Time-frequency and time-scale representations of doubly spread channels

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Abstract

We study the correspondence between time-frequency and time-scale integral operators and determine the mapping between time-scale kernels and time-frequency kernels, which are both used to model communication channels. Time-scale models, which have a physical interpretation in wireless communication, are often approximated as time-frequency models for narrowband communication. In determining this mapping we show that the causal time-scale channel is time-invariant and that the causal time-forward time-frequency channel is time-invariant. A *time-forward* channel is defined as a channel for which the Hardy space and its orthogonal complement are invariant. We derive the form of the equivalent lowpass characterization of the time-varying time-frequency channel and show that the equivalent lowpass characterization only exists for time-forward channels. We show that the mapping between the narrowband channel description based on a time-frequency kernel with constrained support and the wideband channel description based on a time-scale kernel with constrained support does not exist for the channels encountered in realistic time-varying communication scenarios. In light of this result, we develop a canonical time-scale channel model for wideband communication analogous to the canonical time-frequency channel model proposed by Sayeed and Aazhang.

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to Elva

Contents

Chapter 1

Introduction

It is common to assume that a received communication signal is composed of superpositions of the transmitted signal. The superpositions arise from reflections of the signal off scatterers in the environment. In the *time-scale channel model*, each reflection is a delayed and time scaled copy of the transmitted signal. The delays arise from differing path lengths from transmitter to scatterer to receiver, and movement of the transmitter, scatterer, or receiver cause time dilations or contractions. Thus, each reflection is of the form.

$$x_{a,b}(t) = \frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) \tag{1.1}$$

and the received signal is a summation of the reflections characterized by $\mathcal{L}(a, b)$, the *wideband spreading function*[1].

$$y(t) = \iint \mathcal{L}(a, b) \frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) da db. \tag{1.2}$$

In (1.2), we will, for the most part, be interested in $\mathcal{L}$ with support only for $a > 0$ because this constrains the channel only to contain positive time copies of the trans-

[1] We will assume that all integrals are over $(-\infty, \infty)$ unless otherwise specified.

mitted signal. Any $a < 0$ support results in time-reversed signals in the output. We assume the speeds of the objects in the environment are such that the rate of change of path length is less than the speed of propagation of the signal, which prohibits time-reversal of the transmitted signals. However, in order to better illuminate certain analyses, we will consider $\mathcal{L}(a, b)$ with support for all a (including negative a) and all b. We will use $\mathcal{L}_+(a, b)$ to denote wideband channels with support only for positive a, $\mathcal{L}_-(a, b)$ to denote wideband channels with support only for negative a, and $\mathcal{L}(a, b)$ to denote general wideband channels with no support constraint. In all cases, we assume $\mathcal{L}(0, b) = 0, \forall b$ in order to avoid the singularity at $a = 0$.

We call a time-scale channel a *wideband channel* when the wideband spreading function has finite support. We expect finite support for $\mathcal{L}(a, b)$ due to physical limitations of signal propagation. The maximum possible rate of change in path length, which is constrained by the speeds of the objects in the environment, limits the support of $\mathcal{L}(a, b)$ to a narrow range around the $a = 1$ line. Causality and the propagation loss associated with increasing path length effectively limit the support of $\mathcal{L}(a, b)$ to a finite range in the b direction. The support in the a direction causes a spreading in scale of the transmitted signal, and the support in the b direction causes a spreading in time of the transmitted signal; thus the term doubly spread channels.

Many signals and signaling environments satisfy the *narrowband condition*, an assumption under which the time dilations or contractions are modeled as Doppler shifts. Under this assumption, each received reflection of the signal is assumed to be of the form,

$$x_{\tau,\theta}(t) = x(t - \tau)e^{j2\pi\theta t} \tag{1.3}$$

In the *narrowband channel model*, the received signal is a superposition of time delayed and frequency shifted copies of the input and the channel is characterized by the *narrowband spreading function* $S(\theta, \tau)$.

$$y(t) = \iint S(\theta, \tau) x(t - \tau) e^{j 2\pi \theta t} \mathrm{d}\tau \mathrm{d}\theta. \tag{1.4}$$

where $S(\theta, \tau)$ has finite support in θ and τ due to the physical limitations of the channel. This is, the signal is spread in frequency and time. When $S(\theta, \tau)$ has no support constraint, (1.4) is a time-frequency description of a time-varying linear system and the transmitted waveform and environment need not satisfy the narrowband condition.

One of our goals here is to understand which $S(\theta, \tau)$ and $\mathcal{L}(a, b)$ produce the same input-output mapping, regardless as to whether the input is narrowband or not. We hope that such an analysis will shed some light on the nature of time-frequency and time-scale operators and yield a better understanding of time-varying channel models. Specifically, we define operators

$$\mathcal{N}_S\{x\}(t) \doteq \iint S(\theta, \tau) x(t - \tau) e^{j 2\pi \theta t} \mathrm{d}\tau \mathrm{d}\theta \tag{1.5}$$

$$\mathcal{W}_{\mathcal{L}}\{x\}(t) \doteq \iint \mathcal{L}(a, b) \frac{1}{\sqrt{|a|}} x\left(\frac{t - b}{a}\right) \mathrm{d}a \mathrm{d}b \tag{1.6}$$

and we are interested in $(S, \mathcal{L})$ pairs such that

$$\mathcal{N}_S = \mathcal{W}_{\mathcal{L}}. \tag{1.7}$$

We will be interested specifically in $S(\theta, \tau)$ and $\mathcal{L}(a, b)$ with supports consistent with the physical propagation constraints. Indeed, we will see that it is in the limitation of the support of the characterizations that the mapping reveals the differences between the time-frequency narrowband and time-scale wideband models. That is, the narrowband and wideband models are not equivalent specifically for the channels which are typical in mobile communication settings.

In [SA99] a canonical time-frequency channel characterization was proposed which was used to define a delay-Doppler RAKE receiver. a two-dimensional extension of the classic RAKE receiver. which takes advantage of the inherent added channel diversity associated with time-varying narrowband channels. While the narrowband assumption is satisfied in many wireless communication signal environments. [MMH+02] points out that many wireless systems are wideband due to the higher data rates and multiaccess techniques. Thus. we may expect. in light of differences in the narrowband and wideband models. some advantages to the development of a canonical time-scale channel characterization. The hope is that a delay-dilation RAKE receiver based on the canonical time-scale channel characterization will leverage the diversity in wideband signaling environments in the same way that the delay-Doppler RAKE leverages the diversity in narrowband signaling environments. Such a channel model and receiver may be particularly useful for ultra-wide bandwidth signaling (impulse radio) due to the extremely wide transmission signal bandwidth [WS02. CWVM03].

1.1 Previous work

The approach taken in this thesis is different from the traditional interpretation of the narrowband characterization as an approximation of the wideband characterization when applied to narrowband signals. We do not consider the narrowband description of the channel as an approximation of the wideband channel. but rather look at the two descriptions without constraining the properties of the input signal. Also. the focus in this thesis is on the channel model itself, whereas much of the literature focuses on the ambiguity functions (i.e., energy distributions) derived from the model (wideband or narrowband) and not the model itself. For example,

- [Kai96. Kai94] discuss the relationship between the wideband and narrowband ambiguity functions. specifically, how the wideband ambiguity function can be approximated by the narrowband ambiguity function for narrowband signals. [Kai96] generalizes the ambiguity function (and the wideband model) to account for arbitrary transmitter/scatterer/receiver motion (instead of just constant velocity motion) and then shows the reduction to the standard wideband ambiguity function for constant velocity environments and then the approximation as the narrowband ambiguity function when further considering narrowband signals. The focus is on the use of the ambiguity function to link time-frequency and time-scale analysis.

- [BTA98] discusses extending the wideband model characterized by $\mathcal{L}(a, b)$ to one using a three dimensional kernel $\mathcal{L}(a, b, t)$ and sites [You95] which also considers such an extension. [BTA98] claims the advantages of $\mathcal{L}(a, b, t)$ model over the narrowband channel described by $h(t, \tau)$ are that the $\mathcal{L}(a, b, t)$ is valid for longer time intervals when the channel is changing linearly with time and that the processing duration in the narrowband model is limited by the bandwidth and velocities of objects in the environment whereas the processing duration in the extended wideband model is limited by the bandwidth and accelerations of the objects in the environment. Accelerations are also a topic of interest in [Swi69], which argues that the wideband model is better for environments with accelerating objects, instead of constraining all objects to have constant radial velocity as in the narrowband model.

- [Fla88] links the wavelet decomposition of a signal to the short-time Fourier transform representation, although the focus here is on signal analysis and not channel models.

- [DF96], [ZF00], and [ZFL01] introduce a wavelet-based (wireless) channel model. A comparison of their model to the model derived in this work in Chapter 5 is a topic of future research.

- [FS91], [CSB92], [SWD94], [WYS94], [SWY97], [RNFR97], and [SRH98] explore the wideband spreading function channel description and the use of the continuous wavelet transform as a maximum likelihood detector. The narrowband spreading function description in these works is viewed in the traditional way, as an approximation of the wideband channel for narrowband signals.

- The properties of the wideband spreading function from a group theoretic perspective are investigated in [SP95]. Extensions of this work can be found in [IPSBB98b], [IPSBB98a], and [IPSBB02].

1.2 Outline of work

- Chapter 2: Narrowband and Wideband Channel Characterizations - presents background material on time-frequency and time-scale channel models.
  - Discuss Bello's 12 (narrowband) channel characterizations.
  - Examine the form of one path delay-Doppler channels for the 12 characterizations and note the simplicity of the $S(\theta, \tau)$ representation.
  - Display the wideband characterization in time-time and frequency-frequency noting that positive output frequency components depend only on positive input frequency components, and negative output frequency components depend only on negative input frequency components. This frequency dependence is later used as the defining characteristic for time-forward channels.
  - Show the form of one path delay-dilation wideband channels.
- Chapter 3: Narrowband and Wideband Correspondence - derives and discuses the mapping between time-frequency and time-scale kernel operators.

- Examine the representation of time-invariant channels for narrowband ($S(\theta, \tau)$ and $k_3(\theta, \nu)$) and wideband channel ($\mathcal{L}(a, b)$) characterizations.

- Show that the support of the time-invariant characterizations are the $a = 1$ line for $\mathcal{L}(a, b)$, the $\theta = 0$ line for $S(\theta, \tau)$, and the $\theta = \nu$ line for $k_3(\theta, \nu)$.

- Discuss splitting the narrowband and wideband characterizations into time-invariant and time-varying components.

- Discuss the implications of assuming causality for the characterization functions and show that, *the causal wideband channel is time-invariant.* That is, time-varying causal wideband channels do not exist.

- Determine mapping from $\mathcal{L}(a, b)$ to k_0, k_1, k_2, and k_3.

- Discuss the different mapping of the DC component in the wideband and narrowband case. Specifically, in the wideband characterization, the only component that can affect the DC output component is the DC input component. Similarly, the DC input component can only affect the DC output component. This restriction is not present in the narrowband characterization and in order to map the narrowband channels to wideband channels we only consider characterizations which satisfy this constraint.

- Show that there exist narrowband channels with no corresponding wideband channel.

- Show that *the causal time-forward narrowband channel is time-invariant.* That is, time-varying causal time-forward narrowband channels do not exist.

- Determine mapping from $\mathcal{L}(a, b)$ to $S(\theta, \tau)$.

- Determine mapping from $S(\theta, \tau)$ to $\mathcal{L}(a, b)$.

- Examine the form of the one path narrowband and wideband channels using the correspondence.

- Show the instability of the mapping around the important $a = 1$ (time-invariant) line.

- Chapter 4: Characterization of Communication Signals - derive the time-varying narrowband equivalent lowpass characterization.
  - Discuss the standard equivalent lowpass signal/channel characterization.
  - Establish a corresponding equivalent lowpass characterization for the time varying case.
  - Show that the equivalent lowpass time-varying characterization exists only for time-forward channels.

- Chapter 5: Time-Frequency and Time-Scale Canonical Models - develop a general technique for the generation of canonical channel models and demonstrate the application of the technique to time-frequency and time-scale kernel operators.
  - Discuss Sayeed/Aazhang's time-frequency canonical channel characterization which forms the basis for the time-frequency RAKE receiver.
  - Propose a generalization of the canonical channel characterization that allows us to generate canonical channel models based on a pair of projection operators.
  - Show that the time-frequency canonical channel characterization is based on the projection operators in time and frequency.
  - Propose a time-scale canonical channel characterization based on the projection operators in time and scale.

- Summary and Future Work

- Appendix: Doppler effect and Time-frequency duality

Chapter 2

Narrowband and Wideband Channel Characterizations

In this chapter we discuss the time-frequency kernel and time-scale kernel channel models and look at some simple channels to gain some intuition concerning the characterizations. The time-frequency kernel description is a general time-varying linear system characterization. However, in a slight abuse of nomenclature, we will refer to all channels characterizations which can be related to the channel described by $S(\theta, \tau)$ via Fourier transforms and phase factors as narrowband channels. Specifically, in this chapter, we discuss 12 such equivalent characterizations which were first explored as a group in [Bel63]. We call these "narrowband" characterizations because when $S(\theta, \tau)$ has finite support, the characterization is typically used only in narrowband systems. We will only discuss the support condition constraint on $S(\theta, \tau)$ for the narrowband characterizations when relevant, and, in general, consider the more general case where there is no such constraint on the support of $S(\theta, \tau)$. Similarly, we will refer to channel characterizations based on the time-scale kernel, $\mathcal{L}(a, b)$ and $\mathcal{L}^{(2)}(a, \theta)$ as wideband characterizations because they are typically used in a wideband setting [SWD94].

2.1 Narrowband Characterizations

The linear time-varying channel is characterized by the time-varying impulse response $h(t, \tau)$ which denotes the response of the channel at time t to an impulse at time $t-\tau$. The channel input-output relationship is thus.

$$y(t) = \int h(t, \tau)x(t-\tau)\mathrm{d}\tau \tag{2.1}$$

Such notation is used in, for example, [BB99, BPS98, Pro84, Tre71, SA99].

Another possible notation for the time-varying impulse response is

$$y(t) = \int k_0(t, \tau)x(\tau)\mathrm{d}\tau. \tag{2.2}$$

with the interpretation that $k_0(t, \tau)$ is the response of the channel at time t to an impulse at time τ. This is the formulation used in, for example, [Zad50, Ver98, MH02b]. Bello [Bel63] calls $k_0(t, \tau)$ a *kernel system function* and notes the obvious correspondence between the two representations, $h(t, \tau) = k_0(t, t-\tau)$. Bello [Bel63] defines four equivalent representations of the time-varying channel represented by $k_0(t, \tau)$ that map the time or frequency representation of the input into the time or frequency representation of the output. We define these four kernel functions[1],

$$\begin{aligned} y(t) &= \int k_0(t, \tau)x(\tau)\mathrm{d}\tau \qquad & Y(\theta) &= \int k_1(\theta, \tau)x(\tau)\mathrm{d}\tau \\ y(t) &= \int k_2(t, \nu)X(\nu)\mathrm{d}\nu \qquad & Y(\theta) &= \int k_3(\theta, \nu)X(\nu)\mathrm{d}\nu \end{aligned} \tag{2.3}$$

Bello [Bel63] points out that the kernel system functions can be transformed into one another using the Fourier transform. For example, the kernel function that maps the input time domain to the output time domain ($k_0(t, \tau)$) and the kernel function that maps the input time domain to the output frequency domain ($k_1(\theta, \tau)$) are Fourier

[1] We use different notation than Bello.

transforms of one another with respect to the first argument. We can summarize the the relationships between the kernel system functions as follows.

$$\begin{array}{ccc} k_0(t,\tau) & \xrightarrow{\mathcal{F}_{t\to\theta}} & k_1(\theta,\tau) \\ \mathcal{F}_{\nu\to\tau}\Big\uparrow & & \Big\uparrow\mathcal{F}_{\nu\to\tau} \\ k_2(t,\nu) & \xrightarrow[\mathcal{F}_{t\to\theta}]{} & k_3(\theta,\nu) \end{array} \tag{2.4}$$

That is,

$$\begin{aligned} k_0(t,\tau) &= \int k_2(t,\nu)e^{-j2\pi\nu\tau}\mathrm{d}\nu \qquad & k_1(\theta,\tau) &= \int k_0(t,\tau)e^{-j2\pi t\theta}\mathrm{d}t \\ k_2(t,\nu) &= \int k_3(\theta,\nu)e^{j2\pi\theta t}\mathrm{d}\theta \qquad & k_3(\theta,\nu) &= \int k_1(\theta,\tau)e^{j2\pi\tau\nu}\mathrm{d}\tau \end{aligned} \tag{2.5}$$

The direction of the Fourier transform between k_0 and k_2 (and also between k_1 and k_3) is opposite to convention: We take the Fourier transform with respect to a "frequency" variable (ν) and replace it with a "time" variable (τ). This is necessary to be consistent with the kernel functions as defined in (2.3).

Bello [Bel63] provides the following useful interpretation of the kernel system functions.

- The response to input $\delta(t-t_0)$ is time function $k_0(t,t_0)$ with spectrum $k_1(\theta,t_0)$.
- The response to input $e^{j2\pi\theta_0 t}$ is time function $k_2(t,\theta_0)$ with spectrum $k_3(\theta,\theta_0)$.

and also noted, by simple inspection of (2.3), that k_0 and k_3 are time-frequency duals of one another, as are k_1 and k_2. Time-frequency duality is a concept introduced in Bello [Bel64] and discussed briefly in Appendix B.

Despite the simple input-output interpretations, Bello claimed that the kernel system functions often lack intuitive physical interpretations. For this reason, Bello [Bel63] examined eight other system function characterizing the linear time-varying channel. These eight system functions are (2.1),

$$y(t) = \int h(t,\tau)x(t-\tau)\mathrm{d}\tau;$$

its time-frequency dual,

$$Y(\theta) = \int G(\theta,\nu)X(\theta-\nu)\mathrm{d}\nu;$$

the three functions obtained by taking the Fourier transform of $h(t,\tau)$ with respect to t, τ, and both t and τ; and the three functions obtained by taking the inverse Fourier transform of $G(\theta,\nu)$ with respect to θ, ν, and both θ and ν. We define these eight functions (and list Bello's names for them),

$h(t,\tau)$	input delay spread function	$y(t) = \int h(t,\tau)x(t-\tau)\mathrm{d}\tau$	(2.6)
$S(\theta,\tau)$	delay-Doppler spreading function	$S(\theta,\tau) = \int h(t,\tau)e^{-j2\pi t\theta}\mathrm{d}t$	(2.7)
$T(t,\nu)$	time-varying transfer function	$T(t,\nu) = \int h(t,\tau)e^{-j2\pi\tau\nu}\mathrm{d}\tau$	(2.8)
$H(\theta,\nu)$	output Doppler spread function	$H(\theta,\nu)=\iint h(t,\tau)e^{-j2\pi(t\theta+\tau\nu)}\mathrm{d}t\mathrm{d}\tau$	(2.9)
$G(\theta,\nu)$	input Doppler spread function	$Y(\theta)=\int G(\theta,\nu)X(\theta-\nu)\mathrm{d}\nu$	(2.10)
$V(t,\nu)$	Doppler-delay spreading function	$V(t,\nu)=\int G(\theta,\nu)e^{j2\pi\theta t}\mathrm{d}\theta$	(2.11)
$M(\theta,\tau)$	frequency dependent modulation function	$M(\theta,\tau)=\int G(\theta,\nu)e^{j2\pi\nu\tau}\mathrm{d}\nu$	(2.12)
$g(t,\tau)$	output delay spread function	$g(t,\tau)=\iint G(\theta,\nu)e^{j2\pi(\theta t+\nu\tau)}\mathrm{d}\theta\mathrm{d}\nu$	(2.13)

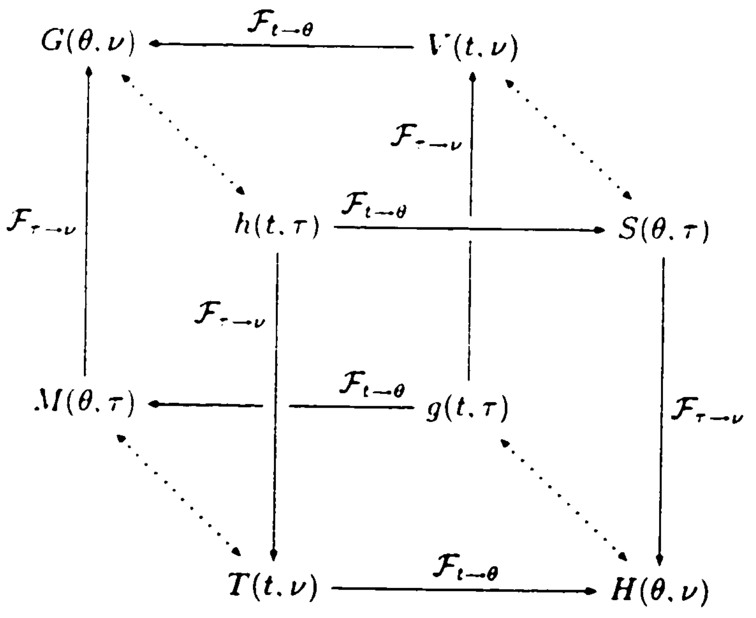
These eight characterizations were also considered in [Kai63]. From (2.6) and (2.10), one can derive the following relationship between the dual characterizations $h(t,\tau)$ and $G(\theta,\nu)$,

$$h(t,\tau) = \iint G(\theta,\nu)e^{j2\pi\theta t}e^{-j2\pi\tau(\theta-\nu)}\mathrm{d}\theta\mathrm{d}\nu \tag{2.14}$$

$$G(\theta,\nu) = \iint h(t,\tau)e^{-j2\pi t\theta}e^{j2\pi\nu(t-\tau)}\mathrm{d}t\mathrm{d}\tau \tag{2.15}$$

In the current literature, $h(t,\tau)$ is usually referred to as the *time-varying impulse response*, (e.g., [BB99, BPS98, Pro84, Tre71, SA99]) and the delay-Doppler spreading function, $S(\theta,\tau)$, is known simply as the *spreading function* (e.g., [BB99, BPS98, Pro84, Tre71, SA99, Ver98]). Unfortunately, $k_0(t,\tau)$ is also commonly referred to as the time-varying impulse response (e.g., [Zad50, Ver98]). We will refer to $k_0(t,\tau)$ as the time-varying impulse response *kernel* to avoid confusion.

