A Computational Signal Algebra Framework for a General Class of Discrete Cohen Distributions

Juan Pablo Soto Quiros and Domingo Rodriguez

***Abstract*—This work seeks to extend the study of the general class of L. Cohen's two dimensional distributions from the space $L^2(\mathbb{R}\times\mathbb{R})$ to the space $l^2(\mathbb{Z}_N\times\mathbb{Z}_N)$. This is accomplished by formulating a computational signal algebra framework which allows the analysis and design of efficient algorithms for discrete Cohen distributions (DCDs). The problem has been addressed following techniques developed by L. Cohen centered on the use of the ambiguity function as an entity generator in the formulation of two dimensional continuous distributions. Kronecker products algebra (KPA) was used as the mathematical language for the formulation of all the DCD computational algorithms.**

***Index Terms*—Discrete Cohen Distribution, Ambiguity Function, Time-Frequency Representation, Kronecker Products Algebra, pMatlab.**

I. INTRODUCTION

In 1953, P.M. Woodward developed the concept of ambiguity function. The ambiguity function (AF) plays an important role in many applications dealing with the analysis of non-stationary signals [1], [2]. It is finding new roles in applications such as the joint time-frequency analysis of multiple input multiple output (MIMO) doubly dispersive channels and phase-coded waveform design for orthogonal frequency division multiplexing (OFDM) radar sensing and communications. L. Cohen developed the general representation of a signal in continuous time and frequency (*two dimensional distribution*) calling it "general class" (GC).

This work is presented as follows: In Section II, we provide a review of the continuous ambiguity function and the general class of Cohen distributions. In Section III, we define the discrete ambiguity function (DAF) and formulate a general class of discrete Cohen distributions (DCDs). In Section IV, we present some properties for the DAF and the DCD. In Section V, we develop a matrix notation to describe algorithm formulations of the DAF and the DCD using Kronecker products algebra. Finally, in Section VI, we present some implementation results.

II. CONTINUOUS AMBIGUITY FUNCTIONS AND THE GENERAL CLASS OF COHEN DISTRIBUTIONS

The *continuous ambiguity function* of $x, y \in L^2(\mathbb{R})$ is defined as a map $A_{x,y} : \mathbb{R}\times\mathbb{R} \to L^2(\mathbb{R}\times\mathbb{R})$ given by

$$A_{x,y}(\tau,\nu) = e^{i\pi\nu\tau}\int x\left(t+\tau\right)y^*\left(t\right)e^{j2\pi\nu t}dt. \qquad (1)$$

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In the case when $x = y$, $A_{x,y}$ becomes A_x. L. Cohen [3], [4] developed the concept of a *general class* for two dimensional distributions. As described by L. Cohen, all time-frequency representations, for $x \in L^2(\mathbb{R})$, can be obtained from the canonical map $C_x : \mathbb{R}\times\mathbb{R} \to L^2(\mathbb{R}\times\mathbb{R})$ expressed as

$$C_x(t,\omega) = \int\int A_x(\tau,\eta)\phi(\tau,\eta)e^{-j2\pi(t\eta+\omega\tau)}\ d\eta\ d\tau, \qquad (2)$$

where the function $\phi \in L^2(\mathbb{R}\times\mathbb{R})$ is termed the kernel of the representation. Previous studies have been conducted for the general class of two dimensional distributions using specific kernel functions. Some known kernel functions are presented in Table 1.

Distribution	$\phi(\tau,\theta)$	$\phi(\tau,\eta)$
Wigner	1	1
Margenau - Hill	$\cos(\tau\theta/2)$	$\cos(\pi\tau\eta)$
Kirwood-Rihanzek	$e^{-\tau\theta/2}$	$e^{-\tau\pi\eta}$
Born - Jordan	$\frac{\sin(\tau\theta/2)}{\tau\theta/2}$	$\frac{\sin(\pi\tau\eta)}{\pi\tau\eta}$
Choi - Williams	$e^{-\alpha(\tau\theta)^2}$	$e^{-4\alpha(\pi\tau\eta)^2}$
Zhao-Atlas-Marks	$e^{-\alpha\tau^2}\ \lvert\tau\rvert\ \frac{\sin(\alpha\theta\tau)}{\alpha\theta\tau}$	$e^{-\alpha\tau^2}\ \lvert\tau\rvert\ \frac{\sin(2\pi\alpha\eta\tau)}{2\pi\alpha\eta\tau}$

Table 1: Commonly Known Kernel Functions

III. DISCRETE AMBIGUITY FUNCTIONS (DAF) AND DISCRETE COHEN DISTRIBUTIONS (DCD)

In [5], M. Richman, et al., presented a discrete formulation for the expression given in (1). This new expression is called the *discrete ambiguity function* (DAF). The DAF, for $x, y \in l^2(\mathbb{Z}_N)$, defined by the map $A_{x,y} : \mathbb{Z}_N\times\mathbb{Z}_N \to l^2(\mathbb{Z}_N\times\mathbb{Z}_N)$ given by

$$A_{x,y}[\tau,\nu] = \rho_{\tau,\nu}\sum_{l=0}^{N-1} x[\langle l+\tau\rangle_N]y^*[l]e^{j\frac{2\pi}{N}\nu l}, \qquad (3)$$

where the expression $\rho_{\tau,\nu}$ is defined in [5]. M. Richman, et al., proceeded to use finite dimensional linear operators to arrive at an operator formulation of the DAF. The procedure is as follows:

