Scattering Function Characterization for Stochastic Linear Time-varying Channels

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Ambiguity Function (AF)

Definition

Woodward [1][a], introduced the ambiguity function in his book *"Probability and Information Theory with Applications to Radar"* for dealing with the radar problem.

The main contribution of this work was to provide a unified method for obtaining, on the same surface, two variables of great importance for the radar problem: the range (expressed as time delay), and Doppler (expressed as frequency shift).

[1] P.M. Woodward, 1964

Ambiguity Function Formulation

In continuous form, the ambiguity function of two signals $x(t)$ and $y(t)$, can be expressed as:

$$\mathcal{A}_{x,y}(\tau,\nu) = \int x\left(t+\frac{\tau}{2}\right) y^*\left(t-\frac{\tau}{2}\right) e^{j2\pi\nu t} dt \qquad (1)$$

Ambiguity function plays an important role in the analysis of non-stationary signals [2][b][3][c].

[2] F. Hlawatsch, B. Bartels
[3] L. Auslander, R. Tolimieri, 1990

Ambiguity Function (AF): Tool of Time-Frequency Analysis

Ambiguity surface allows to visualize the delay (range) and Doppler (radial velocity) variables.

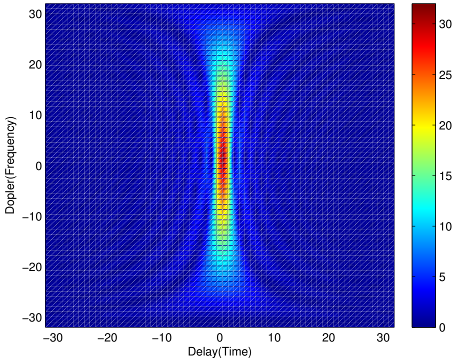

Figure 1: **Delay-Doppler Surface - 2D Representation**

Wigner Distribution (WD)

Definition

The Wigner distribution [4][d] is a quasiprobability distribution. It was introduced by Eugene Wigner in 1932 to study quantum corrections to classical statistical mechanics.
In continuous form, the Wigner distribution of a signal $x(t)$ can be expressed as:

$$\mathcal{W}_x(t, f) = \int_{-\infty}^{\infty} x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) e^{-j2\pi f \tau} d\tau \tag{2}$$

The Wigner distribution is a real function and it can get positive or negative values, preserving time and frequency shifts.

[4] E. Wigner, 1932

Correlation Function (CF)

Definition

In signal processing, the correlation function is a measure of similarity of two waveforms as a function of a time-lag applied to one of them.

For continuous functions, $x(t)$ and $y(t)$, the correlation function is defined as:

$$\mathcal{C}_{x,y}(t) = \int_{-\infty}^{\infty} x(\tau) y^*(t+\tau) d\tau \tag{3}$$

Fourier Operators

Definition

Fourier operators will be very useful in the time-frequency analysis theory. We can define these operators in the follow manner:

$$\mathcal{F}_{v_0}\{f(v_0, v_1, ..., v_{L-1})\} = \int_{-\infty}^{\infty} f(v_0, v_1, ..., v_{L-1}) e^{-j2\pi v_0 \eta} dv_0 \quad (4)$$

$$\mathcal{F}_{v_0}^{-1}\{f(v_0, v_1, ..., v_{L-1})\} = \int_{-\infty}^{\infty} f(v_0, v_1, ..., v_{L-1}) e^{+j2\pi v_0 t} dv_0 \quad (5)$$

Where L is the number of variables of function f. η and t are the output variables in the direct and inverse Fourier domain.