The relationships of the eight functions via duality and the Fourier transform are summarized in the following diagram. Duality is represented by a dotted line.



(2.16)

Using (2.6–2.9), we can derive the following input-output relationships.

$$
\begin{aligned}
y(t) &= \int h(t,\tau)x(t-\tau)\mathrm{d}\tau & \qquad y(t) &= \iint S(\theta,\tau)e^{j2\pi\theta t}x(t-\tau)\mathrm{d}\theta\mathrm{d}\tau \\
y(t) &= \int T(t,\nu)e^{j2\pi\nu t}X(\nu)\mathrm{d}\nu & \qquad Y(\theta) &= \int H(\theta-\nu,\nu)X(\nu)\mathrm{d}\nu
\end{aligned}
\tag{2.17}
$$

and using (2.10–2.13), we can derive the following input-output relationships.

$$Y(\theta) = \int G(\theta, \nu) X(\theta - \nu) \mathrm{d}\nu \qquad Y(\theta) = \iint V(t, \nu) e^{-j2\pi t\theta} X(\theta - \nu) \mathrm{d}t\mathrm{d}\nu$$
$$Y(\theta) = \int M(\theta, \tau) e^{-j2\pi\tau\theta} x(\tau) \mathrm{d}\tau \qquad y(t) = \int g(t - \tau, \tau) x(\tau) \mathrm{d}\tau \tag{2.18}$$

We can relate the eight system functions to the four kernel system functions as follows.

$$k_0(t, \tau) = h(t, t - \tau) = g(t - \tau, \tau) \tag{2.19}$$
$$k_1(\theta, \tau) = \iint S(\nu, t) e^{j2\pi(t+\tau)(\nu-\theta)} \mathrm{d}\nu\mathrm{d}t = M(\theta, \tau) e^{-j2\pi\tau\theta} \tag{2.20}$$
$$k_2(t, \nu) = T(t, \nu) e^{j2\pi t\nu} = \iint V(\tau, \theta) e^{j2\pi(t-\tau)(\theta+\nu)} \mathrm{d}\tau\mathrm{d}\theta \tag{2.21}$$
$$k_3(\theta, \nu) = H(\theta - \nu, \nu) = G(\theta, \theta - \nu) \tag{2.22}$$

$S(\theta, \tau)$ and $V(t, \nu)$ are distinctive in that their input-output characterizations and relations to the kernel system functions involve double integrals. In fact, it is the double integral formulation involving $S(\theta, \tau)$ in (2.17) with the interpretation that the output is a superposition of time-delayed and Doppler-shifted copies of the input that makes $S(\theta, \tau)$ an extremely useful characterization. For completeness, we note the inverse relations.

$$S(\theta, \tau) = \iint k_1(\nu, t) e^{j2\pi(t+\tau)(\nu-\theta)} \mathrm{d}\nu\mathrm{d}t \tag{2.23}$$
$$V(t, \nu) = \iint k_2(\tau, \theta) e^{j2\pi(t-\tau)(\theta+\nu)} \mathrm{d}\tau\mathrm{d}\theta \tag{2.24}$$

Although less commonly used in the literature, $k_3(\theta, \nu)$ plays a pivotal role in understanding the narrowband and wideband characterizations. For this reason, and for later reference. we note the mapping between k_3 and S.

$$k_3(\theta, \nu) = \int S(\theta - \nu, \tau) e^{-j2\pi\tau\nu} d\tau \tag{2.25}$$

$$S(\theta, \tau) = \int k_3(\theta + \nu, \nu) e^{j2\pi\tau\nu} d\nu. \tag{2.26}$$

which can be derived directly from the input-output channel characterizations. In the kernel system formulation of the channel. the outputs could be simply expressed in term of the kernel functions for inputs that were impulses in time and frequency. For the above characterizations. these relations are:

- The response to $\delta(t - t_0)$ is $h(t, t - t_0)$ with spectrum $M(\theta, t_0)e^{-j2\pi\theta t_0}$.
- The response to $e^{j2\pi\theta_0 t}$ is $T(t, \theta_0)e^{j2\pi t\theta_0}$ with spectrum $H(\theta - \theta_0, \theta_0)$.

For clarity. we display just the front face of the cube in (2.16). which details the Fourier transform relationships between the four most commonly used system functions.

$$\begin{array}{ccc} h(t, \tau) & \xrightarrow{\mathcal{F}_{t \to \theta}} & S(\theta, \tau) \\ \mathcal{F}_{\tau \to \nu} \Big\downarrow & & \mathcal{F}_{\tau \to \nu} \Big\downarrow \\ T(t, \nu) & \xrightarrow{\mathcal{F}_{t \to \theta}} & H(\theta, \nu) \end{array} \tag{2.27}$$

2.2 Simple Narrowband Channels

In order to get some intuition concerning the channel characterization functions (both the four kernel functions (2.3) and the additional eight characterizations (2.6) through (2.13)). we examine what form the functions take for simple channels.

Consider the channel which is just a delay.

$$x(t) \rightarrow \boxed{\text{channel}} \rightarrow x(t - \tau_0)$$

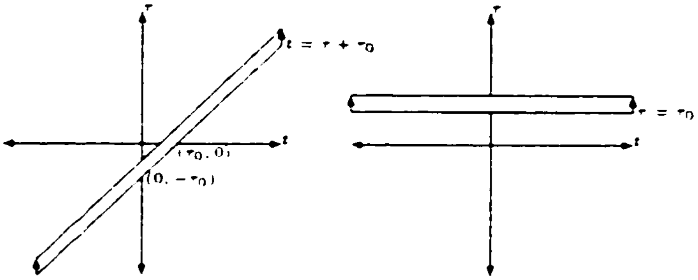


Figure 2.1: $k_0(t, \tau)$ (left) and $h(t, \tau)$ (right) for delay by τ_0 channel.

In the case of the time-varying impulse response kernel, this channel is represented by $k_0(t, \tau) = \delta(t - \tau - \tau_0)$. In the case of the time-varying impulse response, this channel is represented by $h(t, \tau) = \delta(\tau - \tau_0)$. Plots of these two functions are displayed in Figure 2.1.

One useful attribute of a system function is for a visual inspection of the function to readily reveal some physical properties of the channel. In the case of Figure 2.1, we see that, for $k_0(t, \tau)$, a diagonal delta function line crossing through $(0, -\tau_0)$ and $(\tau_0, 0)$ arises from a delay of τ_0. For $h(t, \tau)$, a delay of τ_0 corresponds to a horizontal delta function line τ_0 from the origin. A channel with several reflections (i.e., several different delays), would thus correspond to a system function with several parallel delta function lines. When the channel involves both a simple delay and a Doppler shift, the simple delta function lines for both $k_0(t, \tau)$ and $h(t, \tau)$ are modulated by the Doppler shift. Table 2.1 displays the 12 system functions for the delay and delay-Doppler channels. The system function with the simplest form is $S(\theta, \tau)$ which has the form of the product of delta functions. From this, we interpret a region of localized energy in $S(\theta, \tau)$ centered at (θ_0, τ_0) as arising from an echo path with delay τ_0 and Doppler shift θ_0; see Figure 2.2.

	$y(t) = x(t-\tau_0)$	$y(t) = x(t-\tau_0)e^{j2\pi\theta_0 t}$
$k_0(t,\tau)$	$\delta(t-\tau-\tau_0)$	$\delta(t-\tau-\tau_0)e^{j2\pi\theta_0 t}$
$k_1(\theta,\tau)$	$e^{-j2\pi(\tau+\tau_0)\theta}$	$e^{-j2\pi(\tau+\tau_0)(\theta-\theta_0)}$
$k_2(t,\nu)$	$e^{j2\pi(t-\tau_0)\nu}$	$e^{j2\pi t(\nu+\theta_0)}e^{-j2\pi\tau_0\nu}$
$k_3(\theta,\nu)$	$\delta(\theta-\nu)e^{-j2\pi\tau_0\nu}$	$\delta(\theta-\nu-\theta_0)e^{-j2\pi\tau_0\nu}$
$h(t,\tau)$	$\delta(\tau-\tau_0)$	$\delta(\tau-\tau_0)e^{j2\pi t\theta_0}$
$S(\theta,\tau)$	$\delta(\tau-\tau_0)\delta(\theta)$	$\boldsymbol{\delta(\tau-\tau_0)\delta(\theta-\theta_0)}$
$T(t,\nu)$	$e^{-j2\pi\tau_0\nu}$	$e^{j2\pi\theta_0 t}e^{-j2\pi\tau_0\nu}$
$H(\theta,\nu)$	$e^{-j2\pi\tau_0\nu}\delta(\theta)$	$e^{-j2\pi\tau_0\nu}\delta(\theta-\theta_0)$
$G(\theta,\nu)$	$e^{-j2\pi\tau_0\theta}\delta(\nu)$	$e^{-j2\pi\tau_0(\theta-\theta_0)}\delta(\nu-\theta_0)$
$V(t,\nu)$	$\delta(t-\tau_0)\delta(\nu)$	$e^{j2\pi\tau_0\theta_0}\delta(t-\tau_0)\delta(\nu-\theta_0)$
$M(\theta,\tau)$	$e^{-j2\pi\tau_0\theta}$	$e^{-j2\pi\tau_0(\theta-\theta_0)}e^{j2\pi\tau\theta_0}$
$g(t,\tau)$	$\delta(t-\tau_0)$	$e^{j2\pi\tau_0\theta_0}\delta(t-\tau_0)e^{j2\pi\tau\theta_0}$

Table 2.1: Time-frequency characterization functions for the one path delay and one path delay-Doppler channels. $S(\theta,\tau)$ has a very simple form for the one path delay-Doppler channel.

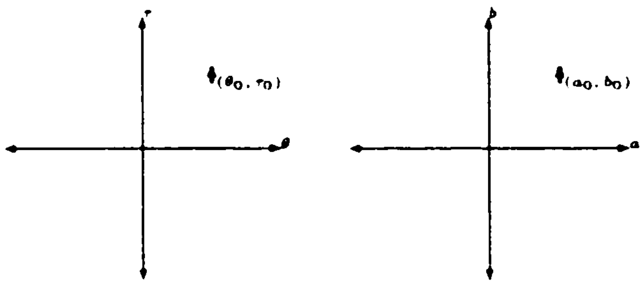


Figure 2.2: $S(\theta,\tau)$ for one path channel with delay τ_0 and Doppler shift θ_0 (left); $\mathcal{C}(a,b)$ for one path channel with delay b_0 and time dilation a_0 (right).

2.3 Wideband Characterization

Starting from the wideband channel characterization,

$$y(t) = \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right)\mathrm{d}a\mathrm{d}b. \tag{2.28}$$

we derive the frequency domain to frequency domain mapping.

$$\begin{aligned} Y(\theta) &= \iiint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right)e^{-j2\pi t\theta}\mathrm{d}a\mathrm{d}b\mathrm{d}t & (2.29a)\\ &= \iiint \mathcal{L}(a,b)\sqrt{|a|}x\left(t'\right)e^{-j2\pi(at'+b)\theta}\mathrm{d}a\mathrm{d}b\mathrm{d}t' & (2.29b)\\ &= \iint \mathcal{L}(a,b)\sqrt{|a|}X\left(a\theta\right)e^{-j2\pi b\theta}\mathrm{d}a\mathrm{d}b & (2.29c) \end{aligned}$$

and defining,

$$\mathcal{L}^{(2)}(a,\theta) = \int \mathcal{L}(a,b)e^{-j2\pi b\theta}\mathrm{d}b. \tag{2.30}$$

we obtain

$$\boxed{Y(\theta) = \int \mathcal{L}^{(2)}(a,\theta)\sqrt{|a|}X\left(a\theta\right)\mathrm{d}a} \tag{2.31}$$

We perform the coordinate transform $\nu = a\theta$ and obtain,

$$Y(\theta) = \begin{cases} \int \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}^{(2)}\left(\frac{\nu}{\theta},\theta\right)X(\nu)\mathrm{d}\nu & : \quad \theta \neq 0 \\ X(0)\iint \sqrt{|a|}\mathcal{L}(a,b)\mathrm{d}a\mathrm{d}b & : \quad \theta = 0 \end{cases} \tag{2.32}$$

When considering $\mathcal{L}_+$, ν and θ must have the same sign for $\mathcal{L}_+^{(2)}(\frac{\nu}{\theta},\theta)$ to be non-zero, and we can express (2.32) as follows,

$$Y(\theta) = \begin{cases} \int_0^{\infty} \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}_+^{(2)}\left(\frac{\nu}{\theta},\theta\right)X(\nu)\mathrm{d}\nu & : \quad \theta > 0 \\ X(0)\iint \sqrt{|a|}\mathcal{L}_+(a,b)\mathrm{d}a\mathrm{d}b & : \quad \theta = 0 \\ \int_{-\infty}^{0} \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}_+^{(2)}\left(\frac{\nu}{\theta},\theta\right)X(\nu)\mathrm{d}\nu & : \quad \theta < 0 \end{cases} \tag{2.33}$$

	$y(t) = x(t - b_0)$	$y(t) = \frac{1}{\sqrt{\lvert a_0 \rvert}} x(\frac{t-b_0}{a_0})$
$\mathcal{L}(a, b)$	$\delta(a-1)\delta(b-b_0)$	$\delta(a-a_0)\delta(b-b_0)$
$\mathcal{L}^{(2)}(a, \theta)$	$\delta(a-1)e^{-j2\pi b_0 \theta}$	$\delta(a-a_0)e^{-j2\pi b_0 \theta}$

Table 2.2: Time-scale characterization functions for the one path delay and one path delay-dilation channels.

Note that positive frequency input components give rise only to positive frequency output components, and negative frequency input components give rise only to negative frequency output components. This property of the wideband input-output mapping will play a large role in understanding the differences between the narrowband and wideband channel models. Indeed, [SP95] noted the significance of this invariance in the difference between time-frequency and time-scale analysis.

Table 2.2 displays the wideband characterization functions for the one path delay and one path delay-dilation channels. In the narrowband case, $S(\theta, \tau)$ took the form of the product of delta functions for the one path delay-Doppler channel; In the wideband case, the one path delay-dilation channel has the product of delta functions formulation. We interpret a region of concentrated energy in $\mathcal{L}(a, b)$ centered at (a_0, b_0) as arising from an echo path with delay b_0 and dilation parameter a_0.

Chapter 3

Narrowband and Wideband Correspondence

In this chapter we study the correspondence between the narrowband and wideband channel models by studying the link between time-frequency and time-scale integral operators. More specifically, we wish to link the narrowband channel model characterized by the dozen system functions discussed in the previous chapter, one of which was described by the operator,

$$\mathcal{N}_S\{x\}(t) = \iint S(\theta, \tau) x(t-\tau) e^{j2\pi\theta t} \mathrm{d}\tau \mathrm{d}\theta \tag{3.1}$$

to the wideband channel description described by the operator,

$$\mathcal{W}_{\mathcal{L}}\{x\}(t) = \iint \mathcal{L}(a, b) \frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) \mathrm{d}a \mathrm{d}b. \tag{3.2}$$

We are interested in the mapping between S and $\mathcal{L}$ for $\mathcal{N}_S = \mathcal{W}_{\mathcal{L}}$.

3.1 Time-invariance

Before we look to the general narrowband-wideband correspondence, we consider time-invariant narrowband and wideband channel characterizations.

3.1.1 Narrowband time-invariance

For time-invariance in the narrowband characterization, for all t_0,

$$y(t - t_0) = \iint S(\theta, \tau) x(t - t_0 - \tau) e^{j2\pi\theta t} \mathrm{d}\tau \mathrm{d}\theta. \tag{3.3}$$

That is, an input delay causes nothing more than an output delay. Shifting the time argument $t \to t + t_0$,

$$y(t) = \iint S(\theta, \tau) x(t - \tau) e^{j2\pi\theta(t+t_0)} \mathrm{d}\tau \mathrm{d}\theta, \ \forall t_0 \tag{3.4}$$

and thus, as (3.4) must be true for all $x \in L^2(\mathbb{R})$, for time-invariant $S(\theta, \tau)$,

$$\int S(\theta, \tau) e^{j2\pi\theta t} \mathrm{d}\theta = \int S(\theta, \tau) e^{j2\pi\theta(t+t_0)} \mathrm{d}\theta, \ \forall t_0. \tag{3.5}$$

Stated in terms of $h(t, \tau) = \int S(\theta, \tau) e^{j2\pi\theta t} \mathrm{d}\theta$, this is,

$$h(t, \tau) = h(t + t_0, \tau), \forall t_0, \tag{3.6}$$

and thus $h(t, \tau)$ is constant for all t for a given τ. Using $S(\theta, \tau) = \int h(t, \tau) e^{-j2\pi\theta t} \mathrm{d}t$, we see that for time-invariant $S(\theta, \tau)$,

$$\operatorname{supp} S(\theta, \tau) = \{(\theta, \tau) : \theta = 0, \forall \tau\}. \tag{3.7}$$

(3.7) is necessary, but not sufficient for time-invariance. We have yet to define the space of valid $S(\theta, \tau)$. For example, if the allowable set of $S(\theta, \tau)$ includes distributions such as,

$$S(\theta, \tau) = \delta'(\theta)\delta(\tau - t_0) \tag{3.8}$$

where $\delta'(\theta)$ is the derivative of the delta-function, then the input-output relation is,

$$\begin{aligned} y(t) &= \iint \delta'(\theta)\delta(\tau - t_0)x(t - \tau)e^{j2\pi\theta t}\mathrm{d}\tau\mathrm{d}\theta && \text{(3.9a)} \\ &= \int \delta'(\theta)x(t - t_0)e^{j2\pi\theta t}\mathrm{d}\theta && \text{(3.9b)} \\ &= -\int \delta(\theta)x(t - t_0)j2\pi t e^{j2\pi\theta t}\mathrm{d}\theta && \text{(3.9c)} \\ &= -j2\pi t x(t - t_0) && \text{(3.9d)} \end{aligned}$$

In moving from (3.9b) to (3.9c) we have used,

$$\int \delta'(\theta)f(\theta)\mathrm{d}\theta = -\int \delta(\theta)f'(\theta)\mathrm{d}\theta = -f'(0). \tag{3.10}$$

(3.9d) is not a channel output with a physical interpretation. We assume the channel produces real-valued output given real-valued input, and this assumption is violated in (3.9d). Also, the multiplicative factor of t has no physical interpretation in the transmitter/scatterer/receiver model we are exclusively considering. For these reasons, we will consider $S(\theta, \tau)$ which do not include derivatives of delta functions. In general, using

$$\int \delta^{(n)}(\theta)f(\theta)\mathrm{d}\theta = (-1)^n f^{(n)}(0), \tag{3.11}$$

for

$$S(\theta, \tau) = \delta^{(n)}(\theta)\delta(\tau - t_0), \tag{3.12}$$

we see the input-output relation is

$$y(t) = (-j2\pi t)^n x(t - t_0). \tag{3.13}$$

(3.13) does not describe a desired input-output relationship for $n \geq 1$ due to the multiplicative factor of t^n and the fact that the channel output is imaginary for real-valued input for odd n. For these reasons, we restrict the space of allowable $S(\theta, \tau)$ to not include distributions with derivatives (of any order) of delta functions. Note however, when $S(\theta, \tau)$ only contains contributions from $\theta = 0$ and does not involve the derivative (of any order) of a delta function (i.e., $S(\theta, \tau) = \delta(\theta) S_0(\tau)$),

$$y(t) = \iint S(\theta, \tau) x(t - \tau) e^{j2\pi\theta t} \mathrm{d}\tau \mathrm{d}\theta \tag{3.14a}$$

$$= \int S_0(\tau) x(t - \tau) \mathrm{d}\tau \tag{3.14b}$$

we obtain the time-invariant channel characterization. This of course makes sense as the Doppler shift is introduced as a model of the effect of motion on narrowband signals. Zero Doppler shift, $\theta = 0$, means no motion, which results in time-invariance. So, the space of allowable $S(\theta, \tau)$ should include distributions with delta functions, but exclude distributions with the derivatives of delta functions or any order [1].

For later reference, we determine the support region for the time-invariant component of $k_3(\theta, \nu)$ using (2.25) which we repeat here,

$$k_3(\theta, \nu) = \int S(\theta - \nu, \tau) e^{-j2\pi\tau\nu} \mathrm{d}\tau. \tag{3.15}$$

$S(\theta - \nu, \tau)$ is time-invariant if its support is on the $\theta - \nu = 0$ line, thus $k_3(\theta, \nu)$ is time-invariant if its support is on the $\theta = \nu$ line.

[1] One such space is the modulation space $M^{1,\infty}$. Modulation spaces were introduced in [Fei83]; Consult the relevant chapters in [Gro00] for illumination concerning many of the properties of modulation spaces and their application in many time-frequency analysis problems.

3.1.2 Wideband time-invariance

Time-invariance in the wideband characterization is defined, for all t_0,

$$y(t - t_0) = \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a} - t_0\right) da db. \tag{3.16}$$

Shifting the time argument $t \rightarrow t + t_0$,

$$y(t) = \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t+t_0-b}{a} - t_0\right) da db. \tag{3.17}$$

which is true when,

$$\frac{t+t_0-b}{a} - t_0 = \frac{t-b}{a}, \quad \forall t_0, \tag{3.18}$$

and thus,

$$(a-1)t_0 = 0. \tag{3.19}$$

So, for time-invariant $\mathcal{L}(a,b)$,

$$\operatorname{supp} \mathcal{L}(a,b) = \{(a,b) : a = 1, \forall b\}. \tag{3.20}$$

If $\mathcal{L}(a,b)$ contains derivatives of delta functions, then the channel output will depend on the derivative of the input signal. As before, this has no physical interpretation, so we exclude distributions with the derivatives of delta functions from the space of allowable $\mathcal{L}(a,b)$. However, $\mathcal{L}(a,b)$ can contain delta functions. For example, consider the case with contributions only when $a = 1$ (i.e., $\mathcal{L}(a,b) = \delta(a-1)\mathcal{L}_1(b)$),

$$y(t) = \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right) da db \tag{3.21a}$$

$$= \int \mathcal{L}_1(b)x(t-b)db \tag{3.21b}$$

characterization	time-invariance support condition
$S(\theta,\tau)$	$\operatorname{supp} S(\theta,\tau) \in \{(\theta,\tau) : \theta = 0\}$
$\mathcal{L}(a,b)$	$\operatorname{supp} \mathcal{L}(a,b) \in \{(a,b) : a = 1\}$
$k_3(\theta,\nu)$	$\operatorname{supp} k_3(\theta,\nu) \in \{(\theta,\nu) : \theta = \nu\}$

Table 3.1: Support of time-invariant component of $S(\theta,\tau)$, $\mathcal{L}(a,b)$, $k_3(\theta,\nu)$.

