On Artificial Neural Networks And Abelian Harmonic Analysis

Domingo Rodríguez
Electrical and Computer Engineering Dept.
University of Puerto Rico-Mayagüez Campus
Mayagüez, PR 00709-5000

Jairo Pertuz-Campo
Physics Department
University of Puerto Rico-Mayagüez Campus
Mayagüez, PR 00709-5000

Abstract

This work deals with the use of artificial neural networks (ANN) for the digital processing of finite discrete time signals. The effort concentrates on the efficient replacement of fast Fourier transform (FFT) algorithms with ANN algorithms in certain engineering and scientific applications. The FFT algorithms are efficient methods of computing the discrete Fourier transform (DFT). The ubiquitous DFT is utilized in almost every digital signal processing application where harmonic analysis information is needed. Applications abound in areas such as audio acoustics, geophysics, biomedicine, telecommunications, astrophysics, etc. To identify more efficient methods to obtain a desired spectral information will result in a reduction in the computational effort required to implement these applications.

Introduction

We define signal processing as the mathematical treatment of signals with the objective of extracting information of relevance to a user. In the case of discrete-time signal processing, the objective is to extract information from a sequence of numbers. We call this process filtering, and call the computational structure which performs this operation a filter. As it is stated by Blahut [11], filtering is the most important task in signal processing. Our objective is to identify ANN structures which will allow us to perform filtering in an efficient manner and with a certain degree of fault-tolerance. We concentrate in a class of filters called linear, shift-invariant finite impulse response (LSI-FIR) systems which Blahut also states are the most important devices in digital signal processing.

We start in the next section by providing a general description of LSI-FIR systems and their importance in signal processing. We then describe the discrete Fourier transform operator, the most important tool for implementing these systems nowadays. We proceed to describe the spectral properties of LSI-FIR systems and then continue describing methods of representing these systems for the purpose of hardware implementations. We conclude by establishing a relationship between LSI-FIR systems and ANN structures. In particular, we discuss multi-layer feed-forward perceptrons as described by [1] and [8].

* This work was partially supported by NSF Grant #CDA-8913486

 0-8194-0693-7/91/$4.00

LSI-FIR Systems

In this section we define the concept of LSI-FIR systems and the substantive role they play in signal processing. We first introduce some mathematical notations and definitions.

Let Z/n denote the set of n nonnegative integers

$$\{0, 1, \ldots, n-1\}. \tag{1}$$

An n-point sequence over the complex field C is the mapping

$$f: Z/n \longrightarrow C \tag{2}$$

The set Z/n is called the indexing set of the sequence of f. The value of the sequence f on $j \in Z/n$ is $f(j)$ and it is usually denoted by f_j.

We denote by the symbol f the n-tuple formed with the values f_j, $j = 0, 1, 2, \ldots, n-1$:

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix} \tag{3}$$

The set of all sequences $f: Z/n \to C$ forms a linear vector space which we denote by $L(Z/n)$. The set $L(Z/n)$ is isomorphic to the n-dimensional complex vector space C^n.

The set of n n-point sequences

$$\left\{\delta_{[k]}\colon\ k = 0, 1, \ldots, n-1\right\}, \tag{4}$$

where

$$\delta_{[k]}(j) = \begin{cases} 1, & j = k; \quad j, k \in Z/n \\ 0, & j \neq k \end{cases}, \tag{5}$$

forms a basis for the space $L(Z/n)$ which we call the standard basis.

We now introduce the shift operator S_n over the space $L(Z/n)$. This operator is the central component in the characterization of LSI-FIR systems.