For $x \in l^2(\mathbb{Z}_N)$, $\gamma, \mu \in \mathbb{Z}$ and $n \in \mathbb{Z}_N$, we define the following linear operators:

- *Translation*: $S_\gamma\{x\}[n] = x[\langle n+\gamma\rangle_N]$
- *Modulation*: $M_\mu\{x\}[n] = e^{j\frac{2\pi}{N}\mu n}x[n]$

For $x, y \in l^2(\mathbb{Z}_N)$, $\langle x, y\rangle$ denotes their inner product and is given by $\langle x, y\rangle = \sum_{n\in\mathbb{Z}_N} x[n]y^*[n]$. Using these definitions, the operator formulation for the DAF follows from (3):

$$A_{x,y}[\tau,\nu] = \rho_{\tau,\upsilon}\langle M_\upsilon\{S_\tau\{x\}\}, y\rangle.$$

This operator formulation of DAF has served as a point of inspiration to develop or general class of discrete Cohen distributions (DCD), expressed in canonical form in the following manner: for $x \in l^2(\mathbb{Z}_N)$, as the map $C_x : \mathbb{Z}_N \times \mathbb{Z}_N \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ as

$$\begin{aligned} C_x[n,k] = \quad & \frac{1}{N}\sum_{\tau=0}^{N-1}\sum_{\nu=0}^{N-1} \rho_{\tau,\upsilon}\langle M_\upsilon\{S_\tau\{x\}\}, x\rangle \\ & \cdot\phi[\tau,\nu] e^{-j\frac{2\pi}{N}(n\nu+k\tau)}, \end{aligned} \tag{4}$$

where ϕ is the kernel in $l^2(\mathbb{Z}_N)$. When $\phi[\tau,\upsilon] = 1$, for $\tau,\upsilon \in \mathbb{Z}_N$, the DCD reduces to the discrete Wigner distribution (DWD) as defined in [5].

Let $\mathcal{F} : \mathbb{Z}_N \times \mathbb{Z}_N \to l^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ be the symmetric discrete Fourier transform in two dimensions (2D); then, (4) defines the symmetric discrete Fourier transform, in 2D, of product of DAF and the kernel [5], i.e.

$$C_x[n,k] = \mathcal{F}\{A_x \cdot \phi\}[k,n] \tag{5}$$

IV. Relating DAF and DCD operations

The concept of the discrete Cohen distributions of a signal was introduced in the previous section through equation (4). In this section we formulate properties of the DCD based on J. J. Benedetto and J. J. Donatelli [6], who presented the following theorem for the DAF of a signal after being effected by translation, modulations, and rotations operators. We first introduce the definition of the rotation operator: For $x \in l^2(\mathbb{Z}_N)$, $\lambda \in \mathbb{C}$, with $|\lambda| = 1$, and $n \in \mathbb{Z}_N$, the rotation operator $R_\lambda\{x\}$ is defined as

$$R_\lambda\{x\}[n] = \lambda x[n]$$

The translation $S_{-\gamma}\{x\}$ and modulation $M_\mu\{x\}$ operators were defined in the previous section.

Theorem 1: Let $x \in l^2(\mathbb{Z}_N)$, $\gamma,\mu,\lambda \in \mathbb{Z}$, with$|\lambda| = 1$, and $\tau,\nu \in \mathbb{Z}_N$:

1) $A_{S_{-\gamma}\{x\}}[\tau,\nu] = e^{-j\frac{2\pi}{N}\gamma\nu}A_x[\tau,\nu]$
2) $A_{M_\mu\{x\}}[\tau,\nu] = e^{j\frac{2\pi}{N}\mu\tau}A_x[\tau,\nu]$
3) $A_{R_\lambda\{x\}}[\tau,\nu] = A_x[\tau,\nu]$

From Theorem 1, and (5), we obtain the following proposition:

Proposition 1: Let $x \in l^2(\mathbb{Z}_N)$ and $\gamma,\mu,\lambda,n,k \in \mathbb{Z}_N$, with $|\lambda| = 1$:

1) $C_{S_{-\gamma}\{x\}}[n,k] = C_x[\langle n+\gamma\rangle_N, k]$
2) $C_{M_\mu\{x\}}[n,k] = C_x[n, \langle k-\mu\rangle_N]$
3) $C_{R_\lambda\{x\}}[n,k] = C_x[n,k]$

V. DCD Computational Frameworks

This section presents a new computational signal algebra framework for the development of discrete Cohen distributions prevalent in diverse applied mathematical and signal processing applications [7], [8], [9], [10], [11], [12].

The sections starts with some basis concepts and definitions from Kronecker products algebra (KPA) [13], [14], [15], a branch of finite dimensional multi-linear algebra, used as the mathematical language to formulate the DCDs. This section, then, continues with the formulation of a computational signal algebra framework for the discrete ambiguity function (DAF). Finally, the section ends with a general Kronecker products based formulation of DCDs under a unifying computational framework.