We obtain the time-invariant channel characterization. This also makes sense as the time dilation/contraction parameter a models the effect of the increasing/decreasing path lengths. The $a = 1$ case is the no motion case which is time-invariant.

We summarize the support conditions for time-invariance for the channel characterization discussed in this section in Table 3.1.

3.1.3 Time-invariant correspondence

As (3.14b) and (3.21b) are true for all $x \in L^2(\mathbb{R})$, we conclude that time-invariant S and $\mathcal{L}$ have the following form and mapping.

$$\mathcal{L}(a,b) = \delta(a-1)f(b) \leftrightarrow S(\theta,\tau) = \delta(\theta)f(\tau). \tag{3.22}$$

This result could have been anticipated from Table 2.1 as we know that the support of the simple one path delay channel represented by $S(\theta,\tau)$ is always on the $\theta = 0$ line

$$S(\theta,\tau) = \delta(\theta)\delta(\tau - \tau_0) \tag{3.23}$$

and similarly, from Table 2.2, the support of the simple one path delay channel represented by $\mathcal{L}(a,b)$ is always on the $a = 1$ line

$$\mathcal{L}(a,b) = \delta(a-1)\delta(b-\tau_0) \tag{3.24}$$

One fact made clear by the derivation of this result is that smooth $S(\theta,\tau)$ or

$\mathcal{L}(a, b)$ have *no* time-invariant components: In order for $S(\theta, \tau)$ or $\mathcal{L}(a, b)$ to possess a time-invariant component, they must have impulsive energy along the $\theta = 0$ and $a = 1$ lines respectively.

3.1.4 Time-invariant/varying split

It will be useful to separate the time-invariant component from the time-varying component in the channel characterizations. We define $S_0(\tau)$ as the impulsive component along $\theta = 0$, if any, of the characterization, and we define $\tilde{S}(\theta, \tau)$ as the characterization with this impulsive component removed. Considering these two components, we have the following division of the narrowband characterization.

$$S(\theta, \tau) = \delta(\theta)S_0(\tau) + \tilde{S}(\theta, \tau) \tag{3.25}$$

We can think of this division as arising from the following split,

$$\begin{aligned} S(\theta, \tau) &= (1 - e^{-\alpha\theta^2} + e^{-\alpha\theta^2})S(\theta, \tau) && (3.26a) \\ &= e^{-\alpha\theta^2}S(\theta, \tau) + (1 - e^{-\alpha\theta^2})S(\theta, \tau) && (3.26b) \end{aligned}$$

when we let $\alpha \to \infty$. Removing the non-impulsive component from $S(\theta, \tau)$ for $\theta = 0$ does not affect the action of the operator as the $\theta = 0$ line has measure zero.

We can similarly split the wideband characterization.

$$\mathcal{L}(a, b) = \delta(a - 1)\mathcal{L}_1(b) + \tilde{\mathcal{L}}(a, b) \tag{3.27}$$

where the first term $\delta(a - 1)\mathcal{L}_1(b)$ captures the impulsive component along $a = 1$, if any, and the second term $\tilde{\mathcal{L}}(a, b)$ is the characterization with this impulsive component removed. As before, we can think of this division as arising from the following split,

$$\mathcal{L}(a,b) = (1 - e^{-\alpha(a-1)^2} + e^{-\alpha(a-1)^2})\mathcal{L}(a,b) \tag{3.28a}$$

$$= e^{-\alpha(a-1)^2}\mathcal{L}(a,b) + (1 - e^{-\alpha(a-1)^2})\mathcal{L}(a,b) \tag{3.28b}$$

when we let $\alpha \rightarrow \infty$. Removing the non-impulsive component from $\mathcal{L}(a,b)$ for $a = 1$ does not affect the action of the operator as the $a = 1$ line has measure zero.

The mapping of the time-invariant component between S and $\mathcal{L}$ is understood.

$$S_0(\tau) = \mathcal{L}_1(\tau). \tag{3.29}$$

The mapping of the time-varying component is the main subject of the remainder of this chapter. To better understand the mapping, we go through the k_3 representation of the channel: This plan is depicted in Figure 3.1.

3.1.5 Summary

In summary,

- The time-invariant component of the channel represented by $\mathcal{L}(a,b)$ has support $(a = 1, b \in (-\infty, \infty))$.
- The time-invariant component of the channel represented by $S(\theta,\tau)$ has support $(\theta = 0, \tau \in (-\infty, \infty))$.
- We can split the channel into time-invariant and time-varying components.

$$S(\theta,\tau) = \delta(\theta)S_0(\tau) + \tilde{S}(\theta,\tau) \tag{3.30}$$

$$\mathcal{L}(a,b) = \delta(a-1)\mathcal{L}_1(b) + \tilde{\mathcal{L}}(a,b) \tag{3.31}$$

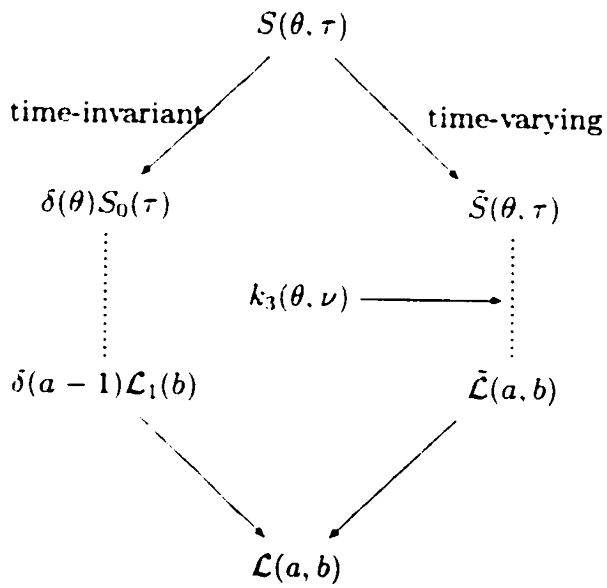


Figure 3.1: $S(\theta, \tau)$ and $\mathcal{L}(a, b)$ can be separated into time-invariant and time-varying components. For the time-invariant component for equivalent channels ($\mathcal{N}_S = \mathcal{W}_{\mathcal{L}}$), $S_0(\tau) = \mathcal{L}_1(\tau)$. For the time-varying component, we will use the frequency-frequency kernel $k_3(\theta, \nu)$ as an intermediary to derive the relationship between $\tilde{S}(\theta, \tau)$ and $\tilde{\mathcal{L}}(a, b)$, the time-varying components of $S(\theta, \tau)$ and $\mathcal{L}(a, b)$.

- For $\mathcal{N}_S = \mathcal{W}_{\mathcal{L}}$, the time-invariant component correspondence is

$$S_0(\tau) = \mathcal{L}_1(\tau). \tag{3.32}$$

3.2 Causality

One common assumption is that the channel is causal, that is, that the output cannot anticipate the input. Stated formally, an operator $\mathcal{O}$ is causal if,

$$x_1(t) = x_2(t),\ \forall t \leq \tau \longrightarrow \mathcal{O}\{x_1\}(t) = \mathcal{O}\{x_2\}(t),\ \forall t \leq \tau. \tag{3.33}$$

That is, identical inputs up to time τ produce identical outputs up to time τ as well. This is true, for example, in the case of $y(t) = \int k_0(t,\tau)x(\tau)d\tau$, when,

$$k_0(t,\tau) = 0,\ \forall t < \tau. \tag{3.34}$$

The causality condition on $\mathcal{L}(a,b)$ is worth discussing, and we will now prove that:

Theorem 1.

The causal time-scale channel is time-invariant.

Proof. From the wideband characterization,

$$y(t) = \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right)dadb, \tag{3.35}$$

looking to the time dependence, we see that the support of $\mathcal{L}$ must satisfy,

$$\text{supp}\,\mathcal{L}(a,b) \subset \left\{(a,b) \quad : \quad t \geq \frac{t-b}{a},\quad \forall t \in (-\infty,\infty)\right\} \tag{3.36}$$

in order to ensure (3.33). Let us first consider (3.36) for $t \in (t_0, t_1)$, $t_0 < t_1$. The allowable support region in this case is,

$$\left\{ (a,b) : \begin{array}{c} a < 0 \text{ and } b \leq -t_0 a + t_0 \text{, or} \\ a > 0 \text{ and } b \geq -t_0 a + t_0 \text{ and } b \geq -t_1 a + t_1 \end{array} \right\} \tag{3.37}$$

and is depicted as the crosshatch region in Figure 3.2.

Now we consider this support region as $t_0 \to -\infty$ and as $t_1 \to \infty$. As $t_0 \to -\infty$, in the $a < 0$ case, we see that,

$$b \leq -t_0 a + t_0 \tag{3.38a}$$

$$b/t_0 \geq -a + 1 \qquad (t_0 < 0) \tag{3.38b}$$

$$a \geq 1 \qquad (t_0 \to -\infty) \tag{3.38c}$$

and thus causal $\mathcal{L}(a,b)$ are zero for $a < 0$. For $a > 0$, we have,

$$b \geq -t_0 a + t_0 \tag{3.39a}$$

$$\frac{b}{t_0} \leq -a + 1 \qquad (t_0 < 0) \tag{3.39b}$$

$$a \leq 1 \qquad (t_0 \to -\infty) \tag{3.39c}$$

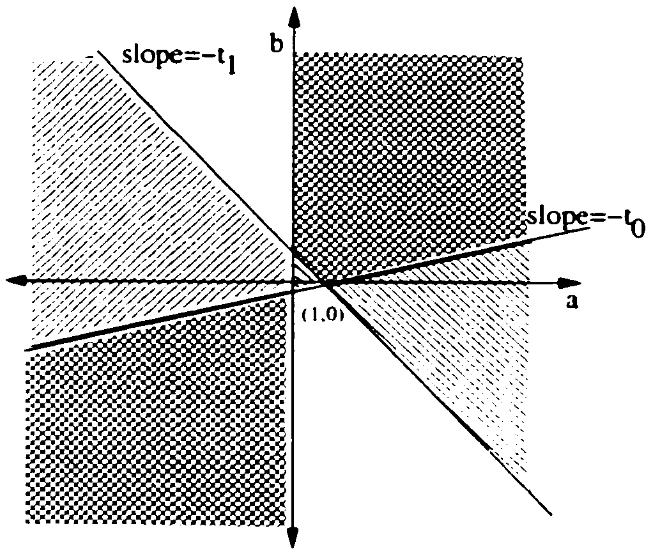
and,

$$b \geq -t_1 a + t_1 \tag{3.40a}$$

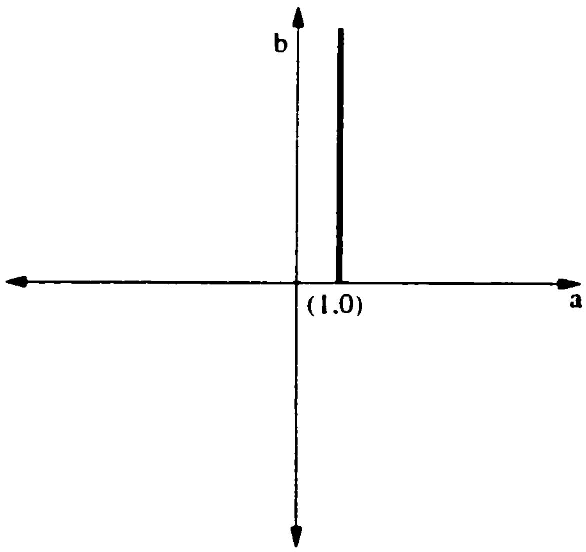
$$\frac{b}{t_1} \geq -a + 1 \qquad (t_1 > 0) \tag{3.40b}$$

$$a \geq 1 \qquad (t_1 \to \infty) \tag{3.40c}$$

Combining (3.40c) and (3.39c) we obtain $a = 1$ and reinserting this back into (3.39a) or (3.40a), we see that $b \geq 0$. Thus the allowable support region becomes the half line $a = 1$, $b \geq 0$. But we have just seen in the previous section that $\mathcal{L}(a,b)$ which



(a) $t \in (t_0, t_1)$



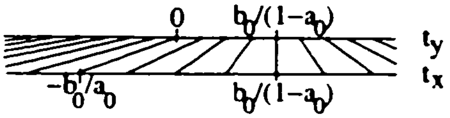
(b) $t \in (-\infty, \infty)$

Figure 3.2: The crosshatch region in plot (a) is the region of allowable support for causal $\mathcal{L}(a, b)$ for $t \in (t_0, t_1)$. As $t_0 \rightarrow -\infty$ the line with slope $-t_0$ rotates counter-clockwise about $(1, 0)$ and it approaches the line $a = 1$. The allowable support region associated with this line is the region below the line for $a < 0$ and the region above the line for $a > 0$; these regions are depicted with negative slope parallel lines. As $t_1 \rightarrow \infty$ the line with slope $-t_1$ rotates clockwise about $(1, 0)$ and it approaches the line $a = 1$. The allowable support associated with this line is the region below the line for $a < 0$ and above the line for $a > 0$; these regions are depicted with positive slope parallel lines. As $t_0 \rightarrow -\infty$ and $t_1 \rightarrow \infty$, the region of allowable support for causal $\mathcal{L}(a, b)$ becomes the half line $a = 1, b > 0$, shown in the plot (b).

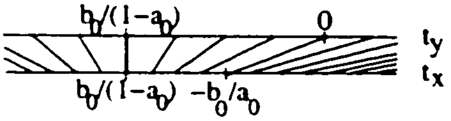
have support along this line are time-invariant. □

Why does causality result in this strong limitation of the channel? As we have seen, this limitation arises from the consideration of the infinite past and infinite future under the constraint of causality. Intuitively, when the channel has a contraction component, that is, support for some $a < 1$, the transmitted signal is being received more quickly than it is being emitted. This is not a problem when considering only the past; the receiver at a point in the past is simply further behind in time relative to the transmitter. Causality requires that the receiver is always behind in time relative to the transmitter. Looking to the future, however, the fact that received waveform is being received more quickly than it is being emitted means that after some point in the future, the received signal has not yet been emitted, which violates causality. This fact is depicted in Figure 3.3. Similarly, if the channel contains a dilation component, support for some $a > 1$, the transmitted signal is being received more slowly than it is being emitted. This is not a problem when considering only the future, the receiver simply gets further and further behind in time relative to the transmitter. Looking back in the past, however, this effect is reversed and before some point in the past, the signal received had not yet been emitted. This is also depicted in Figure 3.3.

This result, while mathematically accurate, is not as limiting as first appears. The normal time intervals of interest at any given time in the communication scenarios we are likely to consider are quite small, which limits the region eliminated from the potential support of $\mathcal{L}(a, b)$ in order for it to be 'locally' time-invariant. Also, the limits on the speeds of the transmitter and receiver and the rapid decay of signal energy with propagation distance limit the region where $\mathcal{L}(a, b)$ could potentially be non-zero far more than this causality constraint does when considering small time intervals. Thus, by redefining the time origin for each processing interval, the region eliminated from consideration will be (for a small interval (t_0, t_1)), essentially just Q2 ($a < 0$ and $b > 0$) and Q4 ($a > 0, b < 0$) and thus does not limit the channel to be



(a) $a_0 < 1$



(b) $a_0 > 1$

Figure 3.3: Causality for the wideband channel. t_y is the time at the receiver and t_x is the time at the transmitter. The lines between t_x and t_y, a depiction of the mapping $t_x = \frac{t_y - b_0}{a_0}$, are shown for $a_0 < 1$ in plot (a) and $a_0 > 1$ plot (b). In both cases, the receiver at time $t_y = 0$ receives the signal sent from the transmitter at time $t_x = -b_0/a_0$. Lines with positive slope are causal; Lines with negative slope are anti-causal. For $a_0 < 1$, there exists some point in the future, $t_x = t_y = \frac{b_0}{1-a_0}$, where the channel becomes anti-causal. For $a < 1$, there exists some point in the past, $t_x = t_y = \frac{b_0}{1-a_0}$, before which the channel was anti-causal.

time-invariant.

As discussed in Appendix A, the wideband model is not truly time-varying, $\mathcal{L}(a,b)$ does not depend on t. It is a snapshot of an environment in which objects are moving with constant radial velocity, but these velocities are constant for all time and the channel transformation that occurs when the transmitter and receiver are collocated cannot be represented using $\mathcal{L}(a,b)$. This restriction coupled with infinite time consideration forces all velocities to be zero, and thus, time-invariance. A more general characterization would be,

$$y(t) = \iint \mathcal{P}(a,b,t) \frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) da db \qquad (3.41)$$

which was proposed in [You95], who also noted that $\mathcal{L}(a,b)$ is a special case of

$k_0(t,\tau) = 0, \forall t < \tau$
$h(t,\tau) = 0, \forall \tau < 0$
$S(\theta,\tau) = 0, \forall \tau < 0$
$\mathcal{L}(a,b) = 0, \forall (a,b)$ except $(a = 1, b \geq 0)$ $\quad (t \in (-\infty,\infty))$
$\mathcal{L}(a,b) = 0, \forall (a,b)$: $a < 0$ and $b > -t_0 a + t_0$, or $a > 0$ and $b < -t_0 a + t_0$ and $b < -t_1 a + t_1$ $\quad (t \in (t_0,t_1))$
$\mathcal{L}_+(a,b) = 0, \forall (a,b)$: $b < -t_0 a + t_0$ or $b < -t_1 a + t_1$ $\quad (t \in (t_0,t_1))$

Table 3.2: Causality constraints for channel characterizations.

real-valued $h(t,\tau)$	real-valued $k_0(t,\tau)$	real-valued $g(t,\tau)$
$S(\theta,\tau) = \overline{S(-\theta,\tau)}$	$k_1(\theta,\tau) = \overline{k_1(-\theta,\tau)}$	$M(\theta,\tau) = \overline{M(-\theta,\tau)}$
$T(t,\nu) = \overline{T(t,-\nu)}$	$k_2(t,\nu) = \overline{k_2(t,-\nu)}$	$V(t,\nu) = \overline{V(t,-\nu)}$
$H(\theta,\nu) = \overline{H(-\theta,-\nu)}$	$k_3(\theta,\nu) = \overline{k_3(-\theta,-\nu)}$	$G(\theta,\nu) = \overline{G(-\theta,-\nu)}$
real-valued $\mathcal{L}(a,b)$	$\mathcal{L}^{(2)}(a,\theta) = \overline{\mathcal{L}^{(2)}(a,-\theta)}$	

Table 3.3: Real-valued channel implications.

$\mathcal{P}(a,b,t)$.

$$\mathcal{L}(a,b) = \mathcal{P}(a,b,t_0) \tag{3.42}$$

which emphasizes the wideband characterizations validity only for a snapshot in time. Incidentally, [You95] also noted we can see the narrowband model as a special case of $\mathcal{P}(a,b,t)$,

$$h(t,\tau) = \mathcal{P}(a,\tau,t)\delta(a-1). \tag{3.43}$$

We summarize the constraints resulting in causality for several of the characterizations in Table 3.2.

3.3 Real-valued channels

We will assume the channels have real-valued impulse responses so that real-valued inputs produce real-valued outputs. We summarize the implications this assumption has for the channel characterizations in Table 3.3.

3.4 Wideband to Narrowband

3.4.1 $\mathcal{L} \longrightarrow$ four narrowband kernels

We first establish the relation from wideband to narrowband, showing that for every time-scale kernel, there exists a corresponding time-frequency kernel. Starting from (3.2),

$$
\begin{aligned}
y(t) &= \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right)\mathrm{d}a\mathrm{d}b && (3.44\text{a})\\
&= \iint \sqrt{|a|}\mathcal{L}(a,t-a\tau)x(\tau)\mathrm{d}a\mathrm{d}\tau && (3.44\text{b})\\
&= \underbrace{\iint \sqrt{|a|}\mathcal{L}(a,t-a\tau)\mathrm{d}a}_{k_0(t,\tau)}\, x(\tau)\mathrm{d}\tau && (3.44\text{c})
\end{aligned}
$$

Therefore,

$$
\boxed{k_0(t,\tau) = \int \sqrt{|a|}\mathcal{L}(a,t-a\tau)\mathrm{d}a} \qquad (3.45)
$$

We remark that when integrating over t, k_0 derived from $\mathcal{L}$ does not depend on τ. That is,

$$
\begin{aligned}
\int k_0(t,\tau)\mathrm{d}t &= \iint \sqrt{|a|}\mathcal{L}(a,t-a\tau)\mathrm{d}a\mathrm{d}t && (3.46\text{a})\\
&= \iint \sqrt{|a|}\mathcal{L}(a,t)\mathrm{d}a\mathrm{d}t && (3.46\text{b})
\end{aligned}
$$

This aspect of the correspondence shows that the DC component of the output depends only on the DC component of the input and is discussed further in the Remarks later in this section.