Let the operator S_n over the space $L(Z/n)$ be defined in the following manner:

$$\begin{aligned} S_n\colon L(Z/n) &\longrightarrow L(Z/n) \\ \delta_{[k]} &\longmapsto S_n\delta_{[k]} = \delta_{[k+1]} \end{aligned}$$

$$S_n\delta_{[n-1]} = \delta_{[n]} = \delta_{[0]} \equiv \delta, \quad S_n^k S_n^m = S_n^{k+m} \quad S_n^k\delta_{[0]} = \delta_{[k]} \tag{6}$$

Given an n-point sequence $f \in L(Z/n)$, we can write this sequence as a linear combination of the set of basis functions $\{\delta_{[k]},\ k \in Z/n\}$ in the following way

$$f = \sum_{0 \leq k < n} f_k \delta_{[k]} \tag{7}$$

where $f_k = f(k)$. We define the inner product $\langle\ ,\ \rangle$ of two sequences $f, g \in L(Z/n)$ as follows

$$\begin{aligned} \langle\ ,\ \rangle : L(Z/n) \times L(Z/n) &\longrightarrow C \\ (f,g) &\longmapsto \langle f, g\rangle = \sum_{0 \le k < n} f_k g_k^* \end{aligned} \tag{8}$$

where $L(Z/n) \times L(Z/n)$ defines the cartesian product of the $L(Z/n)$ with itself, and $g_k^* = (g(k))^*$, the operation $(\ )^*$ implying complex conjugation. Using $\langle f, \delta_{[k]}\rangle = f_k$, we rewrite (7) as

$$f = \sum_{0 \le k < n} \langle f, \delta_{[k]}\rangle \delta_{[k]} \tag{9}$$

Using the shift operator S_n, we can also write $f \in L(Z/n)$ as

$$f = \sum_{0 \le k < n} f_k \delta_{[k]} = \sum_{0 \le k < n} f_k S_n^k \delta_{[0]} \tag{10}$$

LSI-FIR systems are very important in the analysis of digital signals. This stems from the fact that if the response of a given system T, acting on the basis function δ, $\delta \in L(Z/n)$ is known, then the result of the system acting on any other input signal $f \in L(Z/n)$ may be deduced. We make this statement more precise. Let $h_{[0]} = h$ be the sequence obtained by applying a given LSI-FIR system T to the basis function δ:

$$T\delta_{[0]} = T\delta = h_{[0]} = h \tag{11}$$

The result of T acting on any other basis function $\delta_{[k]} \in L(Z/n)$, $k \in Z/n$ is given by

$$T(\delta_{[k]}) = T(S_n^k \delta) = S_n^k(T\delta) = S_n^k h \tag{12}$$

Since any given sequence $f \in L(Z/n)$ can be written uniquely as

$$f = \sum_{k \in Z/n} f(k)\delta_{[k]}, \quad f \in L(Z/n), \tag{13}$$

we have that

$$T(f) = T\Big(\sum_{k \in Z/n} f(k)\delta_{[k]}\Big) = \sum_{k \in Z/n} f(k) T(\delta_{[k]}) = \sum_{k \in Z/n} f(k) S_n^k h \tag{14}$$

The sequence h is usually termed the signal impulse response of the acting system, in this case, the system T. To identify this impulse response sequence h with this particular system, we rename the system T_h and call the action $T_h(f) \equiv h * f$ a linear convolution operation. Our interest is in another type of convolution operation called cyclic convolution. Myers [18] clearly establishes the relationship between linear convolution and cyclic convolution and the importance of cyclic convolution in implementing LSI-FIR systems. Since cyclic convolution is a modulo n operation, it can be viewed as a linear operator acting on the space of signals $f \in L(Z/n)$:

$$\begin{aligned} T_h : L(Z/n) &\longrightarrow L(Z/n) \\ (f) &\longmapsto T_h(f) \end{aligned} \tag{15}$$

where

$$T_h(f) = \sum_{j\in Z/n} h_j S_n^j f \equiv (f) * h \tag{16}$$

This operator T_h can be thought of as representing an LSI-FIR system whose impulse response sequence (function) is h. In this way the system T_h is uniquely characterized by the signal h. Thus, a system T_h acting on the unit sample sequence shifted by j units produces the following response

$$\begin{aligned} T_h(\delta_{[j]})(k) &= (\delta_{[j]} * h)(k) \equiv h_{[j]} \\ &= \sum_{m\in Z/n} h_m S_n^m \delta_{[j]}(k) = \sum_{m\in Z/n} h_m \delta(k-j-m) \\ &= \sum_{m\in Z/n} h_m \delta_{[j-k]}(-m) = h(k-j) = S_n^j h(k) \end{aligned} \tag{17}$$

