u) = \arg\min\{|u-q| : q \in \mathcal{A}_K^\delta\} = q_u$$

PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min |x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then

$$(PCM) \qquad \tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n$$

satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N} \|\sum_{n=1}^{N} (x_n - q_n) e_n\| \leq \frac{d}{N} \frac{\delta}{2} \sum_{n=1}^{N} \|e_n\| = \frac{d}{2}\delta.$$

Not good!

Bennett's white noise assumption

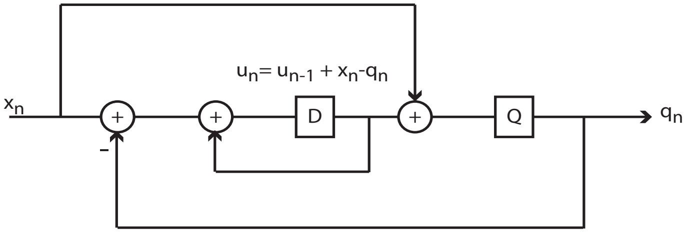
Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A}\,\delta^2 = \frac{(d\delta)^2}{12N}$$

SIGMA-DELTA QUANTIZATION

Given u_0 and $\{x_n\}_{n=1}$

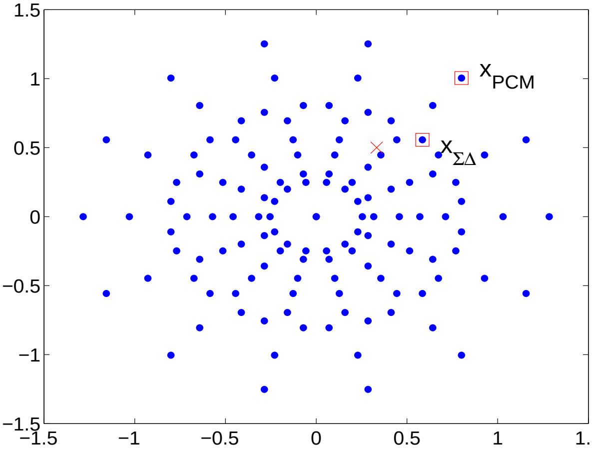
$u_n = u_{n-1} + x_n - q_n$
$q_n = Q(u_{n-1} + x_n)$



First Order ΣΔ

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.



Sigma-Delta quantization – background

- History from 1950s.
- Treatises of Candy, Temes (1992) and Norsworthy, Schreier, Temes (1997).
- PCM for finite frames and $\Sigma\Delta$ for PW_{Ω}:
  Bölcskei, Daubechies, DeVore, Goyal, Güntürk, Kovačevìc, Thao, Vetterli.
- Combination of $\Sigma\Delta$ and finite frames:
  Powell, Yılmaz, and B.
- Subsequent work based on this $\Sigma\Delta$ finite frame theory:
  Bodman and Paulsen; Boufounos and Oppenheim; Jimenez and Yang Wang; Lammers, Powell, and Yılmaz.
- Genuinely apply it.

BPY error estimate

- Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, 2, ..., N\}$. The variation $\sigma(F, p)$ is

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

Theorem

Let $F = \{e_n\}_{n=1}^N$ be an A-FUNTF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ satisfies

$$\|x - \tilde{x}\| \leq \frac{(\sigma(F, p) + 1)d}{N} \frac{\delta}{2}.$$

- Let E_N be the harmonic frame for $\mathbb{R}^d$ (DFT frame for $\mathbb{C}^d$), and let p_N be the identity permutation. Then

$$\forall N,\ \sigma(E_N, p_N) \leq \pi d(d + 1).$$

Complex ΣΔ

Let $\{F_N\}$ be a family of FUNTFs, and p_N be a permutation of $\{1, \ldots, N\}$. Then the frame variation $\sigma(F_N, p_N)$ is a function of N. If $\sigma(F_N, p_N)$ is bounded, then

$$\|x - \widetilde{x}\| = \mathcal{O}(N^{-1}) \textit{ as } N \to \infty.$$

Wang gives an upper bound for the frame variation of frames for $\mathbb{R}^d$, using the results from the Travelling Salesman Problem.