Ambiguity Function and Correlation Function

Definition

Ambiguity function ($\mathcal{A}_h$) and correlation function ($\mathcal{R}_h$) have a mathematical relation. We present this relation below:

$$\mathcal{R}_h(t,\tau) = \int h\left(t',\tau\right) h^*\left(t+t',\tau\right) dt' \quad (6)$$

$$\mathcal{A}_h(\eta,\tau) = \mathcal{F}_t\{\mathcal{R}_h(t,\tau)\} = \int \mathcal{R}_h(t,\tau) e^{-j2\pi\eta t} dt \quad (7)$$

$$\mathcal{A}_h(\eta,\tau) = \int \left[\int h\left(t',\tau\right) h^*\left(t+t',\tau\right) dt'\right] e^{-j2\pi\eta t} dt \quad (8)$$

$$\mathcal{A}_h(\eta,\tau) = \int\int h\left(t',\tau\right) h^*\left(t+t',\tau\right) e^{-j2\pi\eta t} dt' dt \quad (9)$$

Wigner Distribution and Correlation Function Relationship

Definition

Wigner distribution ($\mathcal{W}_h$) and correlation function ($\mathcal{R}_h$) have a mathematical relation. We present this relation below:

$$\mathcal{R}_h(t,\tau) = \int h\left(t',\tau\right) h^*\left(t+t',\tau\right) dt' \quad (10)$$

$$\mathcal{W}_h(t,f) = \mathcal{F}_\tau\{\mathcal{R}_h(t,\tau)\} = \int \mathcal{R}_h(t,\tau) e^{-j2\pi f\tau} d\tau \quad (11)$$

$$\mathcal{W}_h(t,f) = \int \left[\int h\left(t',\tau\right) h^*\left(t+t',\tau\right) dt'\right] e^{-j2\pi f\tau} d\tau \quad (12)$$

$$\mathcal{W}_h(t,f) = \int\int h\left(t',\tau\right) h^*\left(t+t',\tau\right) e^{-j2\pi f\tau} dt' d\tau \quad (13)$$

Inverse relationship

Definition

The inverse Fourier relations in both cases are expressed as:

$$\mathcal{R}_h(t,\tau) = \mathcal{F}_f^{-1}\{\mathcal{W}_h(t,f)\} = \int \mathcal{W}_h(t,f)e^{j2\pi\tau f}df \qquad (14)$$

$$\mathcal{R}_h(t,\tau) = \mathcal{F}_\eta^{-1}\{\mathcal{A}_h(\eta,\tau)\} = \int \mathcal{A}_h(\eta,\tau)e^{j2\pi t\eta}d\eta \qquad (15)$$

Wigner Distribution and Ambiguity Function Relationship

Definition

Wigner distribution ($\mathcal{W}_h$) and ambiguity function ($\mathcal{A}_h$) have a mathematical relation. We present this relation below:

$$\mathcal{A}_h(\eta, \tau) = \mathcal{F}_t\{\mathcal{F}_f^{-1}\{\mathcal{W}_h(t, f)\}\} \tag{16}$$

$$\mathcal{A}_h(\eta, \tau) = \int \left[\int \mathcal{W}_h(t, f) e^{j2\pi\tau f} df \right] e^{-j2\pi\eta t} dt \tag{17}$$

$$\mathcal{A}_h(\eta, \tau) = \int \int \mathcal{W}_h(t, f) e^{-j2\pi[\eta t - \tau f]} df\, dt \tag{18}$$

Wigner Distribution and Ambiguity Function Relationship (2)

Definition

Wigner distribution ($\mathcal{W}_h$) and ambiguity function ($\mathcal{A}_h$) have a mathematical relation. We present this relation below:

$$\mathcal{W}_h(t,f) = \mathcal{F}_\tau\{\mathcal{F}_\eta^{-1}\{\mathcal{A}_h(\eta,\tau)\}\} \quad (19)$$

$$\mathcal{W}_h(t,f) = \int \left[\int \mathcal{A}_h(\eta,\tau) e^{j2\pi t\eta} d\eta \right] e^{-j2\pi f\tau} d\tau \quad (20)$$

$$\mathcal{W}_h(t,f) = \int \int \mathcal{A}_h(\eta,\tau) e^{-j2\pi[f\tau - t\eta]} d\eta d\tau \quad (21)$$