And we conclude

$$T_h(S_n^j \delta[0]) = T_h(S_n^j \delta) = S_n^j h \tag{18}$$

The Discrete Fourier Operator

The space $L(Z/n)$ will be used to represent two related ideas. In the first place, $L(Z/n)$ will be thought of as representing the space of all finite n-point complex sequences with domain Z/n. In the second place, the space $L(Z/n)$ will also be thought of as representing the space of all periodic complex sequences with period n. This latter interpretation will allow us to perform modulo n operations on the indexing set Z/n, turning this set to into an additive group of order n. Evaluations modulo n will become more clear as we delve into the properties of the discrete Fourier transform (DFT) operator. The DFT of an n-point sequence f is defined in this section as a linear operator on the space (Z/n). Before we introduce this definition, we would like to describe some preliminary concepts and definitions.

The indexing set $A = Z/n = 0, 1, \ldots, n-1$ forms an Abelian group with modulo n addition as the internal binary operation. Its dual $\widehat{A}$ is defined as

$$\widehat{A} = \{\chi_k : k \in Z/n\} \equiv \widehat{(Z/n)} \tag{1}$$

where

$$\begin{aligned} \chi_k : Z/n &\longrightarrow C \\ (j) &\longmapsto \chi_k(j) = e^{-2\pi i k\cdot(j)/n} \end{aligned} \tag{2}$$

The functions χ_k are usually termed exponential sequences, characteristic sequences, or, simply, characters.

The set of functions $\widehat{A}$ plays a very important role in the analysis of LSI-FIR systems. They are eigefunctions of this type of systems. For the Fourier operator F_n, we have

$$\begin{aligned} F_n : L(Z/n) &\longrightarrow L(Z/n) \\ \delta_{[j]} &\longmapsto F_n \delta_{[j]} \end{aligned} \tag{3}$$

where

$$F_n \delta_{[j]} = \chi_j, \quad j \in Z/n \tag{4}$$

Allowing F_n to operate on f gives

$$F_n f \equiv \hat{f} = F_n\Big(\sum_{j\in Z/n} f_j\delta_{[j]}\Big) = \sum_{j\in Z/n} f_j F_n \delta_{[j]} = \sum_{j\in Z/n} f_j \chi_j \tag{5}$$

We can use matrices to represent LSI-FIR systems. Since each n-dimensional LSI-FIR system T_h: $L(Z/n) \to L(Z/n)$ represents a linear transformation in the space $L(Z/n)$, T_h is determined by its action on a set of basis vectors (signals) spanning $L(Z/n)$. If we choose as reference the standard basis set $\{\delta_{[j]}: j \in Z/n\}$, then each signal $T_h(\delta_{[j]}) \in L(Z/n)$ can be uniquely expressed as a linear combination of the basis set. We write

$$T_h(\delta_{[k]}) = \sum_{j\in Z/n} h_{j,k}\delta_{[j]} \tag{6}$$

where the set of scalars

$$\{h_{j,k}: j \in Z/n\}, \quad k \in Z/n \tag{7}$$

represents the vector coordinates of the given signal $T_h(\delta_{[k]})$, $k \in Z/n$, with respect to the standard basis set. The signal $T_h(\delta_{[k]})$ can also be written as

$$T_h(\delta_{[k]}) = \sum_{j\in Z/n} T_h(\delta_{[k]})(j)\delta_{[j]} \tag{8}$$

Next, we define the matrix H_n as follows

$$H_n = [h_{j,k}]_{0\leq j,k<n} = [h_{j-k}]_{0\leq j,k<n} \tag{9}$$

The matrix H_n, thus, have the following form

$$H_n = \begin{bmatrix} h_0 & h_{n-1} & h_{n-2} & \dots & h_1 \\ h_1 & h_0 & h_{n-1} & \dots & h_2 \\ h_2 & h_1 & h_0 & \dots & h_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_0 \end{bmatrix} \tag{10}$$