Theorem YW

Let $S = \{v_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ with $d \geq 3$. There exists a permutation p of $\{1, \ldots, N\}$ such that

$$\sum_{j=1}^{N-1} \|v_{p(j)} - v_{p(j+1)}\| \leq 2\sqrt{d+3}N^{1-\frac{1}{d}} - 2\sqrt{d+3}.$$

Complex ΣΔ

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{R}^d$, $|u_0| \leq \delta/2$, and let $x \in \mathbb{R}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. Then, there exists a permutation p of $\{1, 2, \ldots, N\}$ such that the approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\delta d \left((1 - \sqrt{d+3})N^{-1} + \sqrt{d+3}N^{-\frac{1}{d}}\right)$$

This theorem guarantees that

$$\|x - \widetilde{x}\| \leq \mathcal{O}(N^{-\frac{1}{d}}) \text{ as } N \to \infty$$

for FUNTFs for $\mathbb{R}^d$.

A comparison of Σ-Δ and PCM

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Let $x \in \mathbb{C}^d$, $||x|| \leq 1$.

Definition

- $q_{PCM}(x)$ is the sequence to which x is mapped by PCM.
- $q_{\Sigma\Delta}(x)$ is the sequence to which x is mapped by $\Sigma\Delta$.
- $$\mathrm{err}_{PCM}(x) = ||x - \frac{d}{N} L^* q_{PCM}(x)||$$
  $$\mathrm{err}_{\Sigma\Delta}(x) = ||x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)||$$

Fickus question: We shall analyze to what extent $\mathrm{err}_{\Sigma\Delta}(x) < \mathrm{err}_{PCM}(x)$ beyond our results with Powell and Yilmaz.

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Theorem 2

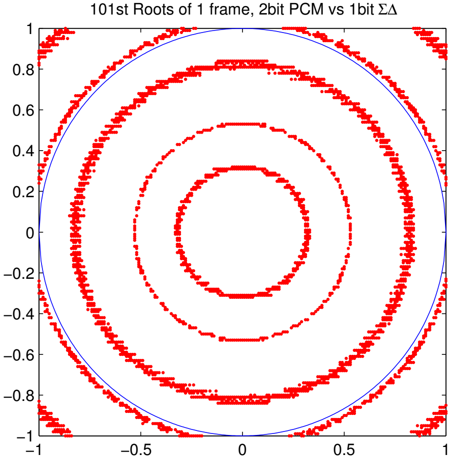
Let $e : [0, 1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = (e(n/N))_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every N. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x) \leq \mathtt{err}_{PCM}(x).$$

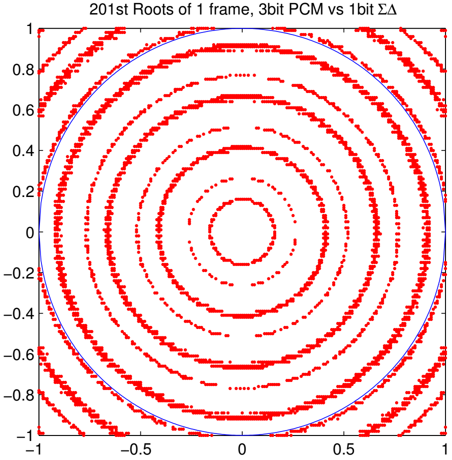
Moreover, a lower bound for N_0 is $d(1 + |e|_{BV})/(\sqrt{d} - 1)$.

Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



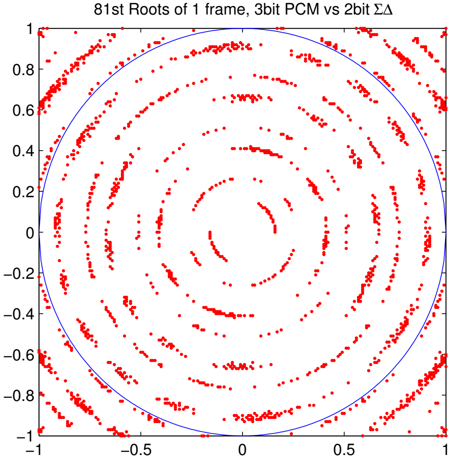
Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 1-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 2-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

CAZAC codes

Discrete ambiguity functions

Let $u : \{0, 1, \ldots, N-1\} \to \mathbb{C}$.