A. Basic Concepts

The *Kronecker product* of two matrix $A \in \mathbb{C}^{N\times N}$ and $B \in \mathbb{C}^{M\times M}$ is $A \otimes B \in \mathbb{C}^{NM\times NM}$ such that:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ \vdots & \dots & \vdots \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix}.$$

The *Hadamard product* of two matrices $A, B \in \mathbb{C}^{N\times N}$ is $A \odot B \in \mathbb{C}^{N\times N}$ such that: $(A \odot B)_{i,j} = A_{i,j} \cdot B_{i,j}$.

Let $N = ML$.

- We define the *N-point stride M permutation matrix* $P_L^N \in \mathbb{C}^{N\times N}$ by the rule $P_L^N(a \otimes b) = b \otimes a$ for $a \in \mathbb{C}^M$ and $b \in \mathbb{C}^L$. Also, $P_L^N P_L^N = I_N$.
- The *commutation theorem of Kronecker products* establishes that if $A \in \mathbb{C}^{M\times M}$ and $B \in \mathbb{C}^{L\times L}$, then $A \otimes B = P_M^N(B \otimes A)P_L^N$.

The *reshape operator* [16] is the linear transformation $\mathcal{V}_{N,M} : \mathbb{C}^{NM\times 1} \to \mathbb{C}^{N\times M}$ such that for $v \in \mathbb{C}^{N\times M}$:

$$v = \begin{bmatrix} v_0 \\ \vdots \\ v_{M-1} \end{bmatrix}, \quad v_k \in l^2(\mathbb{Z}_N), \tag{6}$$

we obtain $\mathcal{V}_{N,M}\{v\} = [v_0 \; ... \; v_{M-1}]_{N\times M}$.

B. DAF Computational Framework

From (3), we obtain $A'_{x,y} \in \mathbb{C}^{N^2\times 1}$ as

$$A'_{x,y} = P_N^{N^2}(\mathcal{P} \odot [I_N \otimes F_N]\, v), \tag{7}$$

where $P_N^{N^2} \in \mathbb{R}^{N^2\times N^2}$ is a stride by N permutation, $F_N \in \mathbb{C}^{N\times N}$ is the finite Fourier transform matrix [17], i.e., $(F_N)_{m,n} = e^{j\frac{2\pi}{N}mn}$, $\mathcal{P} \in \mathbb{C}^{N^2\times 1}$ is defined as

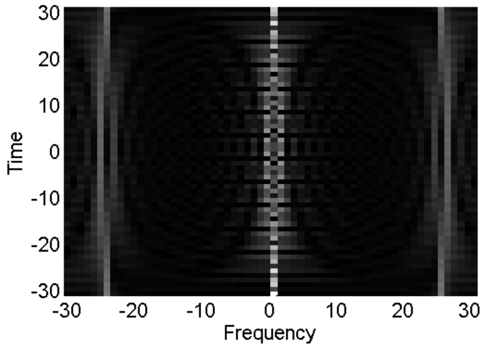
$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_0 \\ \vdots \\ \mathcal{P}_{N-1} \end{bmatrix}, \quad \mathcal{P}_k \in \mathbb{C}^{N\times 1},$$

with $P_k[n] = \rho_{k,n}$, and $v \in \mathbb{C}^{N^2\times 1}$ is defined as

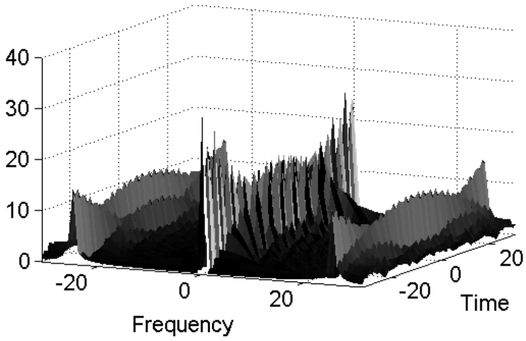
$$v = \begin{bmatrix} v_0 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad v_k \in \mathbb{C}^{N\times 1};$$

such that $v_k[n] = x[\langle n+k\rangle_N]y^*[n]$, for $k,n \in \mathbb{Z}_N$. Then, from (7), we obtain:

$$A'_{x,y} = P_N^{N^2}\begin{bmatrix} H_0 \\ \vdots \\ H_{N-1} \end{bmatrix}, \tag{8}$$

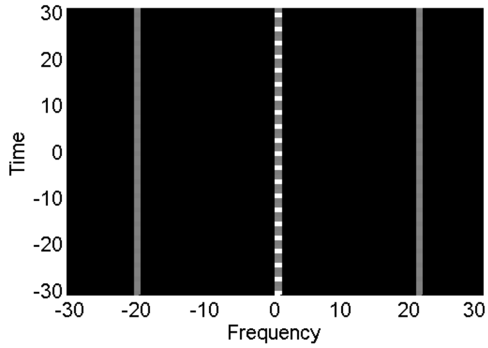


(a)

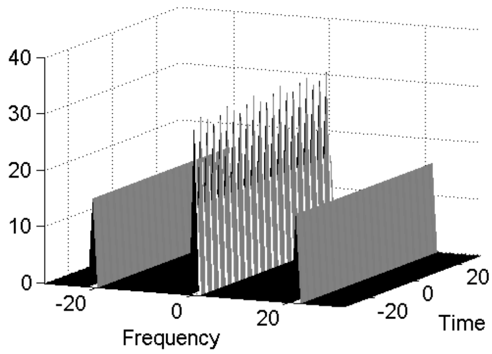


(b)