We notice that the columns of H_n are formed by shifted versions of the coordinate vector representation of the signal h; that is, we can write H_n as

$$H_n = [I_n h, S_n h, S_n^2 h, \dots, S_n^{n-1} h] \tag{11}$$

Spectral Properties of LSI-FIR Systems

In this section we will describe the spectral properties of LSI-FIR systems. A shift invariant linear operator acting on an n-dimensional vector space may be represented in the frequency domain by using the concepts of eigenfunctions (eigenvectors) and eigenvalues. The eigenvalues correspond to the natural frequencies encountered in the spectral representation of the impulse response signal of a given LSI-FIR system. We will be more explicit later on in describing the relationship existing between the eigenvalues (and their associated eigenfunctions) of a given LSI-FIR operator T_h and the frequency components of the associated impulse response sequence h. We start the section describing some properties of the system $T_{\delta_{[1]}}$ which are essentially the same as the properties of the shift operator S_n.

The simplest LSI-FIR system, apart from the trivial system, i.e., the system represented by the identity operator I_n, is the system represented by the shift operator S_n. This system is sometimes called the ***unit-delay*** system because its digital electronics hardware implementation may be accomplished by using a single delay element. We use the same symbol S_n to denote the matrix representation of the shift operator S_n. This matrix representation is now given.

Recalling that

$$T_{\delta_{[1]}} = \sum_{j \in Z/n} \delta_{[1]}(j) S_n^j = S^1 = S_n, \tag{1}$$

we have,

$$T_{\delta[1]}(\delta_{[k]}) = \delta_{[1]} * \delta_{[k]} = S_n \delta_{[k]} = \delta_{[k+1]} \tag{2}$$

The matrix S_n representing the shift operator S_n is obtained by allowing the vector representation (with respect to the standard basis set $\{\delta_{[k]}: k \in Z/n\}$) of the signal $T_{\delta_{[1]}}(\delta_{[k]})$, $k \in Z/n$, become the columns of the matrix S_n:

$$\begin{aligned} S_n &= \left[T_{\delta_{[1]}}(\delta_{[0]}), T_{\delta_{[1]}}(\delta_{[1]}), \ldots, T_{\delta_{[1]}}(\delta_{[n-1]})\right] \\ &= \left[\delta_{[1]}, \delta_{[2]}, \ldots, \delta_{[n-1]}, \delta_{[0]}\right] \end{aligned} \tag{3}$$

where we have separated by commas the columns of S_n for legibility. The matrix S_n becomes

$$S_n = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \tag{4}$$

An important property of the S_n operator matrix is that any LSI-FIR system T_h may be represented by a matrix H_n which can be written as a sum of powers of the matrix S_n premultiplied by a diagonal matrix D_{h_j}:

$$H_n = \sum_{j \in Z/n} D_{h_j} S_n^j = \sum_{j \in Z/n} (h_j \otimes S_n^j) \tag{5}$$

where

$$D_{h_j} = \begin{bmatrix} h_j & & & \\ & h_j & & \\ & & \ddots & \\ & & & h_j \end{bmatrix}, \quad h_j = h(j), \quad j \in Z/n \tag{6}$$

and the symbol $\otimes$ stands for tensor product. Let $A = [a_{k\ell}]$ and $B = [b_{rs}]$ be any two arbitrary matrices (row vectors, column vectors, or scalars also considered). The tensor or Kronecker product between A and B is given by $A \otimes B = [a_{k\ell} \cdot B]$. That is, each entry of this new matrix $A \otimes B$

is obtained by replacing each entry of A by the product of that particular entry times the matrix B. We give the following simple example to illustrate the representation LSI-FIR systems using powers of S_n.