- $u_p : \mathbb{Z}_N \to \mathbb{C}$ is the N-periodic extension of u.
- $u_a : \mathbb{Z} \to \mathbb{C}$ is an aperiodic extension of u:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of u is

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k]\overline{u_p[k]}e^{2\pi ikn/N}.$$

- The *discrete aperiodic ambiguity* function $A_a(u) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ of u is

$$A_a(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_a[m+k]\overline{u_a[k]}e^{2\pi ikn/N}.$$

CAZAC sequences

- $u : \mathbb{Z}_N \to \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)*:

$$\forall m \in \mathbb{Z}_N, \quad |u[m]| = 1, \ \text{(CA)}$$

  and

$$\forall m \in \mathbb{Z}_N \setminus \{0\}, \quad A_p(u)(m, 0) = 0. \ \text{(ZAC)}$$

- Empirically, the (ZAC) property of CAZAC sequences u leads to phase coded waveforms w with low *aperiodic autocorrelation* $\mathcal{A}(w)(t, 0)$.
- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Properties of CAZAC codes

- u CAZAC $\Rightarrow$ u is broadband (full bandwidth).
- There are different constructions of different CAZAC codes resulting in different behavior vis à vis Doppler, additive noises, and approximation by bandlimited waveforms.
- u CA $\Leftrightarrow$ DFT of u is ZAC off dc. (DFT of u can have zeros)
- u CAZAC $\Leftrightarrow$ DFT of u is CAZAC.
- User friendly software: http://www.math.umd.edu/~jjb/cazac

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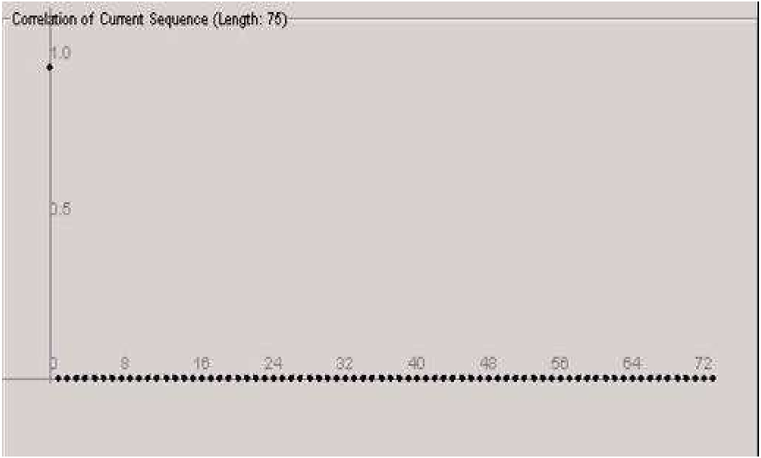
Examples of CAZAC codes