From (3.45), we can derive the relationship between the four kernel system functions and $\mathcal{L}$.

$$k_0(t,\tau) = \int \sqrt{|a|}\mathcal{L}(a, t - a\tau)\mathrm{d}a \tag{3.47a}$$

$$k_1(\theta,\tau) = \int \sqrt{|a|}\mathcal{L}^{(2)}(a,\theta)e^{-j2\pi a\tau\theta}\mathrm{d}a \tag{3.47b}$$

$$k_2(t,\nu) = \int \frac{1}{\sqrt{|a|}}\mathcal{L}^{(2)}(a,\nu/a)e^{j2\pi\nu t/a}\mathrm{d}a \tag{3.47c}$$

$$k_3(\theta,\nu) = \iint k_0(t,\tau)e^{-j2\pi t\theta}e^{j2\pi\tau\nu}\mathrm{d}t\mathrm{d}\tau \tag{3.48a}$$

$$= \iiint \sqrt{|a|}\mathcal{L}(a, t - a\tau)e^{-j2\pi t\theta}e^{j2\pi\tau\nu}\mathrm{d}a\mathrm{d}t\mathrm{d}\tau \tag{3.48b}$$

$$= \iiint \sqrt{|a|}\mathcal{L}(a,b)e^{-j2\pi(b+a\tau)\theta}e^{j2\pi\tau\nu}\mathrm{d}a\mathrm{d}b\mathrm{d}\tau \tag{3.48c}$$

$$= \iiint \sqrt{|a|}\mathcal{L}(a,b)e^{-j2\pi\theta b}e^{j2\pi\tau(\nu-a\theta)}\mathrm{d}a\mathrm{d}b\mathrm{d}\tau \tag{3.48d}$$

$$= \iint \sqrt{|a|}\mathcal{L}(a,b)e^{-j2\pi\theta b}\delta(\nu - a\theta)\mathrm{d}a\mathrm{d}b \tag{3.48e}$$

$$= \begin{cases} \iint \frac{\sqrt{|a|}}{|\theta|}\mathcal{L}(a,b)e^{-j2\pi\theta b}\delta\left(\frac{\nu}{\theta} - a\right)\mathrm{d}a\mathrm{d}b & : \quad \theta \neq 0 \\ \delta(\nu)\iint \sqrt{|a|}\mathcal{L}(a,b)\mathrm{d}a\mathrm{d}b & : \quad \theta = 0 \end{cases} \tag{3.48f}$$

When considering general $\mathcal{L}(a,b)$, we can simplify (3.48f) to,

$$\boxed{k_3(\theta,\nu) = \begin{cases} \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}^{(2)}\left(\frac{\nu}{\theta},\theta\right) & : \quad \theta \neq 0 \\ \delta(\nu)\iint \sqrt{|a|}\mathcal{L}(a,b)\mathrm{d}a\mathrm{d}b & : \quad \theta = 0 \end{cases}} \tag{3.49}$$

which can also be derived directly from (2.32) and the definition of $k_3(\theta,\nu)$. When $\mathcal{L}(a,b)$ has support only for $a > 0$, that is $\mathcal{L}_+(a,b)$, $\delta\left(\frac{\nu}{\theta} - a\right)$ will only be non-zero when θ and ν have the same sign, and thus,

$$\boxed{k_3(\theta,\nu) = \begin{cases} \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}_+^{(2)}\left(\frac{\nu}{\theta},\theta\right) & : \frac{\nu}{\theta} > 0, \quad \theta \neq 0 \\ 0 & : \frac{\nu}{\theta} \leq 0, \quad \theta \neq 0 \\ \delta(\nu)\iint \sqrt{|a|}\mathcal{L}_+(a,b)\mathrm{d}a\mathrm{d}b & : \quad \theta = 0 \end{cases}} \tag{3.50}$$

which can also be derived directly from (2.33) and the definition of $k_3(\theta,\nu)$.

3.4.2 Remarks on $\mathcal{L} \rightarrow k_3$

1. Examining the $\theta = 0$ case in (3.49), we see that $k_3(0,\nu) = 0$, $\forall \nu \neq 0$. Thus, the DC component of the output depends only on the DC component of the input. This is true, in general, for the wideband channel representation, as we can derive from the input-output wideband characterization.

$$Y(0) = \int y(t)\mathrm{d}t = \iiint \frac{1}{\sqrt{|a|}}\mathcal{L}(a,b)x\left(\frac{t-b}{a}\right)\mathrm{d}a\mathrm{d}b\mathrm{d}t \tag{3.51a}$$

$$= \left[\iint \sqrt{|a|}\mathcal{L}(a,b)\mathrm{d}a\mathrm{d}b\right]\int x(t)\mathrm{d}t \tag{3.51b}$$

$$= \left[\iint \sqrt{|a|}\mathcal{L}(a,b)\mathrm{d}a\mathrm{d}b\right]X(0) \tag{3.51c}$$

This restriction does not apply in the narrowband characterization. From the definition of $k_3(\theta,\nu)$, we see that that the DC component of the output in the narrowband characterization depends on all frequency components of the input.

$$Y(0) = \int k_3(0,\nu)X(\nu)\mathrm{d}\nu \tag{3.52a}$$

$$= \int\left[\int S(-\nu,\tau)e^{-j2\pi\nu\tau}\mathrm{d}\tau\right]X(\nu)\mathrm{d}\nu \tag{3.52b}$$

The narrowband channel models time contractions and dilations as frequency shifts, and thus, can shift arbitrary input (non-DC) frequency components down/up to DC, as is clear from (3.52a). The input-output shifting in frequency has a physical realization. For example, a pure tone will shift toward zero frequency for transmitter-receiver pairs moving apart from one another. However, the shifting of an AC input component to DC output has no physical realization under the assumption that the magnitude of the rate of transmitter/receiver path length change is strictly less than the speed of wave propagation. In contrast to the narrowband model, the wideband characterization prohibits such unrealizable mappings. Recalling the frequency-frequency representation of the wideband channel (2.31),

$$Y(\theta) = \int \mathcal{L}^{(2)}(a,\theta)\sqrt{|a|}X(a\theta)\,da \tag{3.53}$$

we see that while input components can be shifted in frequency, as a is strictly larger than 0, no non-zero frequency component can be shifted to DC. As we are interested in physically realizable channel characterizations, we should consider k_3 that forbid this, that is,

$$k_3(0,\nu) = 0,\ \forall \nu \neq 0. \tag{3.54}$$

This can be stated equivalently that a necessary condition for k_3 and $\mathcal{L}$ to have the same action, k_3 must not have support on the θ or ν axis, except for at the origin, $(\theta,\nu) = (0,0)$. This constraint will be shown to be contained in in a more general one (see (3.58)) shortly. It should be pointed out that, as we consider $k_3(\theta,\nu)$ which may have impulsive components (i.e., delta functions), the axis lines do not have measure zero (as would be the case for function spaces) and removing them from the characterization does affect the action of the operator.

2. From the treatment of the DC component, we know that,

$$\boxed{\text{There exist } k_3 \text{ (and thus } S\text{) with no corresponding } \mathcal{L}.} \tag{3.55}$$

We have seen that k_3 with support on the θ or ν axis (with the exception of the origin), have no corresponding $\mathcal{L}$. Now we examine the mapping specifically to $\mathcal{L}_+$. In this case, the DC component mapping is but one difference, and here we characterize all the k_3 that have no corresponding $\mathcal{L}_+$. We see in (3.50) that $k_3(\theta,\nu)$ derived from $\mathcal{L}_+(a,b)$ have support in $\{(\theta,\nu) : \theta\nu > 0 \text{ or } (\theta,\nu) = (0,0)\}$. This constraint came about as a result of the value of a being limited to $(0,\infty)$, which we can see directly from the input-output relationships. In the wideband case,

$$Y(\theta) = \int \mathcal{L}^{(2)}(a,\theta)\sqrt{|a|}X(a\theta)\,da. \tag{3.56}$$

with $a > 0$, positive input frequencies can affect only positive output frequencies and negative input frequencies can affect only negative output frequencies. This constraint is not present in the narrowband case where each output frequency component can depend on any input frequency component, regardless of sign:

$$Y(\theta) = \int k_3(\theta,\nu)X(\nu)d\nu \tag{3.57}$$

Thus, the k_3 that have no corresponding $\mathcal{L}_+$ are those k_3 having some support in Q2 ($\theta < 0, \nu > 0$), Q4 ($\theta > 0, \nu < 0$), or along one of the axis lines ($\theta = 0, \nu \neq 0$ or $\nu = 0, \theta \neq 0$). We can state this fact formally,

$$\boxed{\mathcal{L}(a,b) = 0, \forall a < 0 \rightarrow k_3(\theta,\nu) = 0, \forall \theta\nu \leq 0 \text{ except } (\theta,\nu) = (0,0)} \tag{3.58}$$

This correspondence is displayed in Figure 3.4.

3. But what does the constraint in (3.58) eliminate? Consider a simple one path

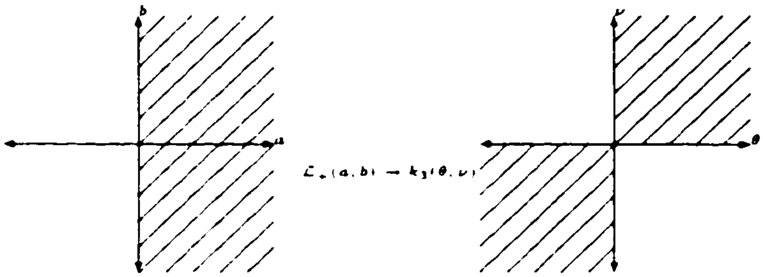


Figure 3.4: If supp $\mathcal{L}(a,b) \subset \{(a,b) : a > 0, \forall b\}$ then the corresponding $k_3(\theta,\nu)$ has support in Q1 and Q3 (and $(\theta,\nu) = (0,0)$).

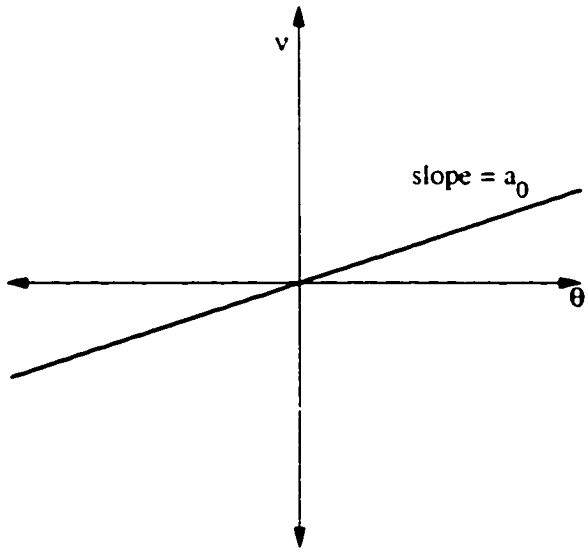


Figure 3.5: The support of $k_3(\theta,\nu)$ for the one path delay-dilation channel.

delay-dilation channel, $\mathcal{L}(a,b) = \delta(a-a_0)\delta(b-b_0)$. The corresponding k_3 is,

$$k_3(\theta,\nu) = \iint \sqrt{|a|}\delta(a-a_0)\delta(b-b_0)e^{-j2\pi\theta b}\delta(\nu - a\theta)dadb \quad (3.59a)$$

$$= \sqrt{|a_0|}e^{-j2\pi\theta b_0}\delta(\nu - a_0\theta) \quad (3.59b)$$

which has support along the line $\nu = a_0\theta$, which is depicted in Figure 3.5.

We see that:

- $a_0 > 0$ results in $k_3(\theta, \nu)$ having support along a line through Q1 and Q3, and
- $a_0 < 0$ results in $k_3(\theta, \nu)$ having support along a line through Q2 and Q4.

From the wideband characterization, we associate $a_0 > 0$ (and thus Q1 and Q3 of k_3) with positive-time copies of the signal, and we associate $a_0 < 0$ (and thus Q2 and Q4 of k_3) with time-reversed copies of the signal, although such an interpretation may only be valid for impulsive channels.

4. From the Fourier transform relation, $\mathcal{F}\{x\}(-t) = X(-\theta)$, we see why Q2 and Q4 of $k_3(\theta, \nu)$ result in time-reversal components of the channel. Considering,

$$k_3(\theta, \nu) = \delta(\theta - \theta_0)\delta(\nu - \nu_0), \tag{3.60}$$

for which the input-output relation is

$$Y(\theta) = \delta(\theta - \theta_0)X(\nu_0), \tag{3.61}$$

we can interpret the four possible quadrant relations in the following way,

- Q1, when $\nu_0 > 0$ and $\theta_0 > 0$, the channel is a Doppler shift by $\theta_0 - \nu_0$,
- Q3, when $\nu_0 < 0$ and $\theta_0 < 0$, the channel is a Doppler shift by $\theta_0 - \nu_0$,
- Q2, when $\nu_0 > 0$ and $\theta_0 < 0$, the channel is a time-reversal and Doppler shift by $\theta_0 + \nu_0$, and
- Q4, when $\nu_0 < 0$ and $\theta_0 > 0$, the channel is a time-reversal and Doppler shift by $\theta_0 + \nu_0$.

Using this interpretation, we define the following concepts:

Definition 1 (Time-forward channel). *A channel is a* time-forward channel *if the support of its $k_3(\theta, \nu)$ representation is exclusively in Q1 and Q3 (and the* *origin):*

$$supp\, k_3(\theta, \nu) \subset \{(\theta, \nu) : \theta\nu > 0 \text{ or } \theta = \nu = 0\} \tag{3.62}$$

Definition 2 (Time-reversal channel). *A channel is a* time-reversal channel *if the support of its $k_3(\theta, \nu)$ representation is exclusively in Q2 and Q4 (and the origin):*

$$supp\, k_3(\theta, \nu) \subset \{(\theta, \nu) : \theta\nu < 0 \text{ or } \theta = \nu = 0\} \tag{3.63}$$

Definition 3 (Non-time-forward channel). *A channel is a* non-time-forward channel *if the support of its $k_3(\theta, \nu)$ representation contains components in Q2 or Q4.*

$$supp\, k_3(\theta, \nu) \cap \{(\theta, \nu) : \theta\nu < 0\} \neq \emptyset \tag{3.64}$$

Note, using these labels, a time-reversal channel is a special case of a non-time-forward channel. $\mathcal{L}_+$ characterizes time-forward channels, and $\mathcal{L}_-$ characterizes time-reversal channels.

5. The support constraint on k_3 has implications for the other narrowband characterizations. We derive the implications for $S(\theta, \tau)$ and conclude that,

Theorem 2.

Causal, time-forward narrowband channels must also be time-invariant.

Proof. The mapping from k_3 to S is (2.26), which we repeat here,

$$S(\theta, \tau) = \int k_3(\theta + \nu, \nu) e^{j2\pi\tau\nu} d\nu. \tag{3.65}$$

For k_3 with support only in Q1, Q3, and the origin, let us consider $S(\theta, \tau)$ as a function of τ. For a given θ, say $\theta = \theta_0$, then $S(\theta_0, \tau)$ is the inverse Fourier transform of k_3 along the line $\nu = \theta - \theta_0$, depicted in Figure 3.6. For time-forward $S(\theta, \tau)$

$$\operatorname{supp} \mathcal{F}\{S(\theta_0, \tau)\}(\nu) \subset \left\{ \nu : \begin{array}{ll} \theta_0 > 0, & \nu < -\theta_0 \text{ or } \nu > 0 \\ \theta_0 < 0, & \nu < 0 \text{ or } \nu > \theta_0 \end{array} \right\} \tag{3.66}$$

If we also want $S(\theta, \tau)$ to be causal, we require (from Table 3.2),

$$S(\theta_0, \tau) = 0, \ \forall \tau < 0. \tag{3.67}$$

However, as Wiener eloquently stated ([Wie49], page 37)[2]:

> No [causal] filter can have infinite attenuation in any finite [nonzero] band. The perfect filter is physically unrealizable by its nature, not merely because of the paucity of means at our disposal. No instrument acting solely on the past has a sufficiently sharp discrimination to separate one [band of frequencies] from another [band of frequencies] with unfailing accuracy.

This quote is an application of the Paley-Weiner theorem.

$$h(t) = 0, \ \forall t < 0 \text{ and } h(t) \in L^2 \longrightarrow \int_{-\infty}^{\infty} \frac{|\ln |H(\theta)||}{1 + \theta^2} d\theta < \infty. \tag{3.68}$$

Using the theorem, we conclude,

$$S(\theta_0, \tau) = 0, \ \forall \theta_0 \neq 0. \tag{3.69}$$

which means that the support of $S(\theta, \tau)$ is on the $\theta = 0$ line, and thus, $S(\theta, \tau)$ must be time-invariant. □

We will see in Chapter 4 that the constraint that the channel is time-forward

[2] Paraphrasing by [Sie86], see page 476.

is necessary for channels which have an equivalent lowpass characterization. an important property for practical implementations of communication receivers. Is the result that causal. time-forward. time-varying narrowband channels do not exist is essentially a restatement of the result in Theorem (1) that there exist no time-varying causal wideband channels? The answer is not clear at this point and further study is needed.

Often results derived from the Paley-Weiner theorem have little practical importance. The fact that perfect bandstop filters are not physically realizable (a conclusion drawn from the Paley-Weiner theorem) has limited practical importance because filters can in fact be designed whose performance is arbitrarily close to that of an ideal filter: 'Infinite attenuation' is not necessary. In fact. the common bandlimited signal assumption itself leads to signals which are necessarily time-infinite. Again. 'bandlimited' is not necessary: 'essentially' bandlimited will suffice. In these examples. mathematical idealizations lead to extreme conclusions which. when the idealizations are relaxed. fade away. Perhaps such is the case here.

6. Causality and time-reversal.

Just because a channel is causal does not mean the channel cannot be a time-reversal channel. Consider the following channel,

$$k_0(t,\tau) = \begin{cases} \delta(t+\tau) & : \quad t > 0 \\ 0 & : \quad \text{otherwise} \end{cases} \tag{3.70}$$

which has output,

$$y(t) = \begin{cases} x(-t) & : \quad t > 0 \\ 0 & : \quad \text{otherwise} \end{cases} \tag{3.71}$$

The channel is causal. but is a time-reversal channel as well.

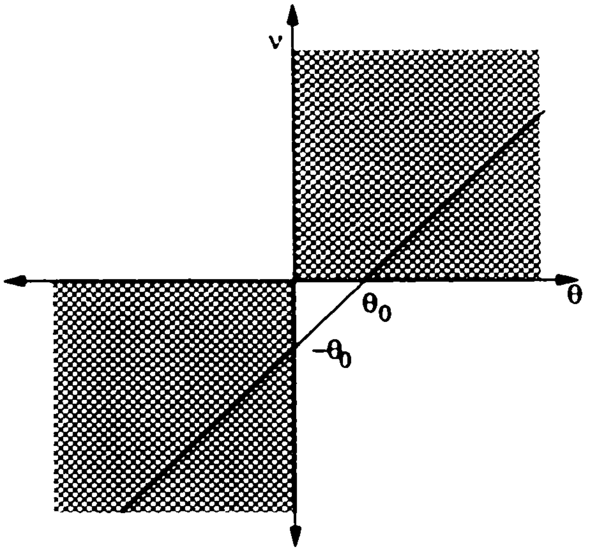


Figure 3.6: The crosshatch region is the allowable support region for time-forward $k_3(\theta, \nu)$. The inverse Fourier transform along the line $\nu = \theta - \theta_0$ yields $S(\theta_0, \tau)$.

7. We show in Chapter 4 that (3.58) is also the necessary and sufficient condition for the existence of a low-pass equivalent representation of the channel.

3.4.3 $\mathcal{L} \longrightarrow$ eight remaining narrowband characterizations

Returning to the mapping from $\mathcal{L}(a, b)$ to the narrowband characterizations, starting from (3.45), the remaining system functions can be related to $\mathcal{L}(a, b)$ as follows,

$$h(t, \tau) = \int \sqrt{|a|}\mathcal{L}(a, (1-a)t + a\tau)\mathrm{d}a \tag{3.72}$$

$$g(t, \tau) = \int \sqrt{|a|}\mathcal{L}(a, t + (1-a)\tau)\mathrm{d}a \tag{3.73}$$

$$M(\theta, \tau) = \int \sqrt{|a|}\mathcal{L}^{(2)}(a, \theta)e^{j2\pi(1-a)\tau\theta}\mathrm{d}a \tag{3.74}$$

$$T(t, \nu) = \int \frac{1}{\sqrt{|a|}}\mathcal{L}^{(2)}(a, \nu/a)e^{j2\pi(\frac{1}{a}-1)t\nu}\mathrm{d}a \tag{3.75}$$

$$H(\theta, \nu) = \begin{cases} \sqrt{\frac{|\nu|}{|\theta+\nu|^3}}\mathcal{L}^{(2)}\left(\frac{\nu}{\theta+\nu}, \theta+\nu\right) & : \theta+\nu \neq 0 \\ \delta(\nu)\iint \sqrt{|a|}\mathcal{L}(a, b)\mathrm{d}a\mathrm{d}b & : \theta+\nu = 0 \end{cases} \tag{3.76}$$

$$G(\theta, \nu) = \begin{cases} \sqrt{\frac{|\theta-\nu|}{|\theta|^3}}\mathcal{L}^{(2)}\left(\frac{\theta-\nu}{\theta}, \theta\right) & : \theta \neq 0 \\ \delta(\theta-\nu)\iint \sqrt{|a|}\mathcal{L}(a, b)\mathrm{d}a\mathrm{d}b & : \theta = 0 \end{cases} \tag{3.77}$$

$$V(t, \nu) = \int \frac{1}{\sqrt{|a|}}\mathcal{L}^{(2)}(a, \nu/a)e^{j2\pi\nu t/a}\mathrm{d}a \tag{3.78}$$

and, taking the Fourier transform of (3.72) with respect to t, we obtain,

$$\boxed{S(\theta, \tau) = \iint \sqrt{|a|}\mathcal{L}(a, (1-a)t + a\tau)e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t} \tag{3.79}$$

Note, when $\mathcal{L}(a, b)$ is time invariant (i.e., $\mathcal{L}(a, b) = \delta(a-1)\mathcal{L}_1(b)$), the mapping in (3.79) results in,

$$S(\theta, \tau) = \mathcal{L}_1(\tau)\delta(\theta) \tag{3.80}$$

which is the time-invariant form of $S(\theta, \tau)$.

To express $S(\theta, \tau)$ directly in terms of $\mathcal{L}(a, b)$, we divide the integral into two parts. one containing the region around $a = 1$ the other for all other a.

$$\begin{aligned} S(\theta,\tau) &= \iint \left(1 - e^{-\alpha(a-1)^2} + e^{-\alpha(a-1)^2}\right) \sqrt{|a|}\mathcal{L}(a,(1-a)t + a\tau)e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t \quad (3.81a) \\ &= \lim_{\alpha\to\infty} \left(\iint e^{-\alpha(a-1)^2}\sqrt{|a|}\mathcal{L}(a,(1-a)t + a\tau)e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t \right. \\ &\quad \left. + \iint \left(1 - e^{-\alpha(a-1)^2}\right)\sqrt{|a|}\mathcal{L}(a,(1-a)t + a\tau)e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t\right) \quad (3.81b) \\ &= \mathcal{L}_1(\tau)\delta(\theta) + \iint \sqrt{|a|}\bar{\mathcal{L}}(a,(1-a)t + a\tau)e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t \quad (3.81c) \end{aligned}$$

We then perform the coordinate transform $b = (1-a)t + a\tau$ and obtain.

$$\boxed{S(\theta,\tau) = \mathcal{L}_1(\tau)\delta(\theta) + \iint \frac{\sqrt{|a|}}{|1-a|}\bar{\mathcal{L}}(a,b)e^{-j2\pi\theta\frac{b-a\tau}{1-a}}\mathrm{d}a\mathrm{d}b} \quad (3.82)$$

Here. we see the time-invariant/time varying split. From our discussion in Section 3.1. the first term in (3.82) contains the time-invariant components of the channel. and thus second term in (3.82) contain the time-varying components.