Example 1: Take $n = 4$. We have

$$D_{h_0} \cdot I_4 = \begin{bmatrix} h_0 & 0 & 0 & 0 \\ 0 & h_0 & 0 & 0 \\ 0 & 0 & h_0 & 0 \\ 0 & 0 & 0 & h_0 \end{bmatrix}, \quad D_{h_1} \cdot S_4 = \begin{bmatrix} 0 & 0 & 0 & h_1 \\ h_1 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_1 & 0 \end{bmatrix}$$

$$D_{h_2} \cdot S_4^2 = \begin{bmatrix} 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_2 \\ h_2 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \end{bmatrix}, \quad D_{h_3} \cdot S_4^3 = \begin{bmatrix} 0 & h_3 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_3 \\ h_3 & 0 & 0 & 0 \end{bmatrix}$$

$$H_4 = \sum_{j \in Z/4} D_{h_j} S_4^j = \sum_{j \in Z/4} (h_j \otimes S_4^j)$$

$$H_4 = \begin{bmatrix} h_0 & h_3 & h_2 & h_1 \\ h_1 & h_0 & h_3 & h_2 \\ h_2 & h_1 & h_0 & h_3 \\ h_3 & h_2 & h_1 & h_0 \end{bmatrix}$$

The matrix F_n, called the DFT matrix and representing the discrete Fourier transform (DFT) operator F_n, is obtained by first determining $F_n(\delta_{[k]})$, $k \in Z/n$:

$$F_n(\delta_{[k]}) = \chi_k, \quad k \in Z/n, \quad \chi_k(j) = e^{-2\pi i k j/n} = w_n^{jk} \tag{7}$$

The matrix F_n is obtained by writing the coordinate vector representation of the signal set $\{F_n(\delta_{[k]}) : k \in Z/n\}$ as the columns of F_n:

$$F_n = [\chi_0, \chi_1, \chi_2, \cdots, \chi_{n-1}] \tag{8}$$

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w_n & w_n^2 & \dots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & \dots & w_n^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w_n^{n-1} & w_n^{n-2} & \dots & w_n \end{bmatrix} \tag{9}$$

We now take an LSI-FIR system T_h, which is represented by the matrix H_n, and obtain the discrete Fourier transform of the system response to a given input signal $f \in L(Z/n)$. Taking the DFT of the response $g = T_h(f)$, and using the cyclic convolution theorem, produces the following result

$$\begin{aligned} F_n(T_h(f)) &= F_n(h) \odot F_n(f) \\ (F_n T_h)(f) &= (F_n(h) \odot F_n)(f) \end{aligned} \tag{10}$$

Here, the symbol $\odot$ implies point-wise multiplication of two vectors in the first equation, and of the vector $F_n(h)$ with each of the columns of the F_n matrix in the second equation. Since the choice of $f \in Z/n$ was arbitrary, we obtain the following important result

$$FT_h = F_n(h) \odot F_n \tag{11}$$

or, emphasizing the diagonalization of T_h by the action of the F_n operator,

$$F_n T_h F_n^{-1} = F_n(h) \odot I_n \tag{12}$$

The expression $F_n(h) \odot I_n$ is denoted by $D_{F(h)} = D_{\widehat{h}}$, where a matrix representation of $D_{\widehat{h}}$ is given by

$$D_{\widehat{h}} = \begin{bmatrix} \widehat{h}_0 & & & & \\ & \widehat{h}_1 & & & \\ & & \widehat{h}_2 & & \\ & & & \ddots & \\ & & & & \widehat{h}_{n-1} \end{bmatrix}, \quad \widehat{h}_j = (F(h))(j), \quad j \in Z/n \tag{13}$$

Thus,

$$F_n T_h F_n^{-1} = D_{\widehat{h}} \tag{14}$$

In essence, we are stating that computing with the filter T_h is equivalent to compute with the structure $F_n^{-1} D_{\widehat{h}} F_n$ which it has been demonstrated is more efficient if we use a fast Fourier transform algorithm to compute the discrete Fourier transform.