Fig. 1. Discrete Ambiguity Function of signal $x[n] = \cos\left(\frac{2\pi}{5}n\right)$ for $n \in \mathbb{Z}_{63}$



(a)



(b)

Fig. 2. Discrete Cohen Distribution (*Wigner Distribution*) of signal $x[n] = \cos\left(\frac{2\pi}{3}n\right)$ for $n \in \mathbb{Z}_{63}$

with $H_k \in l^2(\mathbb{Z}_N)$ such that $H_k = \mathcal{P}_k \odot F_N v_k$. From (8) we obtain the *DAF matrix* $A_{x,y} \in \mathbb{C}^{N\times N}$ as

$$A_{x,y} = \mathcal{V}_{N,N}\{A'_{x,y}\} = [H'_0\ H'_1\ \ldots\ H'_{N-1}]_{N\times N}, \qquad (9)$$

with $H'_k[n] = H_n[k]$.

C. DAF Framework Properties

Allow $x = y$ in (3) above. By Theorem 1, then we obtain:

- **Translation:** Let $A'_{S_{-\gamma}\{x\}} \in \mathbb{C}^{N^2\times 1}$ such that:

  $$A'_{S_{-\gamma}\{x\}} = \Psi \odot A'_x, \qquad (10)$$

  where $\Psi \in \mathbb{C}^{N^2\times 1}$, with

  $$\Psi = \begin{bmatrix} \psi_0^\gamma \\ \vdots \\ \psi_{N-1}^\gamma \end{bmatrix}, \qquad \psi_m^\gamma \in \mathbb{C}^{N\times 1},$$

  and $\psi_m^\gamma[n] = e^{-j\frac{2\pi}{N}\gamma m}$. From (9) and (10), we obtain the *DAF matrix of signal translation* $A_{S_{-\gamma}\{x\}} \in \mathbb{C}^{N\times N}$ as

  $$A_{S_{-\gamma}\{x\}} = \mathcal{V}_{N,N}\{A'_{S_{-\gamma}\{x\}}\} = [u_0\ \ldots\ u_{N-1}]_{N\times N}, \qquad (11)$$

  with $u_k = \psi_k^\gamma \odot H'_k$, $k \in \mathbb{Z}_N$.
- **Modulation:** Let $A'_{M_\upsilon\{x\}} \in \mathbb{C}^{N^2\times 1}$ such that

  $$A'_{M_\mu\{x\}} = (I_N \otimes \theta_\mu)A'_x,$$

  where $\theta_\mu \in \mathbb{C}^{N\times 1}$ with $\theta_\mu[n] = e^{j\frac{2\pi}{N}\mu n}$. $A'_{M_\mu\{x\}}$ may be expressed also as:

  $$A'_{M_\mu\{x\}} = \Theta \odot A'_x, \qquad (12)$$

  with $\Theta \in \mathbb{C}^{N^2\times 1}$ is defined as

  $$\Theta = \begin{bmatrix} \theta_\mu \\ \vdots \\ \theta_\mu \end{bmatrix},$$

  From (9) and (12), we obtain the *DAF matrix of signal modulation* $A_{M_\mu\{x\}} \in \mathbb{C}^{N\times N}$ as

  $$A_{M_\mu\{x\}} = \mathcal{V}_{N,N}\{A'_{M_\mu\{x\}}\} = [w_0\ \ldots\ w_{N-1}]_{N\times N}, \qquad (13)$$

  with $w_k = \theta_\mu \odot H'_k$.

D. Unified DCD Computational Framework

From (4) we may express as $C'_x \in \mathbb{C}^{N^2\times 1}$ as

$$C'_x \ = \tfrac{1}{N}(F_N^* \otimes F_N^*)P_N^{N^2}(A'_x \odot \Phi), \qquad (14)$$

where F_N^* is conjugate matrix of F_N, $\Phi \in \mathbb{C}^{N^2\times 1}$ as

$$\Phi = \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{N-1} \end{bmatrix}, \qquad \phi_m \in \mathbb{C}^{N\times 1},$$

with $\phi_m[n] = \phi[n,m]$ (kernel). We use the Kronecker products property $(F_N^* \otimes F_N^*) = (I_N \otimes F_N^*)(F_N^* \otimes I_N)$, and the commutation theorem of Kronecker products, to express (14) as

$$\begin{aligned} C'_x &= \frac{1}{N}(I_N \otimes F_N^*)P_N^{N^2}(I_N \otimes F_N^*)P_N^{N^2}P_N^{N^2}(A'_x \odot \Phi) \\ &= \frac{1}{N}(I_N \otimes F_N^*)P^{N^2}(I_N \otimes F_N^*)(A'_x \odot \Phi). \qquad (15) \end{aligned}$$