Block Diagram Representation

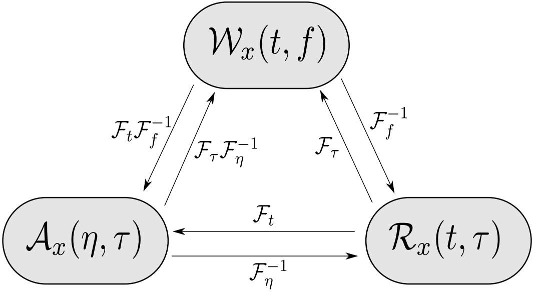

Figure 2: **Relating Wigner Distribution ($\mathcal{W}_x$), Ambiguity Function ($\mathcal{A}_x$), and Correlation Function ($\mathcal{R}_x$).**

Bello's Time-frequency Relations

Bello [5][e] presents relations, Fourier operator based, among system functions: [6][f] [7][g].

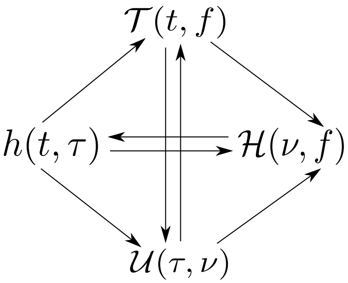

Figure 3: **Bello's System Functions**

[e] [5] P. Bello, 1964
[f] [6] P. Bello, 1963
[g] [7] L. Nguyen, *et al.*, 2001

Delay-Doppler Spread Function

Definition

Function $\mathcal{U}(\tau, \nu)$ is called delay-Doppler spread function, and it has special relevance in this research work. We want to establish a relationship between this function and the other three former system functions.

$$\mathcal{U}(\nu, \tau) = \mathcal{F}_t\{h(t, \tau)\} \tag{22}$$

$$\mathcal{U}(\nu, \tau) = \int h(t, \tau) e^{-j2\pi\nu t} dt \tag{23}$$

$$h(t, \tau) = \mathcal{F}_\nu^{-1}\{\mathcal{U}(\nu, \tau)\} \tag{24}$$

$$h(t, \tau) = \int \mathcal{U}(\nu, \tau) e^{j2\pi t\nu} d\nu \tag{25}$$

Delay-Doppler Spread function. Properties

Definition

Applying Fourier properties, we obtain:

$$\mathcal{F}_t\{h^*(t,\tau)\} = \mathcal{U}^*(-\nu,\tau) \tag{26}$$

Using the correlation function, again, we obtain:

$$\mathcal{R}_h(t,\tau) = \int h\left(t',\tau\right) h^*\left(t+t',\tau\right) dt' \tag{27}$$

Applying the cross-correlation theorem, we obtain:

$$\mathcal{R}_h(t,\tau) = \int |\mathcal{H}_h(\nu,\tau)|^2 e^{j2\pi t} d\nu = \mathcal{F}_\nu^{-1}\{\mathcal{U}_h(\nu,\tau)\} \tag{28}$$

Block Diagram Representation

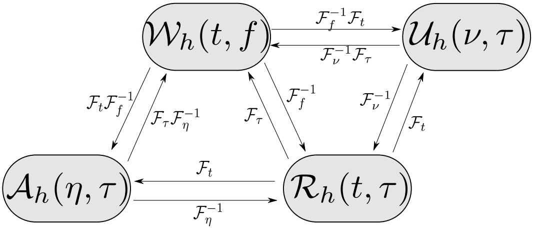

Figure 4: **Relating Wigner Distribution ($\mathcal{W}_x$), Ambiguity Function ($\mathcal{A}_x$), and Correlation Function ($\mathcal{R}_x$), and Delay-Doppler Spread Function ($\mathcal{U}_h$).**

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