Implementation of LSI-FIR Systems

It is important to review the two main methods of representing LSI-FIR filters for the purpose of hardware implementation. We will use some of the linear algebra tools described in the previous sections on LSI-FIR systems. This will allow us to relate the results of this section with with previous works and applications on LSI-FIR systems. The two main methods for representing LSI-FIR filters are usually called the time domain representation and the frequency domain representation. The time domain representation follows readily when we express an LSI-FIR system as a linear combination of powers of the shift operator; that is, the time domain representation of a given LSI-FIR filter T_h is given by

$$T_h = \sum_{j \in Z/n} h_j S_n^j \tag{1}$$

The frequency domain representation is obtained by taking the $\mathcal{Z}$-transform of the impulse response sequence which characterizes the given system T_h. We would like to describe this representation in more details since it is most often used. First, let us discuss some ideas about polynomial functions and introduce the $\mathcal{Z}$-transform of a n-point sequence $h \in L(Z/n)$.

Associate with an n-point sequence

$$h = [h_0 \quad h_1 \quad \ldots \quad h_{n-1}], \tag{2}$$

the $(n-1)$-th degree polynomial p_h in the indeterminate α

$$p_h = h_0 + h_1\alpha + \ldots + h_{n-1}\alpha^{n-1}, \tag{3}$$

and the polynomial function

$$p_h(z^{-1}) = \sum_{j \in Z/n} h_j z^{-1} \equiv H(z), \quad z \in C \tag{4}$$

The polynomial function $p_h(z^{-1})$ belongs to the ordered set (the order induced by the natural order of Z/n)

$$Z(h) = \{p_h((z^{-1})^k): \quad k \in Z/n\} \tag{5}$$

where

$$p_h((z^{-1})^k) = \sum_{j \in Z/n} h_j z^{-jk}, \quad k \in Z/n \tag{6}$$

and the index product $j \cdot k$ is taken modulo n. Since the elements of $Z(h)$ become complex numbers when $z \in C$ is fixed, we can think of this set as an n-point sequence in $L(Z/n)$. Thus, the k-th element of the sequence Z is given by

$$Z(h)(k) = p_h(z^{-k}) \equiv H_k(z), \quad z \in C \text{ fixed} \tag{7}$$

Another way of viewing the set $Z(h)$ is to fixed the index value k (the value of k usually chosen to be one (1)) and allow z to become a variable, taking the entire complex plane C except the origin. The expression $Z(h)$ is then called the Z-transform of the n-point sequence $h \in L(Z/n)$, and is written as

$$Z(h) = p_h(z^{-1}) = \sum_{j \in Z/n} h_j z^{-j} \equiv H(z), \quad z \in C \tag{8}$$

We notice that the set $Z(h)$ now consists of a single element, namely $H(z)$; and by allowing z to take on values on C, $H(z)$ becomes an analytic function on the entire complex plane except at the origin. The function $H(z)$ is usually termed the system function associated with the impulse response sequence h.

We now proceed to obtain the frequency domain representation of an LSI-FIR system T_h. We first recall that given a system T_h, its action on a input sequence $x \in L(Z/n)$ results in the cyclic convolution of this sequence x and the impulse response sequence which characterizes the system T_h. That is, if we call $y \in Z/n$ the output sequence which results when T_h acts on the input sequence x, this sequence is given by

$$y = T_h x = \sum_{j \in Z/n} h_j S_n^j x = x * h \tag{9}$$

We also would like to obtain the Z-transform of the shifted sequence $x_{[j]} = S_n^j x$, which is obtained as follows:

$$\begin{aligned} Z(x_{[j]}) &= \sum_{m \in Z/n} x_{[j]}(m) z^{-m} \\ &= \sum_{m \in Z/n} x(m-j) z^{-m} \\ &= \sum_{k \in Z/n} x_k z^{-(k+j)} \\ &= z^{-j} \sum_{k \in Z/n} x_k z^{-k} = z^{-j} X(z) \end{aligned} \tag{10}$$

where $X(z)$ is the Z-transform of the n-point input sequence x.