From (15), we obtain that the *DCD matrix* $C_x \in \mathbb{C}^{N \times N}$ is

$$C_x = \mathcal{V}_{N,N}\{C'_x\}. \tag{16}$$

E. DCD Framework Properties

By Proposition 1, then we obtain

- **Translation:** Let $C'_{S_{-\gamma}\{x\}} \in \mathbb{C}^{N^2 \times 1}$ such that

$$\begin{aligned} C'_{S_{-\gamma}\{x\}} &= \tfrac{1}{N}(I_N \otimes F_N^*)P_N^{N^2} \\ &\cdot (I_N \otimes F_N^*)(A'_{S_{-\gamma}\{x\}} \odot \Phi). \end{aligned} \tag{17}$$

  From (17), we obtain *DCD matrix of signal modulation* $C_{S_{-\gamma}\{x\}} \in \mathbb{C}^{N \times N}$ such that

$$C_{S_{-\gamma}\{x\}} = \mathcal{V}_{N,N}\{C'_{S_{-\gamma}\{x\}}\}. \tag{18}$$

- **Modulation:** Let $C'_{M_\mu\{x\}} \in \mathbb{C}^{N^2 \times 1}$ as

$$\begin{aligned} C'_{M_\mu\{x\}} &= \tfrac{1}{N}(I_N \otimes F_N^*)P_N^{N^2}(I_N \otimes F_N^*) \\ &\cdot (A'_{M_\mu\{x\}} \odot \Phi). \end{aligned} \tag{19}$$

  From (19), we obtain *DCD matrix of signal translation* $C_{M_\mu\{x\}} \in \mathbb{C}^{N \times N}$ as

$$C_{M_\mu\{x\}} = \mathcal{V}_{N,N}\{C'_{M_\mu\{x\}}\}. \tag{20}$$

VI. IMPLEMENTATION RESULTS

All of the discrete time-frequency distributions presented in this work, under a new computational signal algebra framework, where developed using the numeric computation and graphic visualization software package MATLAB®. Special attention was given to the implementation of the new discrete time-frequency representations presented in equations (9), (11), (13), (16), (20) and (18). The parallel programming environment pMatlab [18] was utilized during the implementation of these discrete time-frequency distributions to seek further insight into the parallel nature of the algorithm formulations. To calculate the stride permutation is used *Proposition 2*:

Proposition 2: Let $v \in \mathbb{C}^{N^2 \times 1}$, as (6). Then,

$$\mathcal{V}_{N,N}\{P_N^{N^2} v\} = (\mathcal{V}_{N,N}\{v\})^T$$

Proof: Let for $i, j = 1, ..., N$. Then,

$$\mathcal{V}_{N,N}\{v\} = [v_0 \; ... \; v_{N-1}]_{N \times N}$$

But, $(\mathcal{V}_{N,N}\{v\})_{ij} = v_j[i] \Rightarrow (\mathcal{V}_{N,N}\{v\})^T_{ij} = v_i[j]$. On the other hand,

$$P_N^{N^2} v = \begin{bmatrix} v'_0 \\ \vdots \\ v'_{N-1}, \end{bmatrix}$$

with $v'_j[i] = v_i[j]$, for $i, j = 1, ..., N$. Then:

$$(\mathcal{V}_{N,N}\{P_N^{N^2} v\})_{ij} = v'_j[i] = v_i[j]$$

■

VII. MIMO TIME-FREQUENCY DISPERSIVE CHANNELS

This section discusses the potential use of DCDs, in particular, the discrete ambiguity function (DAF), to aid in the characterization of multiple input multiple output (MIMO) underwater time-frequency dispersive channels, also known as MIMO underwater doubly dispersive channels. In essence, the DAF is being used as a signal representation tool for the characterization of scattering function of MIMO underwater time-frequency dispersive channels, following the works of T. H. Eggen [19], [20]. We proceed to present an initial matrix model of a MIMO underwater channel using the DAF as a signal representation tool. Thus, we assume $M = SP$ and $N = QR$, with $L, M, N, S, P, Q, R \in \mathbb{Z}_{\mathbb{MN}}$, such that L is the number of scattering point targets, M is the number of transmitter antennas, N is the number of receiver antennas, S is the number of transmitter arrays, P is the number of antennas for each transmitter array, R is the number of receiver arrays, and Q is the number of antennas for each receiver array. We, first, begin by considering the simple case where there is a single transmitter array as well as a single receiver array in the MIMO channel configuration.

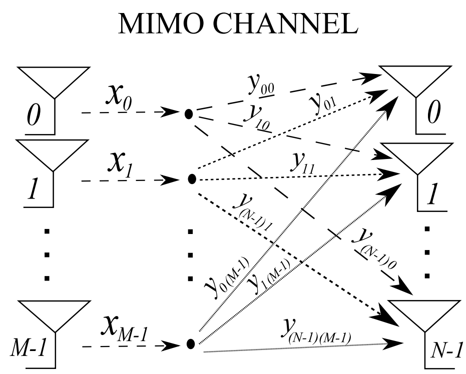


Fig. 3. Underwater Time-Frequency Dispersive Channel for $R = S = 1$

Let $x, y \in C^K$ and $A_{x,y} \in \mathbb{C}^{K^2 NM}$ such that

$$A = (I_{NM} \otimes P_K^{K^2})(I_{KNM} \otimes F_K)(X \odot Y) \tag{21}$$

where $X \in \mathbb{C}^{LNM}$ such that

$$X = \bigsqcup_{m \in \mathbb{Z}_M} X_m, \qquad X_m \in \mathbb{C}^{LN}$$

with $X_k = 1_{KN} \otimes x_k$, where $x_k \in \mathbb{C}^K$ is the transmitting signal (Figure 3), $Y \in \mathbb{C}^{KMN}$ is a vector such that