3.5 Narrowband to Wideband

In the previous section. we characterized the mapping from the wideband representation to the dozen narrowband representations. The natural question arises, what is $\mathcal{L}(a,b)$ in terms of $S(\theta,\tau)$ and the other system functions?

3.5.1 $S \longrightarrow \mathcal{L}$

We use (3.49). which we repeat here. as a starting point.

$$k_3(\theta,\nu) = \begin{cases} \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}^{(2)}\left(\frac{\nu}{\theta},\theta\right) & : \theta \neq 0 \\ \delta(\nu)\iint \sqrt{|a|}\mathcal{L}(a,b)\mathrm{d}a\mathrm{d}b & : \theta = 0 \end{cases} \quad (3.83)$$

Note that, for $k_3(\theta, \nu)$, the $\theta \neq 0$ and $\theta = 0$ cases correspond to the AC and DC output components of the transfer function. First we consider first the DC ($\theta = 0$) case.

As discussed in the first remark in Section 3.4.2, in order for there to be a mapping between k_3 and $\mathcal{L}$, the support of k_3 cannot contain any points or segments on the θ or ν axis except $\theta = \nu = 0$. The $(\theta, \nu) = (0, 0)$ point controls the DC input to DC output mapping. The signals that we are interested in, wireless communication signals, however, have negligible DC energy. Clearly, narrowband signals have their energy concentrated in a small frequency range surrounding the carrier frequency. Even impulse radio signals, also known as 'ultra-wide bandwidth' signals, must have no DC component due to antenna principles[Hus02]. Thus, we assume,

$$X(0) = 0 \tag{3.84}$$

for our signals of interest, and we ignore the $\theta = 0$ case, as it has no effect on the input-output relationship.

Considering the $\theta \neq 0$ case,

$$k_3(\theta, \nu) = \sqrt{\left|\frac{\nu}{\theta^3}\right|}\mathcal{L}^{(2)}\left(\frac{\nu}{\theta}, \theta\right) \tag{3.85}$$

which becomes ($a = \frac{\nu}{\theta}$),

$$\boxed{\mathcal{L}^{(2)}(a, \theta) = \frac{|\theta|}{\sqrt{|a|}} k_3(\theta, a\theta)} \tag{3.86}$$

which is,

$$\int \mathcal{L}(a, b) e^{-j2\pi\theta b} db = \frac{|\theta|}{\sqrt{|a|}} k_3(\theta, a\theta) \tag{3.87}$$

and we invert and obtain

$$\boxed{\mathcal{L}(a,b) = \int \frac{|\theta|}{\sqrt{|a|}} k_3(\theta, a\theta) e^{j2\pi\theta b} \mathrm{d}\theta} \tag{3.88}$$

which maps one of the narrowband characterizations to the wideband characterization under the assumption that the input signal has a negligible DC component.

3.5.2 Remaining narrowband characterizations $\longrightarrow \mathcal{L}$

For $S(\theta,\tau)$, ignoring the DC component as before, starting with (3.88) and using (2.22) to map k_3 to H and then (2.16) to map H to S,

$$\begin{aligned}
\mathcal{L}(a,b) &= \int \frac{|\theta|}{\sqrt{|a|}} k_3(\theta, a\theta) e^{j2\pi\theta b} \mathrm{d}\theta && \text{(3.89a)} \\
&= \int \frac{|\theta|}{\sqrt{|a|}} H((1-a)\theta, a\theta) e^{j2\pi\theta b} \mathrm{d}\theta && \text{(3.89b)} \\
&= \iint \frac{|\theta|}{\sqrt{|a|}} S((1-a)\theta, \tau) e^{-j2\pi\tau a\theta} e^{j2\pi\theta b} \mathrm{d}\theta \mathrm{d}\tau && \text{(3.89c)}
\end{aligned}$$

and finally,

$$\boxed{\mathcal{L}(a,b) = \iint \frac{|\theta|}{\sqrt{|a|}} S((1-a)\theta, \tau) e^{j2\pi\theta(b-a\tau)} \mathrm{d}\theta \mathrm{d}\tau} \tag{3.90}$$

To check, we examine (3.90) for time-invariant $S(\theta,\tau) = \delta(\theta) S_0(\tau)$,

$$\begin{aligned}
\mathcal{L}(a,b) &= \iint \frac{|\theta|}{\sqrt{|a|}} S_0(\tau) \delta((1-a)\theta) e^{j2\pi\theta(b-a\tau)} \mathrm{d}\theta \mathrm{d}\tau && \text{(3.91a)} \\
&= \delta(1-a) \iint S_0(\tau) e^{j2\pi\theta(b-\tau)} \mathrm{d}\theta \mathrm{d}\tau && \text{(3.91b)} \\
&= \delta(1-a) \int S_0(\tau) \delta(b-\tau) \mathrm{d}\tau && \text{(3.91c)} \\
&= \delta(1-a) S_0(b) && \text{(3.91d)}
\end{aligned}$$

and obtain the time-invariant form of $\mathcal{L}(a,b)$.

When we expressed $S(\theta,\tau)$ directly in terms of $\mathcal{L}(a,b)$, moving from (3.79) to

(3.82). it was natural to split the characterization into time-invariant and time varying components. A similar split arises when performing a coordinate transform to express $\mathcal{L}(a, b)$ directly in terms of $S(\theta, \tau)$. Transforming $(1-a)\theta \rightarrow \theta$.

$$\boxed{\mathcal{L}(a, b) = \delta(a-1)S_0(b) + \left\{ \begin{array}{lcr} \iint \frac{|\tilde{\theta}|}{\sqrt{|a|}(1-a)^2}\tilde{S}(\theta, \tau)e^{j2\pi\theta\frac{b-a\tau}{1-a}}\mathrm{d}\theta\mathrm{d}\tau & : & a \neq 1 \\ 0 & : & a = 1 \end{array} \right\}} \tag{3.92}$$

What are the form of the other system functions? The impulse response functions. $k_0(t, \tau)$. $h(t, \tau)$. $g(t, \tau)$. can be related starting from (3.88) and using the relations described by (2.4) and in (2.19).

$$\begin{aligned} \mathcal{L}(a, b) &= \int \frac{|\theta|}{\sqrt{|a|}} k_3(\theta, a\theta)e^{j2\pi\theta b}\mathrm{d}\theta && (3.93a) \\ &= \iiint \frac{|\theta|}{\sqrt{|a|}} k_0(t, \tau)e^{-j2\pi\theta(t-a\tau-b)}\mathrm{d}t\mathrm{d}\tau\mathrm{d}\theta && (3.93b) \\ &= \iiint \frac{|\theta|}{\sqrt{|a|}} h(t, \tau)e^{-j2\pi\theta((1-a)t+a\tau-b)}\mathrm{d}t\mathrm{d}\tau\mathrm{d}\theta && (3.93c) \\ &= \iiint \frac{|\theta|}{\sqrt{|a|}} g(t, \tau)e^{-j2\pi\theta(t+(1-a)\tau-b)}\mathrm{d}t\mathrm{d}\tau\mathrm{d}\theta && (3.93d) \end{aligned}$$

Further simplification of these equations by swapping the order of integration is not possible in this case as.

$$\iiint \frac{|\theta|}{\sqrt{|a|}} |k_0(t, \tau)|\mathrm{d}t\mathrm{d}\tau\mathrm{d}\theta = \infty \tag{3.94}$$

Using (3.88) and the relationships between system functions, the remaining six system functions can be related.

$$
\begin{aligned}
\mathcal{L}(a,b) &= \iint \frac{|\theta|}{\sqrt{|a|}} k_1(\theta,\tau) e^{j2\pi\theta(b+a\tau)} \mathrm{d}\theta \mathrm{d}\tau && \text{(3.95a)} \\
&= \iint \frac{|\theta|}{\sqrt{|a|}} M(\theta,\tau) e^{j2\pi\theta(b+(a-1)\tau)} \mathrm{d}\theta \mathrm{d}\tau && \text{(3.95b)} \\
&= \int \frac{|\theta|}{\sqrt{|a|}} G(\theta,(1-a)\theta) e^{j2\pi\theta b} \mathrm{d}\theta && \text{(3.95c)} \\
&= \iint \frac{|\theta|}{\sqrt{|a|}} V(t,(1-a)\theta) e^{j2\pi\theta(b-t)} \mathrm{d}t \mathrm{d}\theta && \text{(3.95d)} \\
&= \iint \frac{|\theta|}{\sqrt{|a|}} T(t,a\theta) e^{j2\pi\theta(b-t(1-a))} \mathrm{d}t \mathrm{d}\theta && \text{(3.95e)} \\
&= \iint \frac{|\theta|}{\sqrt{|a|}} k_2(t,a\theta) e^{j2\pi\theta(b-t)} \mathrm{d}t \mathrm{d}\theta && \text{(3.95f)}
\end{aligned}
$$

3.6 Simple Narrowband-Wideband Correspondence

We look to some simple channel models and examine the mappings (3.82) and (3.92).

3.6.1 Wideband (delay-dilation) single path

$$\mathcal{L}(a,b) = \delta(a-a_0)\delta(b-b_0) \tag{3.96}$$

It follows from (3.82) that,

$$S(\theta,\tau) = \begin{cases} \frac{\sqrt{|a_0|}}{|1-a_0|} e^{-j2\pi\theta\frac{b_0-a_0\tau}{1-a_0}} & : \; a_0 \neq 1 \\ \delta(\theta)\delta(\tau-b_0) & : \; a_0 = 1 \end{cases} \tag{3.97}$$

and, plugging this into,

$$y(t) = \iint S(\theta,\tau) x(t-\tau) e^{j2\pi\theta t} \mathrm{d}\tau \mathrm{d}\theta. \tag{3.98}$$

we indeed obtain,

$$y(t) = \frac{1}{\sqrt{|a_0|}} x\left(\frac{t-b_0}{a_0}\right) \tag{3.99}$$

We can derive the time-varying impulse response characterization $h(t,\tau)$ for the wideband single path channel,

$$\begin{aligned} h(t,\tau) &= \int S(\theta,\tau) e^{j2\pi\theta t} d\theta & (3.100a) \\ &= \int \frac{\sqrt{|a_0|}}{|1-a_0|} e^{-j2\pi\theta\frac{b_0-a_0\tau}{1-a_0}} e^{j2\pi\theta t} d\theta & (3.100b) \\ &= \frac{\sqrt{|a_0|}}{|1-a_0|}\delta\left(\frac{b_0-a_0\tau-(1-a_0)t}{1-a_0}\right) & (3.100c) \\ &= \sqrt{|a_0|}\delta(b_0-a_0\tau-(1-a_0)t) & (3.100d) \end{aligned}$$

which is also valid when $a_0 = 1$. We can compare this result to that of the single narrowband path (delay by τ_0, Doppler shift by θ_0) channel, $h(t,\tau) = \delta(\tau-\tau_0)e^{j2\pi t\theta_0}$. The wideband path gives rise to a delta function line with slope $\frac{a_0-1}{a_0}$ intersecting the τ-axis at b_0/a_0; The narrowband path gives rise to a modulated delta function line parallel to the t-axis intersecting the τ-axis at τ_0.

3.6.2 Narrowband (delay-Doppler) single path

$$S(\theta,\tau) = \delta(\theta-\theta_0)\delta(\tau-\tau_0) \tag{3.101}$$

It follows that,

$$\mathcal{L}(a,b) = \begin{cases} \frac{|\theta_0|}{\sqrt{|a|(1-a)^2}} e^{j2\pi\theta_0\frac{b-a\tau_0}{1-a}} & : a \neq 1 \\ \delta(\theta_0)\delta(b-\tau_0) & : a = 1 \end{cases} \tag{3.102}$$

and, plugging this into,

$$y(t) = \iint \mathcal{L}(a,b)\frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) da db. \tag{3.103}$$

	one path delay only	one path delay-Doppler, $\theta_0 \neq 0$	one path delay-dilation, $a_0 \neq 1$
$S(\theta, \tau)$	$\delta(\theta)\delta(\tau - t_0)$	$\boldsymbol{\delta(\theta - \theta_0)\delta(\tau - t_0)}$	$\frac{\sqrt{\lvert a_0\rvert}}{\lvert 1-a_0\rvert} e^{-j2\pi\theta\frac{t_0 - a_0\tau}{1-a_0}}$
$\mathcal{L}(a, b)$	$\delta(a-1)\delta(b - t_0)$	$\begin{cases} \frac{\lvert\theta_0\rvert}{\sqrt{\lvert a\rvert(1-a)^2}} e^{j2\pi\theta_0\frac{b-at_0}{1-a}} & : \quad a \neq 1 \\ 0 & : \quad a = 1 \end{cases}$	$\boldsymbol{\delta(a - a_0)\delta(b - t_0)}$
$h(t, \tau)$	$\delta(\tau - t_0)$	$\delta(\tau - t_0)e^{j2\pi t\theta_0}$	$\sqrt{\lvert a_0\rvert}\delta((1-a_0)t + a_0\tau - t_0)$
$k_3(\theta, \nu)$	$\delta(\theta - \nu)e^{-j2\pi t_0\nu}$	$\delta(\theta - \nu - \theta_0)e^{-j2\pi t_0\nu}$	$\sqrt{\lvert a_0\rvert}e^{-j2\pi\theta t_0}\delta(\nu - a_0\theta)$

Table 3.4: Time-frequency and time-scale characterization functions for the one path delay-Doppler and one path delay-dilation channels.

we indeed obtain,

$$y(t) = x(t - \tau_0)e^{j2\pi\theta_0 t} \tag{3.104}$$

3.6.3 The narrowband assumption

The form of the channel characterizations for the simple one-path models (including the time-invariant one path model) are displayed in Table 3.4. Neither (3.97) nor (3.102) are well-localized despite the fact their corresponding generating characterizations are localized. This is an embodiment of the narrowband assumption. We see that the time-frequency spreading function description of the one path delay-dilation channel requires infinite support in time and frequency. Interestingly, the one path delay-Doppler channel gives require infinite support in time and scale for the corresponding wideband description. However, we also see that this mapping is not stable around the $a = 1$ line.

3.7 Main Results and Discussion

The main result of this section is the correspondence,

Theorem 3.

$$S(\theta,\tau) = \iint \sqrt{|a|}\mathcal{L}(a,(1-a)t+a\tau)e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t \tag{3.105}$$

$$\mathcal{L}(a,b) = \iint \frac{|\theta|}{\sqrt{|a|}}S((1-a)\theta,\tau)e^{j2\pi\theta(b-a\tau)}\mathrm{d}\theta\mathrm{d}\tau \tag{3.106}$$

Proof. Even though we have already derived (3.105) and (3.106), we check here that they do indeed invert each other. Plugging (3.106) into (3.105),

$$S(\theta,\tau) = \iiiint \sqrt{|a|}\frac{|\theta'|}{\sqrt{|a|}}S((1-a)\theta',\tau')e^{j2\pi\theta'((1-a)t+a\tau-a\tau')}e^{-j2\pi\theta t}\mathrm{d}a\mathrm{d}t\mathrm{d}\theta'\mathrm{d}\tau' \tag{3.107a}$$

$$= \iiint |\theta'|S((1-a)\theta',\tau')e^{j2\pi\theta' a(\tau-\tau')}\delta(\theta'(1-a)-\theta)\mathrm{d}a\mathrm{d}\theta'\mathrm{d}\tau' \tag{3.107b}$$

$$= \iiint S((1-a)\theta',\tau')e^{j2\pi\theta' a(\tau-\tau')}\delta\left((1-a)-\frac{\theta}{\theta'}\right)\mathrm{d}a\mathrm{d}\theta'\mathrm{d}\tau' \tag{3.107c}$$

$$= \iint S(\theta,\tau')e^{j2\pi(\theta'-\theta)(\tau-\tau')}\mathrm{d}\theta'\mathrm{d}\tau' \tag{3.107d}$$

$$= \int S(\theta,\tau')e^{-j2\pi\theta(\tau-\tau')}\delta(\tau-\tau')\mathrm{d}\tau' \tag{3.107e}$$

$$= S(\theta,\tau) \tag{3.107f}$$

In moving from (3.107b) to (3.107c), we divided by $|\theta'|$, which appears in the first argument of S, and then integrated over a. Alternatively, proceeding from (3.107b), we could have divided by $|1-a|$, which also appears in the first argument of S, and then integrated over θ', in which case,

$$S(\theta,\tau) = \iiint \frac{|\theta'|}{|1-a|}S((1-a)\theta',\tau')e^{j2\pi\theta' a(\tau-\tau')}\delta\left(\theta'-\frac{\theta}{1-a}\right)\mathrm{d}a\mathrm{d}\theta'\mathrm{d}\tau' \tag{3.108a}$$

$$= \iint \frac{|\theta|}{|1-a|^2}S(\theta,\tau')e^{j2\pi\theta\frac{a}{1-a}(\tau-\tau')}\mathrm{d}a\mathrm{d}\tau' \tag{3.108b}$$

$$= \iint |\theta|S(\theta,\tau')e^{j2\pi\theta b(\tau-\tau')}\mathrm{d}b\mathrm{d}\tau' \tag{3.108c}$$

$$= \int |\theta|S(\theta,\tau')\delta(\theta(\tau-\tau'))\mathrm{d}\tau' \tag{3.108d}$$

$$= S(\theta,\tau) \tag{3.108e}$$

We point out the division by $|\theta'|$ in the first derivation and $|1-a|$ (and note θ later) in the second derivation because this division excludes the time-invariant components from the derivation.

Plugging (3.105) into (3.106).

$$
\begin{aligned}
\mathcal{L}(a,b) &= \iiiint \frac{|\theta|}{\sqrt{|a|}} \sqrt{|a'|}\mathcal{L}(a',(1-a')b'+a'\tau)e^{-j2\pi(1-a)\theta b'}e^{j2\pi\theta(b-a\tau)}\mathrm{d}\theta\mathrm{d}\tau\mathrm{d}a'\mathrm{d}b' && (3.109a)\\
&= \iiiint \frac{|\theta|}{\sqrt{|a|}}\frac{\sqrt{|a'|}}{|a'|}\mathcal{L}(a',\tau')e^{-j2\pi(1-a)\theta b'}e^{j2\pi\theta\left(b-a\left(\frac{\tau'-(1-a')b'}{a'}\right)\right)}\mathrm{d}\theta\mathrm{d}\tau'\mathrm{d}a'\mathrm{d}b' && (3.109b)\\
&= \iiint \frac{|\theta|}{\sqrt{|a|}}\frac{\sqrt{|a'|}}{|a'|}\mathcal{L}(a',\tau')\delta(\theta(\frac{a}{a'}-1))e^{j2\pi\theta\left(b-a\left(\frac{\tau'}{a'}\right)\right)}\mathrm{d}\theta\mathrm{d}\tau'\mathrm{d}a' && (3.109c)\\
&= \iiint \frac{1}{\sqrt{|a|}}\frac{\sqrt{|a'|}}{1}\mathcal{L}(a',\tau')\delta\left(a-a'\right)e^{j2\pi\theta\left(b-a\left(\frac{\tau'}{a'}\right)\right)}\mathrm{d}\theta\mathrm{d}\tau'\mathrm{d}a' && (3.109d)\\
&= \iint \mathcal{L}(a,\tau')e^{j2\pi\theta(b-\tau')}\mathrm{d}\theta\mathrm{d}\tau' && (3.109e)\\
&= \int \mathcal{L}(a,\tau')\delta(b-\tau')\mathrm{d}\tau' && (3.109f)\\
&= \mathcal{L}(a,b) && (3.109g)
\end{aligned}
$$

In the above derivation we have divided by a' and θ. The a' division does not matter as $a' > 0$. however the θ division forces us to only consider the time varying components. Thus. (3.105) and (3.106) map the time varying components of the narrowband and wideband channel characterizations to one another. We have seen. however. that the time-invariant component is also correctly mapped. recall (3.80) and (3.91a)-(3.91d). therefore. (3.105) and (3.106) map S to $\mathcal{L}$. □

We break up (3.105) and (3.106) into the time-invariant and time-varying components and express them as.

$$
S(\theta,\tau) = \mathcal{L}_1(\tau)\delta(\theta) + \iint \frac{\sqrt{|a|}}{|1-a|}\tilde{\mathcal{L}}(a,b)e^{-j2\pi\theta\frac{b-a\tau}{1-a}}\mathrm{d}a\mathrm{d}b \qquad (3.110)
$$

$$
\mathcal{L}(a,b) = \delta(a-1)S_0(b) + \left\{ \begin{array}{ll} \iint \frac{|\theta|}{\sqrt{|a|}(1-a)^2}\tilde{S}(\theta,\tau)e^{j2\pi\theta\frac{b-a\tau}{1-a}}\mathrm{d}\theta\mathrm{d}\tau & : \quad a \neq 1 \\ 0 & : \quad a = 1 \end{array} \right\} \qquad (3.111)
$$

which emphasizes the instability inherent in the mapping around the important $a = 1$

case. The channels of interest have support close to and including the $a = 1$ line. But we see here that precisely in this region the time-varying component mapping is not stable. We conclude that the time-frequency kernel channel description and the time-scale kernel channel description models are not-equivalent precisely for the channels that we are interested in modeling. Thus, there may be performance gains associated with developing a model based on the time-scale description, as it is based on a physical interpretation of the channel.

Chapter 4

Characterization of Communication Signals

In this chapter, we show the standard derivation for the equivalent lowpass time-invariant channel characterization, and then we consider the time-varying equivalent lowpass characterization. The result is somewhat surprising in that the low pass equivalent characterization only exists if the channel is time-forward.

4.1 Time-invariant case

This section contains material found in most general signal processing textbooks and roughly follows the relevant section in [Pro84].