If we now take the $\mathcal{Z}$-transform of the cyclic convolution given above Eq. (9) we obtain the following result

$$\begin{aligned} \mathcal{Z}(y) &= \mathcal{Z}(x * y) = \mathcal{Z}\Big(\sum_{j\in Z/n} h_j S_n^j x\Big) \\ &= \sum_{j\in Z/n} h_j \mathcal{Z}(S_n^j x) = \sum_{j\in Z/n} h_j z^{-j} X(z) \\ &= X(z) \sum_{j\in Z/n} h_j z^{-j} = X(z)H(z) \equiv Y(z) \end{aligned} \tag{11}$$

By making an analogy between the expression

$$y = \sum_{j\in Z/n} h_j S_n^j x = x * h \tag{12}$$

and the expression

$$Y(z) = \sum_{j\in Z/n} h_j z^{-j} X(z) = X(z)H(z), \tag{13}$$

the expression

$$H(z) = \sum_{j\in Z/n} h_j z^{-j} \tag{14}$$

is usually termed the "frequency domain representation" of the LSI-FIR system T_h. Thus, the frequency domain representation of a given LSI-FIR system T_h is its associated system function $H(z)$.

The hardware implementation of either the time domain or frequency domain representations of an LSI-FIR system is usually accomplished by identifying either the shift operator S_n in Eq. (12), or the multiplication element z^{-1} in Eq. (13), with a *unit-delay* element device in digital signal processing hardware.

If we allow z in the system function $H(z)$ to take values only on the unit circle, we obtain the Fourier transform of the n-point sequence h. Thus, at $z = e^{i\nu} = e^{2\pi i f}$, we obtain

$$H(e^{i\nu}) = p_h(z^{-1} = e^{-2\pi i f}) = \sum_{j\in Z/n} h_j e^{-2\pi i j f}, \quad 0 \le f < 1 \tag{15}$$

The discrete Fourier transform (DFT) of the n-point sequence h is obtained by setting fixed z in the sequence $\mathcal{Z}(h)$ to the primitive n-th root of unity $e^{2\pi i/n} = w_n^{-1}$. In this way, the k-th term of the discrete Fourier transform of h is given by

$$\hat{h}(k) = F_n(h)(k) = \sum_{j\in Z/n} h_j e^{-2\pi i jk/n} = H_k(e^{2\pi i/n}) \tag{16}$$

Since k takes on values on the set $\{0, 1, 2, \ldots, n-1\}$, the values $(e^{2\pi i/n})^k = w_n^{-k}$ form the set $U(n)$ of the n roots of unity, which are spaced uniformly on the unit circle of the complex plane. This elucidates the known fact that the DFT of an n-point sequence corresponds to the uniform sampling of its $\mathcal{Z}$-transform on the unit circle.

LSI-FIR Systems and ANN Structures

We discussed the role of LSI-FIR systems in signal processing. We know discussed how this systems can be implemented using ANN structures. The fundamental idea is that of the circulant matrix H_n which describes a given LSI-FIR system. Barto [4] has demonstrated how to implement ANN structures using the fast Fourier transform (FFT). This was obtained by transforming the discrete cross-correlation operation describing the global transition function of a network into a cyclic convolution operation and, in turn, computing the convolution operation by indirect methods (as we have discussed previously in the section on spectral properties of LSI-FIR systems) using the fast Fourier transform. In 1989, Culhane, Peckerar, and Marrian [6] presented an electronic circuit for computing the discrete Fourier transform. This circuit was based on a neural net structure.

Recently, Zhang, Jullien, and Miller have demostrated how to compute over finite rings using neural network structures. They present a computational network termed finite ring neural network (FRNN). This work finds some of its importance in the area of signal processing when we are dealing with computation in surrogate fields [11]. Other works in the area of discrete-time signal processing using ANN structures include the work by Widrow, Baudrenghien, Vetterli, and Titchener [3], and the work by Greco, Paoloni, and Ravaioli [17].

Conclusion

We have described in detail the concept of LSI-FIR systems and the role they play in harmonic analysis applications. We have concentrated on the implementation of these systems using ANN structures. In particular, we have concentrated on multi-layer feed-forward perceptrons. A more detail study of these systems in needed to identify other potential implementations using various ANN structures in addition to the multi-layer feed-forward perceptron. We have also concentrated on the theoretical study of these systems and their implementation using ANN structures; but, we have perfomed various successful implementation efforts of the multi-layer feed-forward perceptron using MATLAB environment. To accomplish we have followed the work described in [8].

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