$$
\begin{aligned}
&K = 75 : u(x) = \\
&(1, 1, 1, 1, 1, 1, e^{2\pi i \frac{1}{15}}, e^{2\pi i \frac{2}{15}}, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{4}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{7}{15}}, e^{2\pi i \frac{3}{5}}, \\
&e^{2\pi i \frac{11}{15}}, e^{2\pi i \frac{13}{15}}, 1, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{2}{5}}, e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{4}{5}}, 1, e^{2\pi i \frac{4}{15}}, e^{2\pi i \frac{8}{15}}, e^{2\pi i \frac{4}{5}}, \\
&e^{2\pi i \frac{16}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}, e^{2\pi i}, e^{2\pi i \frac{4}{3}}, e^{2\pi i \frac{5}{3}}, 1, e^{2\pi i \frac{2}{5}}, e^{2\pi i \frac{4}{5}}, e^{2\pi i \frac{6}{5}}, \\
&e^{2\pi i \frac{8}{5}}, 1, e^{2\pi i \frac{7}{15}}, e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{7}{5}}, e^{2\pi i \frac{28}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{13}{15}}, e^{2\pi i \frac{7}{5}}, e^{2\pi i \frac{29}{15}}, \\
&e^{2\pi i \frac{37}{15}}, 1, e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{6}{5}}, e^{2\pi i \frac{9}{5}}, e^{2\pi i \frac{12}{5}}, 1, e^{2\pi i \frac{2}{3}}, e^{2\pi i \frac{4}{3}}, e^{2\pi i \cdot 2}, e^{2\pi i \frac{8}{3}}, \\
&e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{16}{15}}, e^{2\pi i \frac{9}{5}}, e^{2\pi i \frac{38}{15}}, e^{2\pi i \frac{49}{15}}, 1, e^{2\pi i \frac{4}{5}}, e^{2\pi i \frac{8}{5}}, e^{2\pi i \frac{12}{5}}, e^{2\pi i \frac{16}{5}}, \\
&1, e^{2\pi i \frac{13}{15}}, e^{2\pi i \frac{26}{15}}, e^{2\pi i \frac{13}{5}}, e^{2\pi i \frac{52}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{19}{15}}, e^{2\pi i \frac{11}{5}}, e^{2\pi i \frac{47}{15}}, e^{2\pi i \frac{61}{15}})
\end{aligned}
$$

Autocorrelation of CAZAC $K = 75$



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Definition

A *quadratic phase* CAZAC $u : \mathbb{Z}_N \to \mathbb{C}$ is given by

$$u[k] = e^{\pi i P(k)/N}, \quad k = 0, 1, \ldots, N-1,$$

where $P(k)$ is a quadratic polynomial.

Examples:

- *Chu* sequences: $P(k) = k(k-1),\ N$ odd,
- *P4* sequences: $P(k) = k(k-N)$,
- *Wiener* CAZAC sequences: $P(k) = k^2,\ N$ odd.

Rationale and theorem

Theorem 1

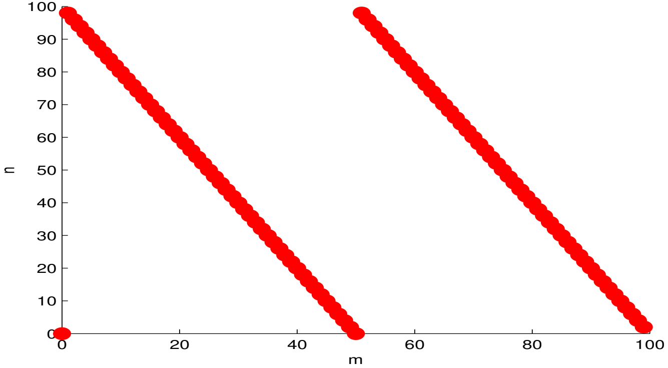
Given $N \geq 1$. Let

$$M = \begin{cases} N, & \text{N odd}, \\ 2N, & \text{N even}, \end{cases}$$

and let ω be a primitive Mth root of unity. Define the Wiener waveform $u : \mathbb{Z}_N \to \mathbb{C}$ by $u(k) = \omega^{k^2}$, $0 \leq k \leq N-1$. Then u is a CAZAC waveform.