$$Y = \bigsqcup_{m \in \mathbb{Z}_M} Y_m, \qquad Y_m \in \mathbb{C}^{K^2 N},$$

$$Y_m = \bigsqcup_{n \in \mathbb{Z}_N} Y_{nm}, \qquad Y_{nm} \in \mathbb{C}^{K^2},$$

$$Y_{nm} = \bigsqcup_{k \in \mathbb{Z}_K} Y_{nm}^k, \qquad Y_{nm}^k \in \mathbb{C}^K,$$

where $Y_{nm}^k[l] = y_{nm}^*[\langle l + k \rangle_N]$, with $l \in \mathbb{Z}_K$ and $y_{nm} \in \mathbb{C}^K$ is the signal at the receiver (Figure 3).

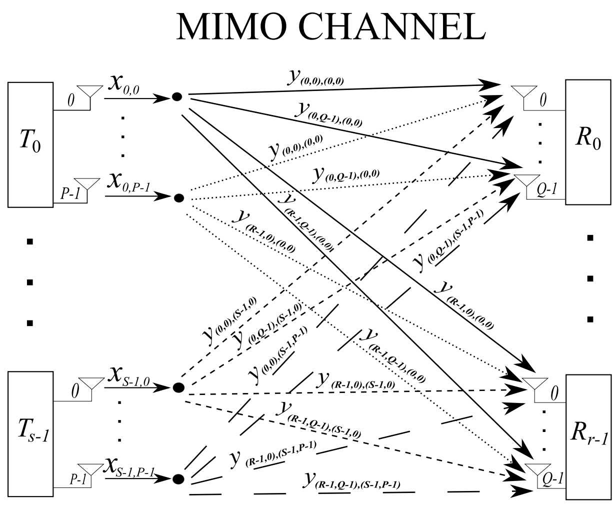


Fig. 4. Underwater Time-Frequency Dispersive Channel for $R, S \in \mathbb{Z}_{\mathbb{MN}}$

We know proceed to discuss the more general MIMO channel configuration described at the beginning of this section.

Let $A \in \mathbb{C}^{L^2NM}$ such that

$$A = (I_{QRSP} \otimes P_K^{K^2})(I_{KQRSP} \otimes F_K)(X \odot Y),$$

where $X \in \mathbb{C}^{K^2NM}$ such that

$$X = \bigsqcup_{s \in \mathbb{Z}_S} X_s, \quad X_s \in \mathbb{C}^{K^2NP},$$

with

$$X_s = \bigsqcup_{p \in \mathbb{Z}_P} X_{s,p}, \quad X_{s,p} \in \mathbb{C}^{K^2N},$$

where $X_{s,p} = 1_{KN} \otimes x_{s,p}$ and $x_{s,p} \in \mathbb{C}^K$ is the transmitting signal (Figure 4); and $Y \in \mathbb{C}^{K^2NM}$ is a vector such that

$$Y = \bigsqcup_{s \in \mathbb{Z}_S} Y_s, \quad Y_s \in \mathbb{C}^{K^2NP},$$

with

$$Y_s = \bigsqcup_{p \in \mathbb{Z}_P} Y_{s,p}, \quad Y_{s,p} \in \mathbb{C}^{K^2N},$$

with

$$Y_{s,p} = \bigsqcup_{r \in \mathbb{Z}_R} Y_{r,(s,p)}, \quad Y_{r,(s,p)} \in \mathbb{C}^{K^2Q},$$

with

$$Y_{r,(s,p)} = \bigsqcup_{q \in \mathbb{Z}_Q} Y_{(r,q),(s,p)}, \quad Y_{(r,q),(s,p)} \in \mathbb{C}^{K^2},$$

with

$$Y_{(r,q),(s,p)} = \bigsqcup_{k \in \mathbb{Z}_K} Y^k_{(r,q),(s,p)}, \quad Y^k_{(r,q),(s,p)} \in \mathbb{C}^K,$$

such that $Y^k_{(r,q),(s,p)}[l] = y^*_{(r,q),(s,p)}[\langle l + k \rangle_K]$, where $y_{(r,q),(s,p)} \in \mathbb{C}^K$ are receivers (Figure 4).

A series of computational steps follows in order to arrive at the desired result in an algorithmic manner:

Step 1 : Let $E_s \in \mathbb{C}^{K^2NP}$ such that

$$E_s = (I_{QRP} \otimes P_K^{K^2})(I_{KQRP} \otimes F_K)(X_s \odot Y_s),$$

then $A = \bigsqcup_{s \in \mathbb{Z}_S} E_s$.

Step 2 : Let

$$B = \mathcal{V}_{K^2QRP,S}\{A\} = [E_0 \ ... \ E_{S-1}]_{K^2QRP \times S}.$$

Step 3 : Let $D_{s,p} \in \mathbb{C}^{K^2N}$ with

$$D_{s,p} = (I_{QR} \otimes P_K^{K^2})(I_{KQR} \otimes F_K)(X_{s,p} \odot Y_{s,p}),$$

then let $\mathcal{C}_s\{B\} = E_s$ such that $E_s = \bigsqcup_{p \in \mathbb{Z}_P} D_{s,p}$.