A signal $x(t)$ can be represented in the frequency domain as,

$$X(\theta) = \int x(t)e^{-j2\pi t\theta}\mathrm{d}t, \tag{4.1}$$

which can be represented in terms of the amplitude ($|X(\theta)|$) and phase ($\phi(\theta)$) of $x(t)$,

$$X(\theta) = |X(\theta)|e^{j\phi(\theta)} \tag{4.2}$$

If $x(t)$ is real (which we assume to be true),

$$X(\theta) = \overline{X(-\theta)}. \tag{4.3}$$

So, for real-valued signals, the positive and negative frequencies contain redundant information. Thus, an equivalent characterization of real-valued $x(t)$, called the *analytic signal* (introduced in [Gab46]), is,

$$X_+(\theta) = 2u(\theta)X(\theta), \tag{4.4}$$

where $u(\theta)$ is the unit step function. In the time domain, $X_+(\theta)$ can be expressed,

$$x_+(t) = \mathcal{F}^{-1}\{X_+(\theta)\} = x(t) + j\tilde{x}(t), \tag{4.5}$$

where, $\tilde{x}$ is the Hilbert transform of $x(t)$,

$$\tilde{x}(t) = \mathcal{F}^{-1}\{\mathcal{H}(\theta)X(\theta)\} \tag{4.6}$$

where,

$$\mathcal{H}(\theta) = \begin{cases} -j & : \quad \theta > 0 \\ 0 & : \quad \theta = 0 \\ j & : \quad \theta < 0 \end{cases} \tag{4.7}$$

The modulus of the analytic signal, $|x_+(t)|$, is called the *envelope of* $x(t)$. We can shift $X_+(\theta)$ in down in frequency so that it is centered on $\theta = 0$ and define the *equivalent lowpass signal*,

$$\boxed{X_l(\theta) = X_+(\theta + \theta_c) = 2u(\theta + \theta_c)X(\theta + \theta_c)} \tag{4.8}$$

where θ_c is the carrier frequency for bandpass signals[1]. In the time domain.

$$x_l(t) = x_+(t)e^{-j2\pi\theta_c t}. \tag{4.9}$$

The equivalent lowpass signal, also known as the *complex envelope*, can be expressed in terms of the inphase ($x_i(t)$) and quadrature ($x_q(t)$) components.

$$x_l(t) = x_i(t) + jx_q(t) \tag{4.10}$$

Using (4.5) and (4.9), we see that,

$$x(t) + j\hat{x}(t) = e^{j2\pi\theta_c t}(x_i(t) + jx_q(t)). \tag{4.11}$$

Examining the real and imaginary parts, we can see the direct relationship between $x(t)$ and the inphase/quadrature components.

$$x(t) = x_i(t)\cos(2\pi\theta_c t) - x_q(t)\sin(2\pi\theta_c t) \tag{4.12}$$

$$\hat{x}(t) = x_i(t)\sin(2\pi\theta_c t) + x_q(t)\cos(2\pi\theta_c t) \tag{4.13}$$

Clearly from (4.11),

$$x(t) = \text{Re}\{(x_i(t) + jx_q(t))e^{j2\pi\theta_c t}\}. \tag{4.14}$$

and, using (4.10), we see that,

$$\boxed{x(t) = \text{Re}\{x_l(t)e^{j2\pi\theta_c t}\}} \tag{4.15}$$

[1] While it is customary to choose θ_c to be the carrier frequency, all of the results in this section still apply for arbitrary $\theta_c > 0$. Indeed, we can also shift real signals which were not originally bandpass assuming they have a negligible DC component. If the signal is lowpass, then it is reasonable (although not necessary) to choose θ_c to be half the highest frequency. One advantage of choosing θ_c to be the carrier frequency (or half the highest frequency for lowpass signals), is that then the direct application of the Sampling Theorem results in the smallest (i.e., most efficient) sampling rate capable of characterizing the signal.

In the frequency domain, this relationship can be expressed,

$$X(\theta) = \frac{1}{2}[\overline{X_l(-\theta-\theta_c)} + X_l(\theta-\theta_c)] \tag{4.16}$$

Now we consider the characterization of the channel. We consider here the time-invariant linear channel: In the next section we discuss the extension to the time-varying case. The input-output relationship in the time invariant case is described by,

$$y(t) = \int h(t-\tau)x(\tau)d\tau \tag{4.17}$$

We assume $h(t)$ is real, and thus, $H(\theta) = \overline{H(-\theta)}$, and so we define the lowpass channel,

$$\boxed{H_l(\theta-\theta_c) = \begin{cases} H(\theta) & : \quad \theta > 0 \\ 0 & : \quad \theta \leq 0 \end{cases}} \tag{4.18}$$

which we can state equivalently,

$$\boxed{H_l(\theta) = u(\theta+\theta_c)H(\theta+\theta_c)} \tag{4.19}$$

It follows from $h(t)$ being real-valued that,

$$\overline{H_l(-\theta-\theta_c)} = \begin{cases} \overline{H(-\theta)} & : \quad \theta < 0 \\ 0 & : \quad \theta \geq 0 \end{cases} . \tag{4.20}$$

and thus,

$$H(\theta) = \overline{H_l(-\theta-\theta_c)} + H_l(\theta-\theta_c) \tag{4.21}$$

$$h(t) = \overline{h_l(t)}e^{-j2\pi\theta_c t} + h_l(t)e^{j2\pi\theta_c t} \tag{4.22}$$

and thus,

$$\boxed{h(t) = 2\text{Re}\{h_l(t)e^{j2\pi\theta_c t}\}} \tag{4.23}$$

We can write (4.17) in the frequency domain and expand using (4.16) and (4.21).

$$\begin{aligned} Y(\theta) &= H(\theta)X(\theta) & (4.24a) \\ &= \frac{1}{2}[(\overline{H_l(-\theta-\theta_c)} + H_l(\theta-\theta_c))(\overline{X_l(-\theta-\theta_c)} + X_l(\theta-\theta_c))] & (4.24b) \\ &= \frac{1}{2}[\overline{H_l(-\theta-\theta_c)X_l(-\theta-\theta_c)} + H_l(\theta-\theta_c)X_l(\theta-\theta_c)] & (4.24c) \end{aligned}$$

In moving from (4.24b) to (4.24c), we used the fact that,

$$\underbrace{H_l(\theta-\theta_c)}_{=0 \text{ if } \theta\leq 0}\underbrace{\overline{X_l(-\theta-\theta_c)}}_{=0 \text{ if } \theta\geq 0} = 0, \quad \forall\nu \tag{4.25}$$

and

$$\underbrace{\overline{H_l(-\theta-\theta_c)}}_{=0 \text{ if } \theta\geq 0}\underbrace{X_l(\theta-\theta_c)}_{=0 \text{ if } \theta\leq 0} = 0, \quad \forall\nu \tag{4.26}$$

which is true by definition, because,

$$X_l(\theta) = 0 \quad : \quad \theta < -\theta_c \tag{4.27}$$

and

$$H_l(\theta) = 0 \quad : \quad \theta < -\theta_c. \tag{4.28}$$

Defining,

$$\boxed{Y_l(\theta) = H_l(\theta)X_l(\theta)} \tag{4.29}$$

we obtain from (4.24c),

$$\boxed{Y(\theta) = \frac{1}{2}[\overline{Y_l(-\theta-\theta_c)} + Y_l(\theta-\theta_c)]} \tag{4.30}$$

from which we see.

$$\boxed{y(t) = \mathrm{Re}\{y_l(t)e^{j2\pi\theta_c t}\}} \tag{4.31}$$

In summary, we can analyze the linear time-invariant system,

$$Y(\theta) = H(\theta)X(\theta) \tag{4.32}$$

by looking to the equivalent lowpass characterization,

$$Y_l(\theta) = H_l(\theta)X_l(\theta). \tag{4.33}$$

which has the advantages of being carrier frequency independent and efficient to represent (from a sampling perspective).

4.2 Time-varying case

We wish to establish a similar result for the time-varying case. Our goal is to express this input-output relationship,

$$Y(\theta) = \int k_3(\theta,\nu)X(\nu)\mathrm{d}\nu. \tag{4.34}$$

for the equivalent lowpass Y_l and X_l signals,

$$Y_l(\theta) = \int k_3^l(\theta,\nu)X_l(\nu)\mathrm{d}\nu. \tag{4.35}$$

What is the form of $k_3^l(\theta,\nu)$? The result is perhaps surprising in that such an equivalent characterization is possible only when $k_3(\theta,\nu)$ is a time-forward channel. Previously, we saw the time-forward constraint arise from the consideration only of positive scale ($a > 0$) in the wideband channel model. In the time-invariant case, the channel was automatically time-forward due to time-invariance. We discuss these points in Section 4.3.

Theorem 4. *For a time-frequency channel to have an equivalent lowpass characterization, it must be time-forward.*

Proof. From the definition of Y_l,

$$\begin{aligned}
Y(\theta) &= \frac{1}{2}\left[\overline{Y_l(-\theta-\theta_c)} + Y_l(\theta-\theta_c)\right] && (4.36a)\\
&= \frac{1}{2}\left[\int \overline{k_3^l(-\theta-\theta_c,\nu)X_l(\nu)}\mathrm{d}\nu + \int k_3^l(\theta-\theta_c,\nu)X_l(\nu)\mathrm{d}\nu\right] && (4.36b)\\
&= \int \overline{k_3^l(-\theta-\theta_c,\nu)}u(\nu+\theta_c)\overline{X(\nu+\theta_c)}\mathrm{d}\nu + \int k_3^l(\theta-\theta_c,\nu)u(\nu+\theta_c)X(\nu+\theta_c)\mathrm{d}\nu && (4.36c)\\
&= \int \overline{k_3^l(-\theta-\theta_c,\nu)}u(\nu+\theta_c)X(-\nu-\theta_c)\mathrm{d}\nu + \int k_3^l(\theta-\theta_c,\nu)u(\nu+\theta_c)X(\nu+\theta_c)\mathrm{d}\nu && (4.36d)\\
&= \int \overline{k_3^l(-\theta-\theta_c,\nu-\theta_c)}u(\nu)X(-\nu)\mathrm{d}\nu + \int k_3^l(\theta-\theta_c,\nu-\theta_c)u(\nu)X(\nu)\mathrm{d}\nu && (4.36e)\\
&= \int \left[\overline{k_3^l(-\theta-\theta_c,-\nu-\theta_c)}u(-\nu) + k_3^l(\theta-\theta_c,\nu-\theta_c)u(\nu)\right]X(\nu)\mathrm{d}\nu && (4.36f)
\end{aligned}$$

and we obtain,

$$\boxed{k_3(\theta,\nu) = \overline{k_3^l(-\theta-\theta_c,-\nu-\theta_c)}u(-\nu) + k_3^l(\theta-\theta_c,\nu-\theta_c)u(\nu)} \qquad (4.37)$$

From (4.35) and the fact that $Y_l(\theta) = 0,\ \forall\theta < -\theta_c$, we conclude that,

$$k_3^l(\theta,\nu) = 0,\ \forall\theta < \theta_c \qquad (4.38)$$

g and thus,

$$\boxed{k_3(\theta,\nu) = \begin{cases} \overline{k_3^l(-\theta-\theta_c,-\nu-\theta_c)} & : \ \nu < 0,\ \theta < 0 \\ k_3^l(\theta-\theta_c,\nu-\theta_c) & : \ \nu > 0,\ \theta > 0 \\ 0 & : \ \text{otherwise} \end{cases}} \qquad (4.39)$$

So $k_3(\theta,\nu)$ can only be non-zero in Q1 and Q3. □

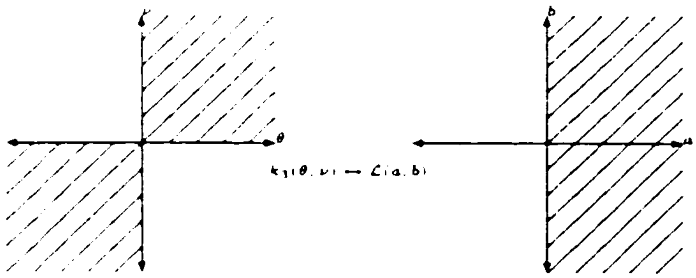


Figure 4.1: For $k_3(\theta, \nu)$ to have an equivalent lowpass characterization, it can only be non-zero zero in Q1 and Q3. This corresponds to $\mathcal{C}(a, b)$ only being non-zero in Q1 and Q4; which can be interpreted as enforcing no time-reversals.

Note that, $k_3(\theta, \nu) = \overline{k_3(-\theta, -\nu)}$, as it should because by definition of the lowpass signals we have forced $x(t)$ and $y(t)$ to be real-valued functions.

Note that from (4.35), $k_3^l(\theta, \nu)$ is never evaluated for $\nu < -\theta_c$, and thus we have the freedom to define $k_3^l(\theta, \nu)$ as we please for $\nu < -\theta_c$. Choosing $k_3^l(\theta, \nu) = 0, \ \forall \nu < -\theta_c$ allows us to define,

$$\boxed{k_3^l(\theta, \nu) = u(\theta + \theta_c)u(\nu + \theta_c)k_3(\theta + \theta_c, \nu + \theta_c)} \tag{4.40}$$

4.3 Remarks

The equivalent lowpass characterization exists only for $k_3(\theta, \nu) = 0, \ \forall \theta\nu < 0$. That is, $k_3(\theta, \nu)$ must be zero in Q2 and Q4. We have seen this constraint before as arising from the support constraint on $\mathcal{C}_+$, see (3.58), and used it as the defining characteristic for time-forward channels.

In the time-invariant case, this constraint did not arise in the derivation of the equivalent lowpass characterization because the constraint must be satisfied due to time-invariance. That is, the time-invariant characterization,

$$y(t) = \int h(t-\tau)x(\tau)\mathrm{d}\tau \tag{4.41}$$

has

$$k_0(t,\tau) = h(t-\tau) \tag{4.42}$$

and thus, in this case,

$$k_3(\theta,\nu) = \iint h(t-\tau)e^{-j2\pi t\theta}e^{j2\pi\tau\nu}\mathrm{d}t\mathrm{d}\tau \tag{4.43a}$$

$$= H(\theta)\delta(\nu-\theta) \tag{4.43b}$$

which lives on the $\nu = \theta$ line, and thus clearly satisfies $k_3(\theta,\nu) = 0$, $\theta\nu < 0$. Thus the time-invariant channel is a time-forward channel.

Chapter 5

Time-Frequency and Time-Scale Canonical Models

5.1 Canonical Time-Frequency Model

We begin with the derivation of the canonical model associated with the standard RAKE receiver. The classic expression of the sampling theorem for a signal $X(\nu)$ with support $(-W/2, W/2)$ is,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{W}\right) \frac{\sin\left(\pi W\left(t - \frac{n}{W}\right)\right)}{\pi W\left(t - \frac{n}{W}\right)}. \tag{5.1}$$

We can obtain an alternative formulation of the sampling theorem by defining, $g(t) = x(\alpha - t)$, which satisfies

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{W}\right) \frac{\sin\left(\pi W\left(t - \frac{n}{W}\right)\right)}{\pi W\left(t - \frac{n}{W}\right)} \tag{5.2}$$

and thus,

$$x(\alpha - t) = \sum_{n=-\infty}^{\infty} x\left(\alpha - \frac{n}{W}\right) \frac{\sin\left(\pi W\left(t - \frac{n}{W}\right)\right)}{\pi W\left(t - \frac{n}{W}\right)} \tag{5.3}$$

With $(\alpha, t) \rightarrow (t, \tau)$, we obtain [Tre71],

$$x(t-\tau) = \sum_{n=-\infty}^{\infty} x\left(t - \frac{n}{W}\right) \frac{\sin\left(\pi W\left(\tau - \frac{n}{W}\right)\right)}{\pi W\left(\tau - \frac{n}{W}\right)}. \tag{5.4}$$

Following [Tre71], substituting (5.4) into the time-varying impulse response channel characterization (2.1), we obtain,

$$\begin{aligned} y(t) &= \int h(t,\tau) x(t-\tau) \mathrm{d}\tau & (5.5a) \\ &= \int h(t,\tau) \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right) \frac{\sin\left(\pi W\left(\tau-\frac{n}{W}\right)\right)}{\pi W\left(\tau-\frac{n}{W}\right)} \mathrm{d}\tau & (5.5b) \\ &= \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right) \underbrace{\left[\int h(t,\tau) \frac{\sin\left(\pi W\left(\tau-\frac{n}{W}\right)\right)}{\pi W\left(\tau-\frac{n}{W}\right)} \mathrm{d}\tau\right]}_{=h_n(t)} & (5.5c) \\ &= \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right) h_n(t) & (5.5d) \\ &\approx \sum_{n=0}^{L:=\lceil T_m/W \rceil} x\left(t-\frac{n}{W}\right) h_n(t) & (5.5e) \end{aligned}$$

where the approximation is made based on the assumption that the channel is causal and has finite multipath spread, T_m. That is, $h(t,\tau) = 0, \forall \tau < 0, \tau > T_m$. Under this assumption, the approximation (5.5e) corresponds to $h_n(t)$ for which the mainlobe of the sinc function overlaps with the support of the time-varying impulse response. The tapped-delay line in (5.5e) forms the basis for the classic RAKE receiver, where each of the $h_n(t)$ are (usually) assumed to be independent.

Now, we move to the Time-Frequency RAKE which was originally derived in [SA99]. Alternative, but similar models are explored in [GT98, TV00, MB02]. The path we take in this derivation is essentially the same as that in [SA99]. We look at only the $(0, T)$ portion of the received waveform, that is, $y(t)1_{(0,T)}(t)$. Starting from

(5.5c), we insert the $(0,T)$ portion assumption and obtain

$$y(t)1_{(0,T)}(t) = \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right) 1_{(0,T)}(t) \left[\int h(t,\tau)\text{sinc}\left(W\left(\tau-\frac{n}{W}\right)\right) \mathrm{d}\tau\right] \quad (5.6a)$$

$$= \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right) \left[\int h(t,\tau)1_{(0,T)}(t)\text{sinc}\left(W\left(\tau-\frac{n}{W}\right)\right) \mathrm{d}\tau\right] \quad (5.6b)$$

Now we expand the $h(t,\tau)1_{(0,T)}(t)$ term as a Fourier series.

$$h(t,\tau)1_{(0,T)}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}\left[\int_0^T h(t',\tau)e^{-j2\pi kt'/T}\mathrm{d}t'\right] e^{j2\pi kt/T} \quad (5.7a)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\left[\int_{-\infty}^{\infty} h(t',\tau)1_{(0,T)}(t')e^{-j2\pi kt'/T}\mathrm{d}t'\right] e^{j2\pi kt/T} \quad (5.7b)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\left[\int_{-\infty}^{\infty} S(\theta,\tau)T\text{sinc}\left(\left(\frac{k}{T}-\theta\right)T\right) e^{-j\pi(k-T\theta)}\mathrm{d}\theta\right] e^{j2\pi kt/T} \quad (5.7c)$$

(5.7c) if valid for $t \in (0,T)$.

Plugging (5.7c) into (5.6b) we obtain,

$$y(t) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x\left(t-\frac{n}{W}\right) e^{j2\pi kt/T}\hat{S}\left(\frac{k}{T},\frac{n}{W}\right) \quad (5.8)$$

where,

$$\hat{S}(\theta,\tau) := \iint S(\theta',\tau')\text{sinc}\left((\tau-\tau')W\right)\text{sinc}\left((\theta-\theta')T\right)e^{-j\pi(\theta-\theta')T}\mathrm{d}\theta'\mathrm{d}\tau' \quad (5.9)$$

(5.8) is valid for the $(0,T)$ received portion of bandlimited signals.

Under the path scatterer interpretation we assume that the channel introduces a maximum delay spread of T_m and maximum Doppler spread of B_d, that is, $S(\theta,\tau)$ has support in $(-B_d,B_d) \times (0,T_m)$. In the smoothed version of $S(\theta,\tau)$ in (5.9), if we consider only the terms in (5.8) where the mainlobe of the smoothing kernel (which has size $(-1/T,1/T)$-by-$(-1/W,1/W)$) overlaps with the support of $S(\theta,\tau)$, we need only sum over $n = 0,\ldots,N$ where $N = \lceil WT_m \rceil$ and $k = -K,\ldots,K$ where

$K = \lceil TB_d \rceil$. We thus obtain the canonical representation of the time-frequency channel model.

$$y(t) = \sum_{n=0}^{\lceil WT_m \rceil} \sum_{k=-\lceil TB_d \rceil}^{\lceil TB_d \rceil} x\left(t - \frac{n}{W}\right) e^{j2\pi kt/T} \hat{S}\left(\frac{k}{T}, \frac{n}{W}\right) \tag{5.10}$$

5.2 Restatement

The double sum time-frequency channel formulation (5.8) was obtained by assuming,

- the input signal is bandpass with bandwidth W, and
- the output signal is analyzed only over a time interval of length T.

With these assumptions in mind, we define the following two projection operators,

$$P_T x(t) \doteq 1_{[0,T]}(t) x(t) \tag{5.11}$$

and,

$$Q_W x(t) \doteq \mathcal{F}^{-1}\{1_{[-W/2,W/2]}(\omega)\mathcal{F}\{x(t)\}(\omega)\}, \tag{5.12}$$

and using the following two operators, translation operator,

$$T_\tau x(t) \doteq x(t - \tau), \tag{5.13}$$

and modulation operator,

$$M_\nu x(t) \doteq x(t) e^{j2\pi\nu t}, \tag{5.14}$$

we can rewrite (5.8) as,

$$P_T \mathcal{N}_S Q_W = \sum_{m,n} c_{m,n} P_T M_{\frac{1}{T}}^{m} T_{\frac{1}{W}}^{n} Q_W \tag{5.15}$$

where the $c_{m,n} = \hat{S}\left(\frac{m}{T}, \frac{n}{W}\right)$. Restating the channel operator in this setting, we can ask the question, what general properties of the operators allow us to express the channel operator as a double summation of transformed input waveforms. In the next section, we determine properties of the operators used in the expansion that are sufficient conditions for the existence of such an expansion. Our goal is to develop an analogous time-scale canonical channel model. That is, in Section 5.5 we propose projections P and Q such that,

$$PW_L Q = \sum_{m,n} c_{m,n} P D_{a_0}^m T_{t_0}^n Q \tag{5.16}$$

for some choice of dilation and translation spacing parameters (a_0 and t_0), where $c_{m,n}$ depend on $\mathcal{L}$, and D is the dilation operator,

$$D_a x(t) = \frac{1}{\sqrt{|a|}} x\left(\frac{t}{a}\right). \tag{5.17}$$

5.3 Generalization

For the statement of the general theorem, we require the following definition.

Definition 4 (paired-up operators). *P and U are paired-up operators with generator e_0 iff,*

1. *P is an orthogonal projection in $L^2(\mathbb{R})$*
2. *U is unitary in $L^2(\mathbb{R})$*
3. *$PU = UP$*
4. *$\exists e_0 \in Ran\,P$ s.t. $\{U^m e_0 : m \in \mathbb{Z}\}$ is an orthonormal basis for $Ran\,P$*

Using two different pairs of paired-up operators, the following theorem gives a sufficient condition for the type of channel expansion described by [SA99].