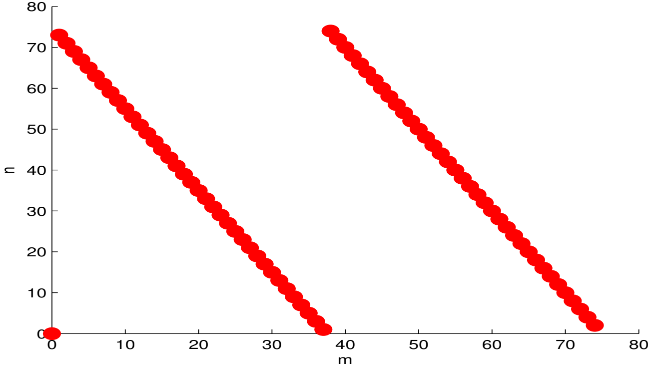
Wiener CAZAC ambiguity domain

$K = 100, j = 2$



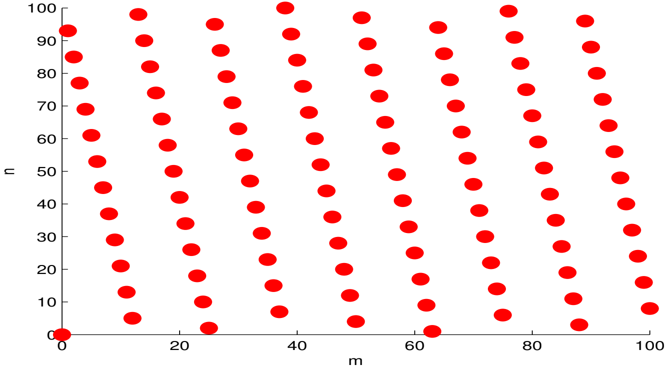
Wiener CAZAC ambiguity domain

$K = 75, j = 1$



Wiener CAZAC ambiguity domain

$K = 101, j = 4$



Definition

Let N be a prime number. A *Björck CAZAC sequence* of length N is

$$u[k] = e^{i\theta_N(k)}, \quad k = 0, 1, \dots, N-1,$$

where, for $N = 1 \pmod 4$,

$$\theta_N(k) = \arccos\left(\frac{1}{1+\sqrt{N}}\right)\left(\frac{k}{N}\right),$$

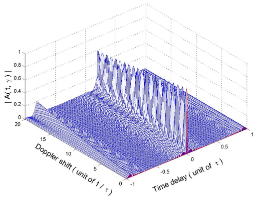
and, for $N = 3 \pmod 4$,

$$\theta_N(k) = \frac{1}{2}\arccos\left(\frac{1-N}{1+N}\right)\left[(1-\delta_k)\left(\frac{k}{N}\right)+\delta_k\right].$$

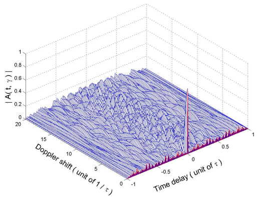
δ_k is Kronecker delta and $\left(\frac{k}{N}\right)$ is Legendre symbol.

Quadratic and Björck ambiguity comparison

- Waveforms associated to Chu-Zadoff and P4 CAZACs are known for their low sidelobes at zero Doppler shift, but their ambiguity functions exhibit strong coupling in the time-frequency plane.
- Waveforms associated to Björck CAZACs can more effectively decouple the effect of time and frequency shifts. However, at zero Doppler shift, their sidelobe behavior is less desirable than quadratic phase CAZACs.
- These differences led to our concatenation idea.



Chu-Zadoff 101



Björck 101

A vector-valued ambiguity function

Outline

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General problem and STFT theme

- Establish the theory of vector-valued ambiguity functions to estimate v in terms of ambiguity data.
- First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.
- Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).

STFT and ambiguity function

Short time Fourier transform – STFT

- The narrow band cross-correlation ambiguity function of v, w defined on $\mathbb{R}$ is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi is\gamma}\,ds.$$

- $A(v,w)$ is the STFT of v with window w.
- The *narrow band radar ambiguity function* $A(v)$ of v on $\mathbb{R}$ is

$$A(v)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi is\gamma}\,ds$$

$$= e^{\pi it\gamma}\int_{\mathbb{R}} v\left(s+\frac{t}{2}\right)\overline{v\left(s-\frac{t}{2}\right)}e^{-2\pi is\gamma}\,ds, \text{ for } (t,\gamma)\in\mathbb{R}^2.$$

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Goal

- Let v be a phase coded waveform with N lags defined by the code u.
- Let u be N-periodic, and so $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$, where $\mathbb{Z}_N$ is the additive group of integers modulo N.
- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ is

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi i k n/N}.$$

Goal

Given a vector valued N-periodic code $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, construct the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A^1_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A^d_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Multiplication problem

- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Multiplication problem

To characterize sequences $\{E_k\} \subseteq \mathbb{C}^d$ and multiplications $*$ so that

$$A^1_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

Ambiguity function assumptions

There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists \, \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ such that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} c_j d_k E_j * E_k.$$

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Remark

- In the previous DFT example, $*$ is intrinsically related to the "addition" defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.
- Alternatively, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.
- Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\bullet$ with the $E_m * E_n = E_{m \bullet n}$ property.
- If $\bullet$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \bullet n}$ property.