Step 4 : Let

$$C = \mathcal{V}_{K^2QR,P}\{E_s\} = [D_{s,0} \ ... \ E_{s,P-1})]_{K^2QR \times P}.$$

Step 5 : Let $J_{r,(s,p)} \in \mathbb{C}^{K^2Q}$ with

$$J_{r,(s,p)} = (I_Q \otimes P_K^{K^2})(I_{KQ} \otimes F_K)((1_{KQ} \otimes x_{s,p}) \odot Y_{r,(s,p)}),$$

then let $\mathcal{C}_p\{C\} = D_{s,p}$ such that $D_{s,p} = \bigsqcup_{r \in \mathbb{Z}_R} J_{r,(s,p)}$.

Step 6 : Let

$$Q = \mathcal{V}_{K^2Q,R}\{D_{s,p}\} = [J_{0,(s,p)} \ ... \ J_{R-1,(s,p)}]_{K^2Q \times R}.$$

Step 7 : Let $G_{(r,q),(s,p)} \in \mathbb{C}^{K^2}$ with

$$Q_{(r,q),(s,p)} = P_K^{K^2}(I_K \otimes F_K)((1_K \otimes x_{s,p}) \odot Y_{(r,q),(s,p)}),$$

then let $\mathcal{C}_p\{Q\} = J_{r,(s,p)}$ such that $J_{r,(s,p)} = \bigsqcup_{q \in \mathbb{Z}_Q} G_{(r,q),(s,p)}$.

Step 8 : Let

$$W = \mathcal{V}_{K^2,Q}\{J_{r,(s,p)}\} = [G_{(r,0),(s,p)} \ ... \ G_{(r,Q-1),(s,p)}]_{K^2 \times Q}$$

Step 9 : Let $H^k_{(r,q),(s,p)} \in \mathbb{C}^K$ with

$$H^k_{(r,q),(s,p)} = F_K(x_{s,p} \odot Y^k_{(r,q),(s,p)}),$$

then let $\mathcal{C}_p\{W\} = G_{(r,q),(s,p)}$ such that $G_{(r,q),(s,p)} = P_K^{K^2} \bigsqcup_{k \in \mathbb{Z}_K} H^k_{(r,q),(s,p)}$.

Step 10: Let

$$O = \mathcal{V}_{K,K}\{G_{(r,q),(s,p)}\} = [H'^0_{(r,q),(s,p)} \ ... \ H'^{K-1}_{(r,q),(s,p)}]_{K \times K}.$$

with $H'^k_{(r,q),(s,p)}[l] = H^l_{(r,q),(s,p)}[k]$.

Step 11: Thus, with the vector A, the DAF matrix of $x_(s,p), y_(r,q)(s,p)$, $A_{x_(s,p),y_(r,q)(s,p)} \in \mathbb{C}^{K,K}$ is obtained of

$$A_{x_(s,p),y_(r,q)(s,p)} = \mathcal{V}_{K,K}\{\mathcal{C}_q\{\mathcal{V}_{K^2,Q}\{\mathcal{C}_r\{\mathcal{V}_{K^2Q,R}\{\mathcal{C}_p\{\mathcal{V}_{K^2QR,P}\{\mathcal{C}_s\{\mathcal{V}_{K^2QRP,S}\{A\}\}\}\}\}\}\}\}\}.$$

VIII. Conclusion

This work presented a unifying computational signal algebra framework for the formulation of discrete Cohen distributions under the language of Kronecker products algebra. The work is based on the work of L. Cohen on the general formulation of continuous time-frequency distributions. It is also based on the work of M. Richman, W. Parks and R. Shenoy on discrete-time, discrete-frequency, time-frequency representations. Special attention was given to the work of R. Tolimieri and L. Auslander on the group theoretic properties of the ambiguity function as well as the work of J. J. Benedetto and J. J. Donatelli on vector-valued ambiguity functions.

As part of future works associated with the results presented on this article, we contemplate the formulation of a holomorphic oriented geometric signal algebra framework for non-Abelian signal-based information processing for underwater communication signal processing applications. This proposed geometric signal algebra framework intends to shed new insights and understanding of how to characterize information flow from MIMO underwater communication systems, envisioned here as physical resources, to what has been termed the underwater cyberspace infosphere (UCI), namely, all the computational infrastructure and informational resources available in cyberspace for underwater surveillance monitoring under a cyber-physical systems setting. Along these lines, the proposed work deals with the acquisition of sensor-based underwater signals from spatially distributed observable entities, in a generalized observational environment.

This proposed geometric signal algebra framework tries to inquire into a new modality for the treatment of theses sensor-based signals in order to seek understanding of how to combine syntactical and structural methods for underwater signal description, such as space-time and time-frequency representations, with semantics and semiotic aspects of the signals, resulting in a new signal representation paradigm. Holomorphic signals, linear operator mappings, manifolds, and tensor fields are used as functional primitives or basic atoms for the structural foundation of the proposed geometric signal algebra framework.