Theorem 5. *If* (P,U) *and* (Q,V) *are both paired-up operators with generator elements* e_0 *and* f_0 *respectively,* H *is a bounded operator, and* $\exists c_{m,n}$ *such that*

$$\sum_{m,n} c_{m,n} \left\langle V^{n+k} f_0, U^{l-m} e_0 \right\rangle = \left\langle H V^k f_0, U^l e_0 \right\rangle, \quad \forall k, l, \tag{5.18}$$

then,

$$PHQ = P\left(\sum_{m,n} c_{m,n} U^m V^n\right) Q \tag{5.19}$$

Proof. First we expand out PQ using the orthonormal basis and unitary properties of the paired-up operators,

$$P = \sum_m \langle \cdot, U^m e_0 \rangle \, U^m e_0 \tag{5.20}$$

and

$$Q = \sum_n \langle \cdot, V^n f_0 \rangle \, V^n f_0. \tag{5.21}$$

we derive,

$$PQx = \sum_m \langle Qx, U^m e_0 \rangle \, U^m e_0 \tag{5.22a}$$

$$= \sum_m \left\langle \sum_n \langle x, V^n f_0 \rangle \, V^n f_0, U^m e_0 \right\rangle U^m e_0 \tag{5.22b}$$

$$= \sum_{m,n} \langle x, V^n f_0 \rangle \, \langle V^n f_0, U^m e_0 \rangle \, U^m e_0. \tag{5.22c}$$

We use this to determine,

$$P\left(\sum_{m,n} c_{m,n} U^m V^n\right) Qx = \sum_{m,n} c_{m,n} U^m PQV^n x \tag{5.23a}$$

$$= \sum_{m,n} c_{m,n} U^m \left(\sum_{k,l} \langle V^l f_0, U^k e_0\rangle \langle V^n x, V^l f_0\rangle U^k e_0\right) \tag{5.23b}$$

$$= \sum_{m,n,k,l} c_{m,n} \langle V^l f_0, U^k e_0\rangle \langle x, V^{-n} V^l f_0\rangle U^m U^k e_0 \tag{5.23c}$$

$$= \sum_{u,s}\left(\sum_{m,n} c_{m,n} \langle V^{n+u} f_0, U^{s-m} e_0\rangle\right) \langle x, V^u f_0\rangle U^s e_0 \tag{5.23d}$$

where the commuting property of paired-up operators was used in (5.23a). (5.22c) was used in moving from (5.23a) to (5.23b), and the unitary property of V was used in moving from (5.23b) to (5.23c). Now, looking to the LHS of (5.19), we use expand using the orthonormal basis and obtain,

$$PHQx = \sum_s \langle HQx, U^s e_0\rangle U^s e_0 \tag{5.24a}$$

$$= \sum_s \left\langle H\left(\sum_u \langle x, V^u f_0\rangle V^u f_0\right), U^s e_0\right\rangle U^s e_0 \tag{5.24b}$$

$$= \sum_{s,u} \langle x, V^u f_0\rangle \langle HV^u f_0, U^s e_0\rangle U^s e_0 \tag{5.24c}$$

$$= \sum_{u,s} h_{u,s} \langle x, V^u f_0\rangle U^s e_0 \tag{5.24d}$$

Given H, we then compute,

$$h_{u,s} \doteq \langle HV^u f_0, U^s e_0\rangle \tag{5.25}$$

which we use to solve,

$$\sum_{m,n} c_{m,n} \langle V^{n+u} f_0, U^{s-m} e_0\rangle = h_{u,s}, \quad \forall u, s \tag{5.26}$$

for $c_{m,n}$. These $c_{m,n}$ satisfy (5.19). □

5.3.1 Solving the coefficient equation

We now discuss the form of the solution to (5.18). We define

$$a_{k,l} \doteq \langle V^k f_0, U^l e_0 \rangle \tag{5.27}$$

and define

$$\tilde{c}_{m,n} = c_{n,-m} \tag{5.28}$$

which allows us to express (5.18) as,

$$\begin{aligned} h_{u,s} &= \sum_{m,n} c_{m,n} \langle V^{n+u} f_0, U^{s-m} e_0 \rangle && (5.29a)\\ &= \sum_{m,n} \langle V^{u-n} f_0, U^{s-m} e_0 \rangle \tilde{c}_{n,m} && (5.29b)\\ &= (a \star \tilde{c})_{u,s} && (5.29c) \end{aligned}$$

where

$$(a \star \tilde{c})_{u,s} \doteq \sum_{k,l} a_{u-k,s-l} \tilde{c}_{k,l} = \sum_{k,l} a_{k,l} \tilde{c}_{u-k,s-l} \tag{5.30}$$

Expressing h, a, and $\tilde{c}$ in the Z-transform domain,

$$\begin{aligned} A(z_1, z_2) &\doteq \textstyle\sum_{k,l} z_1^k z_2^l a_{k,l} = \displaystyle\sum_{k,l} z_1^k z_2^l \langle V^k f_0, U^l e_0 \rangle && (5.31)\\ H(z_1, z_2) &\doteq \textstyle\sum_{k,l} z_1^k z_2^l h_{k,l} = \displaystyle\sum_{k,l} z_1^k z_2^l \langle H V^k f_0, U^l e_0 \rangle && (5.32)\\ \tilde{C}(z_1, z_2) &\doteq \textstyle\sum_{k,l} z_1^k z_2^l \tilde{c}_{k,l} && (5.33) \end{aligned}$$

we can write (5.29c) as,

$$H = A\tilde{C} \tag{5.34}$$

and solve for $\tilde{C}$

$$\tilde{C}(z_1, z_2) = \frac{H(z_1, z_2)}{A(z_1, z_2)}. \tag{5.35}$$

In terms of $c_{m,n}$, this is,

$$c_{m,n} = Z^{-1}\left(\frac{H(z_1, z_2)}{A(z_1, z_2)}\right)_{n,-m} \tag{5.36}$$

where

$$Z^{-1}\left(F(z_1, z_2)\right)_{m,n} = \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} F\left(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}\right) \tag{5.37}$$

We can express (5.36) as a convolution of coefficients by defining

$$\hat{A}(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}) = \frac{1}{A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})} \tag{5.38}$$

and

$$\hat{a}_{m,n} = \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} \hat{A}\left(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}\right). \tag{5.39}$$

and we can obtain the $c_{m,n}$ using

$$c_{m,n} = \tilde{c}_{n,-m} = (\hat{a} \star h)_{n,-m}. \tag{5.40}$$

We will use (5.40) to determine the coefficients in practice.

5.4 Revisiting time-frequency

The example we have seen so far of the application of this theorem had,

- $(P, U, e_0) = (P_T, M_{\frac{1}{T}}, \frac{1}{\sqrt{T}} 1_{[0,T]}(t))$
- $(Q, V, f_0) = (Q_W, T_{\frac{1}{W}}, \sqrt{W}\mathrm{sinc}(Wt))$

for the operator $H = \mathcal{N}_S$ of the form.

$$Hx(t) = \iint S(\theta, \tau) e^{j2\pi\theta t} x(t - \tau) \mathrm{d}\theta \mathrm{d}\tau. \tag{5.41}$$

Modulation and translation operators were a natural fit with our channel description. $\mathcal{N}_S$, which describes the channel as a (continuous) summation of time and frequency shifts of the input signal.

The plan of attack to calculate the coefficients $c_{m,n}$ is to:

1. calculate $a_{m,n}$
2. use $a_{m,n}$ to obtain $A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})$
3. use $A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})$ to obtain $\hat{a}_{m,n}$
4. calculate $h_{k,l}$
5. use $\hat{a}_{m,n}$ and $h_{k,l}$ to obtain $c_{m,n}$ via (5.40)

And we begin,

$$\begin{aligned} a_{m,n} &= \langle V^M f_0, U^n e_0 \rangle && (5.42a) \\ &= \int \mathrm{sinc}\left(W\left(t - \frac{m}{W}\right)\right) e^{-j2\pi\frac{t}{T}n} \frac{\sqrt{W}}{\sqrt{T}} 1_{[0,T]}(t)\mathrm{d}t && (5.42b) \\ &= \sqrt{\frac{W}{T}} \int_0^T e^{-j2\pi\frac{nt}{T}} \mathrm{sinc}(Wt - m)\mathrm{d}t && (5.42c) \\ &\left(= \sqrt{WT} \int_0^1 e^{-j2\pi nt} \mathrm{sinc}(WTt - m)\mathrm{d}t\right) && (5.42d) \end{aligned}$$

For $\theta_1, \theta_2 \in [0, 1]$,

$$A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}) = \sum_{m,n} a_{m,n} e^{j2\pi\theta_1 m} e^{j2\pi\theta_2 n} \quad (5.43a)$$

$$= \sum_{m,n} \sqrt{\frac{W}{T}} \int_0^T e^{-j2\pi\frac{nt}{T}} \mathrm{sinc}(Wt - m) e^{j2\pi\theta_1 m} e^{j2\pi\theta_2 n} \mathrm{d}t \quad (5.43b)$$

$$\left(\sum_n e^{j2\pi n(\theta_2 - \frac{t}{T})} = \sum_n \delta(\theta_2 - \frac{t}{T} + n) = T \sum_n \delta(t - T\theta_2 - Tn) \right) \quad (5.43c)$$

$$= \sum_m \sqrt{WT} \mathrm{sinc}(WT\theta_2 - m) e^{j2\pi m\theta_1} \quad (5.43d)$$

$$\left(\mathrm{sinc}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi x\omega} \mathrm{d}\omega \right) \quad (5.43e)$$

$$= \sqrt{WT} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi\omega WT\theta_2} \underbrace{\sum_m e^{j2\pi m(\theta_1 - \omega)}}_{\sum_m \delta(\theta_1 - \omega - m)} \mathrm{d}\omega \quad (5.43f)$$

$$= \begin{cases} \sqrt{WT} e^{j2\pi WT\theta_1\theta_2} & : \theta_1 \in (0, \frac{1}{2}) \\ \sqrt{WT} e^{j2\pi WT(\theta_1 - 1)\theta_2} & : \theta_1 \in (\frac{1}{2}, 1) \end{cases} \quad (5.43g)$$

$$\hat{A}(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}) = \frac{1}{\sqrt{WT}} \begin{cases} e^{-j2\pi WT\theta_1\theta_2} & : \theta_1 \in (0, \frac{1}{2}) \\ e^{-j2\pi WT(\theta_1 - 1)\theta_2} & : \theta_1 \in (\frac{1}{2}, 1) \end{cases} \quad (5.44)$$

Substituting (5.44) into (5.39),

$$\hat{a}_{m,n} = \frac{1}{\sqrt{WT}} \int_0^1 \mathrm{d}\theta_2 \left[\int_0^{\frac{1}{2}} \mathrm{d}\theta_1 e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} e^{-j2\pi WT\theta_1\theta_2} + \int_{\frac{1}{2}}^1 \mathrm{d}\theta_1 e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} e^{-j2\pi WT(\theta_1 - 1)\theta_2} \right] \quad (5.45a)$$

$$= \frac{1}{\sqrt{WT}} \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_2 n} \left[\frac{e^{-j\pi(m+WT\theta_2)} - 1}{-j2\pi(m + WT\theta_2)} + e^{j2\pi WT\theta_2} \frac{e^{-j2\pi(m+WT\theta_2)} - e^{-j\pi(m+WT\theta_2)}}{-j2\pi(m + WT\theta_2)} \right] \quad (5.45b)$$

$$= \frac{1}{\sqrt{WT}} \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_2 n} \frac{e^{-j\pi(m+WT\theta_2)} - e^{j\pi(m+WT\theta_2)}}{-j2\pi(m + WT\theta_2)} \quad (5.45c)$$

$$= \frac{1}{\sqrt{WT}} \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_2 n} \mathrm{sinc}(WT\theta_2 + m) \quad (5.45d)$$

$$h_{k,l} = \sum_{k,l} z_1^k z_2^l \langle HV^{k} f_0, U^{l} e_0 \rangle \tag{5.46a}$$

$$= \int \iint S(\theta,\tau) e^{j2\pi\theta t} V^{k}[f_0](t-\tau) \mathrm{d}\theta \mathrm{d}\tau\, e^{-j2\pi\frac{t}{T}l} \frac{1}{\sqrt{T}} 1_{[0,T]}(t)) \mathrm{d}t \tag{5.46b}$$

$$= \sqrt{\frac{W}{T}} \iiint \mathrm{d}\theta \mathrm{d}\tau \mathrm{d}t\, 1_{[0,T]}(t) e^{j2\pi t(\theta - \frac{l}{T})} \mathrm{sinc}(Wt - k - W\tau) S(\theta,\tau) \tag{5.46c}$$

$$c_{m,n} = (\hat{a} * h)_{n,-m} = \sum_{k,l} \hat{a}_{n-k,-m-l} h_{k,l} \tag{5.47}$$

$$= \sum_{k,l} \frac{1}{T} \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_2(-m-l)} \mathrm{sinc}(WT\theta_2 + n - k) \iiint_0^T \mathrm{d}\theta \mathrm{d}\tau \mathrm{d}t e^{j2\pi t(\theta - \frac{l}{T})} \mathrm{sinc}(Wt - k - W\tau) S(\theta,\tau) \tag{5.48}$$

$$= \iint \mathrm{d}\theta \mathrm{d}\tau S(\theta,\tau) \underbrace{\int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}\theta_2 \sum_{k,l} e^{-j2\pi\theta_2(-m-l)} \mathrm{sinc}(WT\theta_2 + n - k) e^{j2\pi Tt(\theta - \frac{l}{T})} \mathrm{sinc}(WTt - k - W\tau)}_{E} \tag{5.49}$$

$$E = \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}\theta_2 \sum_k \underbrace{\left(\sum_l e^{j2\pi l(\theta_2 - t)}\right)}_{\delta(\theta_2 - t)} \mathrm{sinc}(WT\theta_2 + n - k)\mathrm{sinc}(WTt - k - W\tau)e^{j2\pi Tt\theta}e^{j2\pi\theta_2 m} \tag{5.50a}$$

$$= \int_0^1 \mathrm{d}t \sum_k \mathrm{sinc}(WTt + n - k)\mathrm{sinc}(WTt - k - W\tau)e^{j2\pi t(T\theta + m)} \tag{5.50b}$$

$$= \int_0^1 e^{j2\pi t(T\theta + m)}\mathrm{d}t \sum_k \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi s_1(WTt + n - k)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi s_2(WTt - k - W\tau)}\mathrm{d}s_1\mathrm{d}s_2 \tag{5.50c}$$

$$= \int_0^1 e^{j2\pi t(T\theta + m)}\mathrm{d}t \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\sum_k e^{-j2\pi k(s_1 + s_2)}}_{\delta(s_1 + s_2)} e^{j2\pi s_1(WTt + n)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi s_2(WTt - W\tau)}\mathrm{d}s_1\mathrm{d}s_2 \tag{5.50d}$$

$$= \int_0^1 e^{j2\pi t(T\theta + m)}\mathrm{d}t \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi s(WTt + n)}e^{j2\pi s(WTt - W\tau)}\mathrm{d}s \tag{5.50e}$$

$$= \int_0^1 e^{j2\pi t(T\theta + m)}\mathrm{d}t\,\mathrm{sinc}(n + W\tau) \tag{5.50f}$$

$$= e^{j\pi(T\theta + m)}\mathrm{sinc}(T\theta + m)\mathrm{sinc}(n + W\tau) \tag{5.50g}$$

$$\tag{5.50h}$$

And we obtain.

$$c_{m,n} = \iint S(\theta, \tau)e^{j\pi(T\theta+m)}\mathrm{sinc}(T\theta + m)\mathrm{sinc}(n + W\tau)\mathrm{d}\theta\mathrm{d}\tau \tag{5.51}$$

which are precisely the coefficients derived in [SA99].

5.5 Time-scale canonical model

We now develop the time-scale canonical characterization. For other possible extensions to time-scale, see the approach in [DF96], [ZF00], and [ZFL01] using wavelet packet modulation.

5.5.1 The scale projection

We use the following projection operator in scale space,

$$P = U_1^{-1}\left(1_{\left[0, \frac{1}{\ln a_0}\right]} \oplus 0\right) U_1 \tag{5.52}$$

where,

$$U_1 = (\mathcal{F} \oplus \mathcal{F})\, U \tag{5.53}$$

where,

$$U_1 : x \xrightarrow{U} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{\mathcal{F}\oplus\mathcal{F}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{5.54}$$

for,

$$x_1(t) = e^{\frac{t}{2}} x(e^t) \quad x_2(t) = e^{\frac{t}{2}} x(-e^t) \tag{5.55}$$

$$x(t) = \frac{1}{\sqrt{t}} x_1(\ln t), t > 0 \quad x(t) = \frac{1}{\sqrt{|t|}} x_2(\ln(-t)), t < 0 \tag{5.56}$$

and,

$$U_1^{-1} : \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \to x = \frac{1}{\sqrt{|t|}}\left(x_1(\ln(t))1_{(0,\infty)} + x_2(\ln(-t))1_{(-\infty,0)}\right) \tag{5.57a}$$

$$= \begin{cases} \frac{1}{\sqrt{t}} x_1(\ln(t)) & : \quad t > 0 \\ \frac{1}{\sqrt{|t|}} x_2(\ln(-t)) & : \quad t < 0 \end{cases} \tag{5.57b}$$

where, $x, x_1, x_2, X_1, X_2 \in L^2$.

5.5.2 The scale generator

Using the characteristic function in scale space (Ω_1, Ω_2), $\Omega_1 = \left[-\frac{1}{2\ln a_0}, \frac{1}{2\ln a_0}\right]$, $\Omega_2 = \emptyset$, leads to the generator,

$$e_0(t) = \begin{cases} \frac{1}{\sqrt{\ln a_0}}\frac{1}{\sqrt{t}}\text{sinc}\left(\frac{\ln |t|}{\ln a_0}\right) & : \quad t > 0 \\ 0 & : \quad t < 0 \end{cases} \tag{5.58}$$

It can be shown that $(P, U, e_0) = (U_1^{-1}\left(1_{[0, \frac{1}{\ln a_0}]} \dotplus 0\right) U_1, D_{a_0}, e_0)$ are paired-up.

5.5.3 Time-scale paired-up operators

For the time-scale model, we use the following paired-up operators,

- $(P, U, e_0) = (U_1^{-1}\left(1_{[0, \frac{1}{\ln a_0}]} \dotplus 0\right) U_1, D_{a_0}, e_0(t)$ from (5.58))
- $(Q, V, f_0) = (Q_{\frac{1}{t_0}}, T_{t_0}, \frac{1}{\sqrt{t_0}}\text{sinc}(\frac{t}{t_0}))$

to decompose the wideband channel corresponding to the operator $H = \mathcal{W}_\mathcal{L}$ of the form,

$$Hx(t) = \iint \mathcal{L}(a, b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right) dadb \tag{5.59}$$

into a discrete double summation,

$$P\mathcal{W}_L Q = \sum_{m,n} c_{m,n} P D_{a_0}^m T_{t_0}^n Q. \tag{5.60}$$

5.5.4 The coefficients

Following the same plan of attack,

1. calculate $a_{m,n}$
2. use $a_{m,n}$ to obtain $A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})$

3. use $A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})$ to obtain $\hat{a}_{m,n}$

4. calculate $h_{k,l}$

5. use $\hat{a}_{m,n}$ and $h_{k,l}$ to obtain $c_{m,n}$ via (5.40)

$$a_{m,n} = \langle T_{t_0} f_0, D_{a_0} e_0 \rangle = \int f_0(t - m t_0) \frac{1}{a_0^{\frac{n}{2}}} e_0\left(\frac{t}{a_0^n}\right) \mathrm{d}t \tag{5.61a}$$

$$= \sqrt{\frac{1}{t_0 \ln a_0}} \int_0^{\infty} \frac{1}{\sqrt{t}} \mathrm{sinc}\left(\frac{t}{t_0} - m\right) \mathrm{sinc}\left(\frac{\ln|t|}{\ln a_0} - n\right) \mathrm{d}t \tag{5.61b}$$

$$A(\theta_1, \theta_2) = \sum_{m,n} a_{m,n} e^{j2\pi\theta_1 m} e^{j2\pi\theta_2 n} \tag{5.62a}$$

$$= \sqrt{\frac{1}{t_0 \ln a_0}} \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{t}} \left[\sum_m e^{j2\pi\theta_1 m} \mathrm{sinc}\left(\frac{t}{t_0} - m\right)\right] \left[\sum_n e^{j2\pi\theta_2 n} \mathrm{sinc}\left(\frac{\ln|t|}{\ln a_0} - n\right)\right] \tag{5.62b}$$

$$\sum_m e^{j2\pi\theta_1 m} \mathrm{sinc}\left(\frac{t}{t_0} - m\right) = \sum_m e^{j2\pi\theta_1 m} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi\omega\left(\frac{t}{t_0} - m\right)} \mathrm{d}\omega \tag{5.63a}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi\omega\frac{t}{t_0}} \underbrace{\sum_m e^{-j2\pi m(\omega - \theta_1)}}_{\sum_m \delta(\omega - \theta_1 - m)} \mathrm{d}\omega \tag{5.63b}$$