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$A_p^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.
- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.
- The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}.$$

- Consequently, $A_p^1(u)$ can be well-defined.

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Vector-valued ambiguity function $A_p^d(u)$

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- The following definition is clearly *motivated* by the STFT.

Definition

$A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ is defined by

$$A_p^d(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}.$$

Outline

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STFT formulation of $A_p(u)$

- The discrete periodic ambiguity function of $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$ can be written as

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k)\rangle,$$

  where $\tau_(m)u(k) = u(m+k)$ is translation by m and $F^{-1}(u)(k)) = \check{u}(k)$ is Fourier inversion.
- As such we see that $A_p(u)$ has the form of a STFT.
- We shall develop a vector-valued DFT theory to *verify* (not just *motivate*) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$.

DFT frames and the vector-valued DFT

Definition

Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of u is

$$\forall\ n \in \mathbb{Z}_N, \quad F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) multiplication.

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Vector-valued Fourier inversion theorem

- Inversion process for the vector-valued case is analogous to the 1-dimensional case.
- We must define a new multiplication in the frequency domain to avoid divisibility problems.
- Define the weighted multiplication $(*) : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ by $u(*)v = u * v * \omega$ where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that each $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$.
- For the following theorem assume $d << N$ or N prime.

Theorem - Vector-valued Fourier inversion

The vector valued Fourier transform F is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

N prime implies F is unitary.

General setting

- If $(G, \bullet)$ is a finite group with representation $\rho : G \longrightarrow GL(\mathbb{C}^d)$, then there is a frame $\{E_n\}_{n \in G}$ and bilinear multiplication, $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, such that $E_m * E_n = E_{m \bullet n}$. Thus, we can develop $A_p^d(u)$ theory in this setting.
- Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms $v : \mathbb{R} \longrightarrow \mathbb{C}^d$, defined by $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, as

$$v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT,(k+1)T)},$$

in terms of $A_p^d(u)$.

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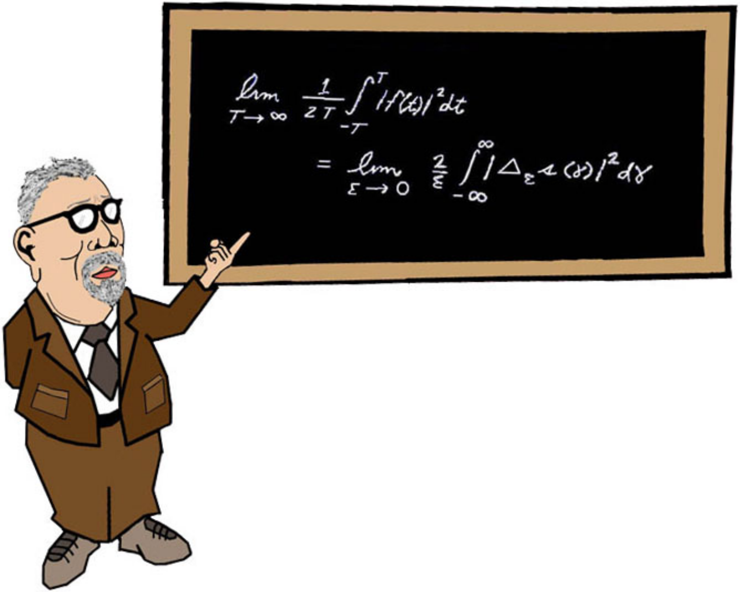
$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$
$= \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon s(\gamma)|^2 d\gamma$

That’s all folks!

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