A preliminary approach at this geometric signal algebra framework deals with treatment of syntactic and semantic information associated a given underwater signal model. The syntactic information is manipulated in finite dimensional multi-linear signal algebra spaces where space-time and time-frequency representations become vector entities. The semantic information is explored by retrieving qualitative and quantitative information of these vector entities when embedded in smooth manifolds. A first attempt at linking syntactic and semantic information is being made by restricting the type of underwater signals to multi-component polynomial phase type signals which can then be modeled as holomorphic signals.

Dealing with finite length discrete holomorphic signals allows for introducing a finite dimensional linear operator approach for the treatment of these signals as well as analysis techniques using signal tensor algebra. In this context, tensor signal processing becomes a subset of signal tensor algebra, which, in turns, is a subset of signal tensor analytics (sigtetics). Finally, some aspects of non-additive divergence measures are being used to study information functionals in the context of information-based complexity to aid in the linear time varying channel identification process.

References

[1] F. Hlawatsch and G. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations," *Signal Processing Magazine, IEEE*, vol. 9, no. 2, pp. 21 –67, Apr. 1992.

[2] L. Auslander and R. Tolimier, *On finite Gabor expansion of signals*. London, UK: Springer-Verlag, 1990, pp. 13–23.

[3] L. Cohen, "Generalized Phase-Space Distribution Functions," *Journal of Mathematical Physics*, vol. 7, no. 5, pp. 781–786, 1966.

[4] ——, *Time-frequency analysis: theory and applications*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1995.

[5] M. Richman, T. Parks, and R. Shenoy, "Discrete-time, discrete-frequency, time-frequency analysis," *Signal Processing, IEEE Transactions on*, vol. 46, no. 6, pp. 1517 –1527, Jun. 1998.

[6] J. Benedetto and J. Donatelli, "Ambiguity function and frame-theoretic properties of periodic zero-autocorrelation waveforms," *Selected Topics in Signal Processing, IEEE Journal of*, vol. 1, no. 1, pp. 6 –20, june 2007.

[7] L. Auslander and R. Tolimieri, "Radar ambiguity functions and group theory," *SIAM J. Math. Anal.*, vol. 16, pp. 577–601, 1985.

[8] R. Tolimieri and R. S. Orr, "Poisson summation, the ambiguity function, and the theory of weyl-heisenberg frames," *Journal of Fourier Analysis and Applications*, vol. 1, pp. 233–247, 1994.

[9] A. Bonamia, G. Garrigósb, and P. Jaminga, "Discrete radar ambiguity problems," *Applied and Computational Harmonic Analysis*, vol. 23, no. 3, pp. 388 – 414, 2007.

[10] G. San Antonio, D. Fuhrmann, and F. Robey, "Mimo radar ambiguity functions," *Selected Topics in Signal Processing, IEEE Journal of*, vol. 1, no. 1, pp. 167 –177, june 2007.

[11] D. Wu and J. Morris, "Discrete cohen's class of distributions," in *Time-Frequency and Time-Scale Analysis, 1994., Proceedings of the IEEE-SP International Symposium on*, oct 1994, pp. 532 –535.

[12] ——, "Time-frequency representations using a radial butterworth kernel," in *Time-Frequency and Time-Scale Analysis, 1994., Proceedings of the IEEE-SP International Symposium on*, oct 1994, pp. 60 –63.

[13] C. F. van Loan, "The ubiquitous kronecker product," *J. Comput. Appl. Math.*, vol. 123, pp. 85–100, November 2000.

[14] J. R. Johnson, R. W. Johnson, D. Rodriguez, and R. Tolimieri, "A methodology for designing, modifying, and implementing fourier transform algorithms on various architectures," *Circuits, Systems, and Signal Processing*, vol. 9, pp. 449–500, 1990.

[15] D. Rodriguez, J. Seguel, and E. Cruz, "Algebraic methods for the analysis and design of time-frequency signal processing algorithms," in *Circuits and Systems, 1993., ISCAS '93, 1993 IEEE International Symposium on*, may 1993, pp. 196 –199 vol.1.

[16] C. D. M. Martin and C. F. Van Loan, "A jacobi-type method for computing orthogonal tensor decompositions," *SIAM J. Matrix Anal. Appl.*, vol. 30, pp. 1219–1232, September 2008.

[17] M. An, A. K. Brodzik, and R. Tolimieri, *Ideal Sequence Design in Time-Frequency Space*. Boston: Birkhäuser, 2009.

[18] N. Bliss, J. Kepner, H. Kim, and A. Reuther, "pmatlab: Parallel matlab library for signal processing applications," *Acoustics Speech and Signal Processing 2007 ICASSP 2007 IEEE International Conference on*, vol. 4, pp. IV–1189 – IV–1192, 2007.

[19] T. Eggen, A. Baggeroer, and J. Preisig, "Communication over doppler spread channels. part i: Channel and receiver presentation," *Oceanic Engineering, IEEE Journal of*, vol. 25, no. 1, pp. 62 –71, jan 2000.

[20] T. Eggen, J. Preisig, and A. Baggeroer, "Communication over doppler spread channels. ii. receiver characterization and practical results," *Oceanic Engineering, IEEE Journal of*, vol. 26, no. 4, pp. 612 –621, oct 2001.