For $\theta_1, \theta_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\sum_m e^{j2\pi\theta_1 m} \mathrm{sinc}\left(\frac{t}{t_0} - m\right) = e^{j2\pi\theta_1 \frac{t}{t_0}} \tag{5.64a}$$

$$\sum_n e^{j2\pi\theta_2 n} \mathrm{sinc}\left(\frac{\ln|t|}{\ln a_0} - n\right) = e^{j2\pi\theta_2 \frac{\ln|t|}{t_0}} \tag{5.64b}$$

$$\mathcal{A}(\theta_1,\theta_2) = \sqrt{\frac{1}{t_0 \ln a_0}} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}} e^{j2\pi\left(\theta_1 \frac{t}{t_0} + \theta_2 \frac{\ln |t|}{\ln a_0}\right)} \tag{5.65a}$$

$$= \sqrt{\frac{1}{t_0 \ln a_0}} \int_0^\infty t^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 \frac{t}{t_0}} \mathrm{d}t \tag{5.65b}$$

$$= \sqrt{\frac{1}{t_0 \ln a_0}} t_0^{\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} \int_0^\infty t^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 t} \mathrm{d}t \tag{5.65c}$$

$$\tilde{a}_{m,n} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} \frac{1}{\mathcal{A}(\theta_1,\theta_2)} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \tag{5.66a}$$

$$= t_0 \sqrt{\ln a_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \frac{t_0^{j2\pi\frac{\theta_2}{\ln a_0}} e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n}}{\int_0^\infty t^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 t} \mathrm{d}t} \tag{5.66b}$$

$$h_{u,s} = \langle H T_{t_0}^{u} f_0, D_{a_0}^{s} e_0 \rangle \tag{5.67a}$$

$$= \int \left(\iint \sqrt{|a|} \mathcal{L}(a,b) \frac{1}{\sqrt{|a|}} (T_{ut_0} f_0) \left(\frac{t-b}{a}\right) \mathrm{d}a \mathrm{d}b \right) \frac{1}{a_0^{s/2}} e_0 \left(\frac{t}{a_0^s}\right) \mathrm{d}t \tag{5.67b}$$

$$= \frac{1}{a_0^{s/2}} \int_0^\infty \mathrm{d}t \iint \mathrm{d}a \mathrm{d}b \mathcal{L}(a,b) \frac{1}{\sqrt{t_0}} \operatorname{sinc}\left(\frac{t-b}{at_0} - u\right) \frac{1}{\sqrt{\ln a_0}} \frac{1}{\sqrt{\frac{t}{a_0^s}}} \operatorname{sinc}\left(\frac{\ln \frac{t}{a_0^s}}{\ln a_0}\right) \tag{5.67c}$$

$$= \frac{1}{\sqrt{t_0 \ln a_0}} \int \mathrm{d}a \int \mathrm{d}b \mathcal{L}(a,b) \int_0^\infty \mathrm{d}t \frac{1}{\sqrt{t}} \operatorname{sinc}\left(\frac{t-b}{at_0} - u\right) \operatorname{sinc}\left(\frac{\ln t}{\ln a_0} - s\right) \tag{5.67d}$$

$$c_{m,n} = \sum_{k,l} \hat{a}_{-n-k,m-l} h_{k,l} \tag{5.68a}$$

$$= \sum_{k,l} \sqrt{t_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \frac{t_0^{j2\pi \frac{\theta_2}{\ln a_0}} e^{-j2\pi\theta_1(-n-k)} e^{-j2\pi\theta_2(m-l)}}{\int_0^\infty t^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 t} \mathrm{d}t} \cdot \tag{5.68b}$$

$$\int \mathrm{d}a \int \mathrm{d}b \mathcal{L}(a,b) \int_0^\infty \mathrm{d}t \frac{1}{\sqrt{t}} \mathrm{sinc}\left(\frac{t-b}{at_0} - k\right) \mathrm{sinc}\left(\frac{\ln t}{\ln a_0} - l\right) \tag{5.68c}$$

$$= \iint \mathrm{d}a\mathrm{d}b \mathcal{L}(a,b) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \int_0^\infty \mathrm{d}t \frac{\sqrt{t_0}}{\sqrt{t}} \frac{t_0^{j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 n} e^{-j2\pi\theta_2 m}}{\int_0^\infty \tau^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\tau} \mathrm{d}\tau} \cdot$$

$$\underbrace{\left[\sum_k e^{j2\pi\theta_1 k} \mathrm{sinc}\left(\frac{t-b}{at_0} - k\right)\right]}_{e^{j2\pi\theta_1 \frac{t-b}{at_0}}} \underbrace{\left[\sum_l e^{j2\pi\theta_2 l} \mathrm{sinc}\left(\frac{\ln t}{\ln a_0} - l\right)\right]}_{e^{j2\pi\theta_2 \frac{\ln t}{\ln a_0}}} \tag{5.68d}$$

$$= \iint \mathrm{d}a\mathrm{d}b \mathcal{L}(a,b) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \int_0^\infty \mathrm{d}t \frac{\sqrt{t_0}}{\sqrt{t}} \frac{t_0^{j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\left(\frac{t-b}{at_0}+n\right)} e^{j2\pi\theta_2\left(\frac{\ln t}{\ln a_0}-m\right)}}{\int_0^\infty \tau^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\tau} \mathrm{d}\tau} \tag{5.68e}$$

$$= \iint \mathrm{d}a\mathrm{d}b \mathcal{L}(a,b) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 t_0^{\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\left(n-\frac{b}{at_0}\right)} e^{-j2\pi\theta_2 m} \cdot D \tag{5.68f}$$

$$D = \left[\frac{\int_0^\infty t^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\frac{t}{at_0}} \mathrm{d}t}{\int_0^\infty \tau^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\tau} \mathrm{d}\tau}\right] \tag{5.69a}$$

$$= \frac{at_0 \int_0^\infty (at_0)^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} s^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 s} \mathrm{d}s}{\int_0^\infty \tau^{-\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\tau} \mathrm{d}\tau} \tag{5.69b}$$

$$= (at_0)^{\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} \tag{5.69c}$$

$$c_{m,n} = \iint \mathrm{d}a\mathrm{d}b\mathcal{L}(a,b) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 t_0^{\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1\left(n-\frac{b}{at_0}\right)} e^{-j2\pi\theta_2 m} (at_0)^{\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} \quad (5.70a)$$

$$= \iint \mathrm{d}a\mathrm{d}b\mathcal{L}(a,b) \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 e^{j2\pi\theta_1\left(n-\frac{b}{at_0}\right)}\right] \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_2 (at_0^2)^{\frac{1}{2}+j2\pi\frac{\theta_2}{\ln a_0}} e^{-j2\pi\theta_2 m}\right] \quad (5.70b)$$

$$= \iint \mathrm{d}a\mathrm{d}b\mathcal{L}(a,b) \mathrm{sinc}\left(n-\frac{b}{at_0}\right) \sqrt{a}t_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_2 e^{j2\pi\theta_2\left(\frac{\ln(at_0^2)}{\ln a_0}-m\right)} \quad (5.70c)$$

$$= \iint \mathrm{d}a\mathrm{d}b\mathcal{L}(a,b)\sqrt{a}t_0 \mathrm{sinc}\left(n-\frac{b}{at_0}\right) \mathrm{sinc}\left(\frac{\ln(at_0^2)}{\ln a_0}-m\right) \quad (5.70d)$$

The canonical time-scale model is then,

$$y(t) = \sum_{m,n} \frac{c_{m,n}}{a^{m/2}} x\left(\frac{t-nba^m}{a^m}\right) \quad (5.71)$$

for the $c_{m,n}$ defined in (5.70d).

Chapter 6

Summary and Future Work

6.1 Summary

We have considered the mapping between time-frequency integral kernels and time-scale integral kernels. Both of these kernel operators are often used to model time-varying communication channels. We determined the time-invariant correspondence between the two kernels. and determined the general mapping between the time-frequency and time-scale operators by separating out the time-invariant component and relating the remaining time-varying component. In the determination of the time-varying component mapping, we came to the surprising conclusion that causality. when considering infinite time. forces the time-scale channel description to be time-invariant. That is, causal time-varying time-scale channels do not exist. In practice. when considering smaller time intervals, this was not seen to be much of a limitation. We also showed a similar result for time-forward time-frequency kernel channels, which are channels whose negative and positive frequencies do not interact. The result was that causal time-varying time-frequency channels do not exist. Or. stated alternatively. causal time-varying time-frequency channels must also be time-invariant. We also proved that in order for a time-frequency kernel channel to have an equivalent lowpass characterization, it must be time-forward. Further study is required for the interpretation and practical implications of these results. The mapping between the time-frequency integral kernels and time-scale integral kernels was shown not to be stable precisely in the region of interest for the channels we are encounter in practical wireless settings. Sayeed and Aazhang have developed a canonical time-frequency representation of the doubly spread channel which has proved useful for the exploitation of the diversity of such channels. We developed a generalization of this canonical model and showed their time-frequency canonical model as an application of this generalization, which was also applied in a time-scale setting to derive a time-scale canonical description of the channel. We hope that further study of this time-scale description will yield similar benefits for wideband signals that Sayeed and Aazhang demonstrated in the narrowband setting.

6.2 Future Work

There are many items to consider for the continuation of the work presented here:

1. Develop the delay-dilation RAKE receiver based on the time-scale canonical channel model.

2. Examine a simple communication scenario where the delay-dilation RAKE yields performance gains over the delay-Doppler RAKE and conventional receivers.

3. Repeat the information theoretic analysis for the delay-dilation RAKE receiver similar to that which was done for delay-Doppler RAKE receiver [SA99].

4. Consider specifically ultra-wideband communication and the time-scale canonical channel model.

5. Explore the results of Sayeed/Aazhang on multipath-Doppler diversity and repeat the analysis for the multipath-multiscale diversity arising in the wideband channel.

6. Develop a canonical time-scale multiantenna wideband channel model similar to that proposed in [Say02] (and [ECS+98]).

7. Rather than just time-frequency or just time-scale, how about a non-unique representation such as time-frequency-scale. Is there anything to be gained (more efficient utilization of diversity)?

8. Examine further the mapping between $S(\theta, \tau)$ and $\mathcal{L}(a, b)$.

9. Determine how the support criteria on the spreading function relates to limits on the acceleration/velocity and the duration of the valid processing interval.

10. Determine the practical implications of the causal time-forward narrowband channel being time-invariant.

11. Is there a corresponding underspread/overspread theory (see: [MH02b, MH02a]) for the time-scale canonical model?

Appendix

A The Doppler effect

A.1 Classic treatment

Consider a source, $x(t)$ located at the origin moving to the right with velocity v_x: The position of the source is described by,

$$p_x(t) = v_x t \tag{A.1}$$

Consider a receiver, $y(t)$ located initially at a distance d_0 from the origin along the positive x-axis moving to the right with velocity v_y: The position of the receiver is described by,

$$p_y(t) = v_y t + d_0 \tag{A.2}$$

We define τ to be the time delay such that the source signal traveling with speed c_0 emitted at time $t - \tau$ reaches the receiver at time t. Clearly, the distance that the signal travels must equal the difference between the current receiver position and position of the source τ seconds ago. This can be described,

$$c_0\tau = |p_y(t) - p_x(t-\tau)| \tag{A.3}$$

$$= |v_y t + d_0 - v_x(t-\tau)| \tag{A.4}$$

Solving for τ,

$$\text{Case 1:} \quad p_y(t) > p_x(t-\tau), \quad \tau = \frac{(v_y - v_x)t + d_0}{c_0 - v_x}, \tag{A.5}$$

$$\text{Case 2:} \quad p_y(t) < p_x(t-\tau), \quad \tau = \frac{(-v_y + v_x)t - d_0}{c_0 + v_x}. \tag{A.6}$$

Substituting these back into the case definitions, we obtain (assuming $|v_x| < c_0$),

$$\text{Case 1:} \quad p_y(t) > p_x(t), \quad \tau = \frac{(v_y - v_x)t + d_0}{c_0 - v_x}, \tag{A.7}$$

$$\text{Case 2:} \quad p_y(t) < p_x(t), \quad \tau = \frac{(-v_y + v_x)t - d_0}{c_0 + v_x}. \tag{A.8}$$

The input output mapping is,

$$y(t) = x(t-\tau) \tag{A.9}$$

$$= \left\{ \begin{array}{lll} x\left(\frac{t - \frac{d_0}{c_0 - v_y}}{\frac{1 - v_x/c_0}{1 - v_y/c_0}}\right) & : & p_y(t) > p_x(t) \\ x\left(\frac{t + \frac{d_0}{c_0 + v_y}}{\frac{1 + v_x/c_0}{1 + v_y/c_0}}\right) & : & p_y(t) < p_x(t) \end{array} \right\} \tag{A.10}$$

where the change in Doppler effect from contraction to expansion occurs when the transmitter and receiver are collocated ($p_y(t) = p_x(t)$).

In the above derivation we ignored relativistic considerations, which is reasonable assuming that, $|v_x| \ll c$, $|v_y| \ll c$, and $c_0 \ll c$, where c is the speed of light. Note the lack of symmetry with v_x and $-v_y$. That is, the effect of x moving toward stationary y with speed v is different than the effect of y moving toward stationary x with speed v. We see in the next section that this is not the case when $c_0 = c$. That is, when the speed of signal propagation is the speed of light, it is impossible to for either x or y to determine whether x is moving toward stationary y or y is moving toward stationary x. Also note that $v_x = v$ and $v_x = -v$ do not shift the frequency by the same amount; This difference will remain true even when $c_0 = c$.

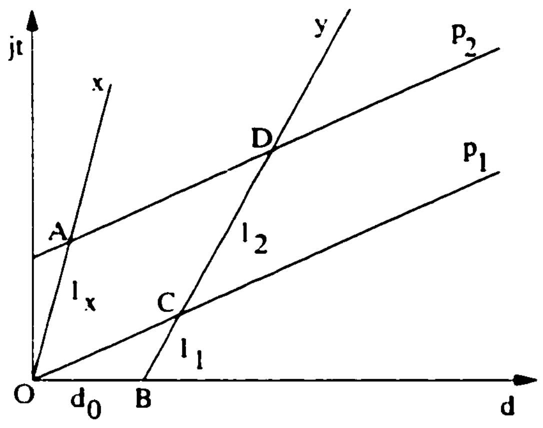


Figure A.1: Space-time diagram for the derivation of the Doppler effect.

A.2 Relativistic treatment

For electromagnetic waves, $c_0 = c$, the speed of light, and we must account for relativistic effects that we ignored in the derivation of (A.10). Consider the same setup as before, which is now depicted in the space-time diagram in Figure A.1. We have,

$$OA: \quad t = \frac{c}{v_x} d \tag{A.11}$$

$$OC: \quad t = \frac{c}{c_0} d \tag{A.12}$$

$$BD: \quad t = \frac{c}{v_y}(d - d_0) \tag{A.13}$$

$$AD: \quad t = \frac{c}{c_0} d + b \tag{A.14}$$

where b is yet to be determined. Without loss of generality, we define $l_x = 1$ and we are interested in $l_x/l_2 = 1/l_2$ which is the time dilation factor.

For simplicity, we consider here only the case where x is located to the left of y. The complementing case can be determined by swapping the signs on the velocities and the original separation distance, as was the case in the non-relativistic result. Using (A.11) and distance as measured in Minkowski space for the space-time diagram (for example, see Chapter 14 in [KK73]).

$$\sqrt{t_A^2 - d_A^2} = 1 \tag{A.15}$$

we determine the location of point A.

$$(d_A, t_A) = \left(\frac{v_x}{\sqrt{c^2 - v_x^2}}, \frac{c}{\sqrt{c^2 - v_x^2}} \right) \tag{A.16}$$

Now we can determine the t-intercept in (A.14).

$$b = \frac{1 - \frac{v_x}{c_0}}{\sqrt{1 - \frac{v_x^2}{c^2}}} \tag{A.17}$$

The intersection of AD and BD yields,

$$(d_D, t_D) = \left(\frac{\frac{bv_y}{c} + d_0}{1 - \frac{v_y}{c_0}}, \frac{cd_0 + bc_0}{c_0 - v_y} \right) \tag{A.18}$$

Intersecting OC and BD,

$$(d_C, t_C) = \left(\frac{c_0 d_0}{c_0 - v_y}, \frac{cd_0}{c_0 - v_y} \right) \tag{A.19}$$

Then the length from C to D is,

$$l_2 = \sqrt{(t_D - t_C)^2 - (d_D - d_C)^2} \tag{A.20}$$

$$= \sqrt{\left(\frac{bc_0}{c_0 - v_y} \right)^2 - \left(\frac{c_0 bv_y/c}{c_0 - v_y} \right)^2} \tag{A.21}$$

$$= \frac{b}{1 - v_y/c_0} \sqrt{1 - (v_y/c)^2} \tag{A.22}$$

$$= \frac{1 - \frac{v_x}{c_0}}{1 - \frac{v_y}{c_0}} \cdot \frac{\sqrt{1 - (\frac{v_y}{c})^2}}{\sqrt{1 - (\frac{v_x}{c})^2}} \tag{A.23}$$

The first term in (A.23) is the same as the scale factor in (A.10). the second term is the relativistic correction.

When $c = c_0$.

$$l_2 = \sqrt{\frac{1 - \frac{v_x}{c}}{1 + \frac{v_x}{c}} \frac{1 + \frac{v_y}{c}}{1 - \frac{v_y}{c}}} \tag{A.24}$$

and defining $v = \frac{v_y - v_x}{1 - v_x v_y / c^2}$.

$$l_2 = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = \frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{A.25}$$

the input-output relation takes the form.

$$y(t) = x\left(\frac{t}{\frac{1 - v/c}{\sqrt{1 - v^2/c^2}}} - d_1\right) \tag{A.26}$$

Noting that $y(t_c) = x(0)$. we solve for d_1. and obtain,

$$y(t) = x\left(\frac{t - \frac{cd_0}{c_0(1 - v_y/c_0)}}{\frac{1 - v/c}{\sqrt{1 - v^2/c^2}}}\right) \tag{A.27}$$

Unlike in (A.10), we see that the Doppler effect electromagnetic transmission depends only on the relative velocity of transmitter and receiver and not on the individual velocities.

A.3 Approximation

Despite the different forms of the Doppler shift for source moving toward receiver, receiver moving toward source, and taking into account relativistic effects, we now show that all three of these scenarios have approximately the same scaling factor. We consider the following three cases where the transmitter and receiver are approaching one another with speed v_0: (A.10) with $v_x = v_0$ and $v_y = 0$, (A.10) with $v_x = 0$ and

$v_y = -v_0$, and (A.25) with $v = v_0$.

$$1 - \frac{v_0}{c_0} = 1 - \frac{v_0}{c_0} \quad (A.28)$$

$$\frac{1}{1 + \frac{v_0}{c_0}} = 1 - \frac{v_0}{c_0} + (\frac{v_0}{c_0})^2 - \cdots \quad (A.29)$$

$$\sqrt{\frac{1 - \frac{v_0}{c_0}}{1 + \frac{v_0}{c_0}}} = 1 - \frac{v_0}{c_0} + \frac{1}{2}(\frac{v_0}{c_0})^2 - \cdots \quad (A.30)$$

As $\frac{v_0}{c_0} \ll 1$, in general, if we ignore second and higher order terms, we see that all three have time scaling factor equal to $1 - \frac{v_0}{c_0}$.

B Time-Frequency Duality

Bello defines time-frequency dual functions as follows.

Definition 5 (Time-frequency dual functions).

- *The direct dual of* $x(t)$ *is* $X(\theta) = \int x(t)e^{-j2\pi t\theta}dt$.
- *The reflection dual of* $X(\theta)$ *is* $x(t) = \int X(\theta)e^{j2\pi t\theta}d\theta$.

The interpretation of an function argument as a "time" or "frequency" variable is arbitrary with respect to duality. A function of time, say $x(t)$, still has a reflection dual, namely $X(-\theta)$. The reflection dual of a time function was used in (2.4) to stay consistent with the definitions in (2.3).

Time-frequency dual operators relate inputs and outputs that are dual functions. For this definition, it is useful to consider the interpretation that inputs and outputs are functions of either time and frequency. For example, the integral operator associated with $k_0(t, \tau)$ in (2.3) is an operator which is interpreted as mapping a time-domain function to another time-domain function. Indeed, the four kernel functions each have a unique interpretation mapping from either the time or frequency domain into either the time or frequency domain. We define the general kernel operator as $\mathcal{L}_k x = \int k(\alpha, \beta) x(\beta) \mathrm{d}\beta$. One set of dual characterizations, for example, is then,

$$y = \mathcal{L}_{k_0} x \longleftrightarrow \mathcal{F} y = \mathcal{L}_{k_3} \mathcal{F} x \tag{B.1}$$

as one maps time-to-time, the other frequency-to-frequency, and both have identical integral kernel formulations. We use $\mathcal{L}_{tt}$ to denote an integral kernel operator that maps time-to-time (such as $\mathcal{L}_{k_0}$). Similarly, we use $\mathcal{L}_{ff}$, $\mathcal{L}_{tf}$, and $\mathcal{L}_{ft}$ to denote integral kernel operators that map from frequency-to-frequency, time-to-frequency, and frequency-to-time, respectively. Clearly, $\mathcal{L}_k$ can be used to represent all four types, the only difference is in the interpretation of the inputs and outputs. We define time-frequency dual operators:

Definition 6 (Time-frequency dual operators).

- *The dual of operator* $\mathcal{L}_{tt}$ *is* $\mathcal{L}_{ff} = \mathcal{F} \mathcal{L}_{tt} \mathcal{F}^{-1}$
- *The dual of operator* $\mathcal{L}_{ff}$ *is* $\mathcal{L}_{tt} = \mathcal{F}^{-1} \mathcal{L}_{ff} \mathcal{F}$
- *The dual of operator* $\mathcal{L}_{tf}$ *is* $\mathcal{L}_{ft} = \mathcal{F}^{-1} \mathcal{L}_{tf} \mathcal{F}^{-1}$
- *The dual of operator* $\mathcal{L}_{ft}$ *is* $\mathcal{L}_{tf} = \mathcal{F} \mathcal{L}_{ft} \mathcal{F}$

From the definition, it is clear,

- $\mathcal{L}_{tt}$ and $\mathcal{F} \mathcal{L}_{tt} \mathcal{F}^{-1}$ are time-frequency dual operators, and
- $\mathcal{L}_{tf}$ and $\mathcal{F}^{-1} \mathcal{L}_{tf} \mathcal{F}^{-1}$ are time-frequency dual operators.

Using this definition, we can derive $\mathcal{L}_{k_0}^{D}$, the dual operator of $\mathcal{L}_{k_0}$, as follows,

$$\mathcal{L}_{k_0}^{D} = \mathcal{F} \mathcal{L}_{k_0} \mathcal{F}^{-1} \tag{B.2}$$

$$\mathcal{F}\mathcal{L}_{k_0}\mathcal{F}^{-1}x = \int e^{-j2\pi t\theta}\left[\int k_0(t,\tau)\left[\int x(\nu)e^{j2\pi\nu\tau}d\nu\right]d\tau\right]dt \quad \text{(B.3a)}$$

$$= \int\left[\iint k_0(t,\tau)e^{j2\pi(\tau\nu-t\theta)}dtd\tau\right]x(\nu)d\nu \quad \text{(B.3b)}$$

Defining,

$$k_0^D(\theta,\nu) = \iint k_0(t,\tau)e^{j2\pi(\tau\nu-t\theta)}dtd\tau \quad \text{(B.4)}$$

we see that,

$$\mathcal{L}_{k_0}^D = \mathcal{L}_{k_0^D} \quad \text{(B.5)}$$

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