10 Discrete-Time Fourier Series

Recommended Problems

P10.1

Consider a discrete-time system with impulse response

$$h[n] = (\tfrac{1}{2})^n u[n]$$

Determine the response to each of the following inputs:

(a) $x[n] = (-1)^n = e^{j\pi n}$ for all n

(b) $x[n] = e^{j(\pi n/4)}$ for all n

(c) $x[n] = \cos\left(\dfrac{\pi n}{4} + \dfrac{\pi}{8}\right)$ for all n

P10.2

Consider the following two periodic sequences:

$$\tilde{x}_1[n] = 1 + \sin\left(\frac{2\pi n}{10}\right) \tag{P10.2-1}$$

$$\tilde{x}_2[n] = 1 + \sin\left(\frac{20\pi}{12}n + \frac{\pi}{2}\right) \tag{P10.2-2}$$

(a) Determine the period of $x_1[n]$ and of $x_2[n]$.

(b) Determine the sequence of Fourier series coefficients a_{1k} for $x_1[n]$ and a_{2k} for $x_2[n]$.

(c) In each case, the sequence of Fourier series coefficients is periodic. Determine the period of the sequence a_{1k} and the sequence a_{2k}.

P10.3

Determine the Fourier series coefficients for the three periodic sequences shown in Figures P10.3-1 to P10.3-3. Since these three sequences all have the same nonzero values over one period, we suggest that you first determine an expression for the envelope of the Fourier series coefficients and then sample this envelope at the appropriate spacings in each case.

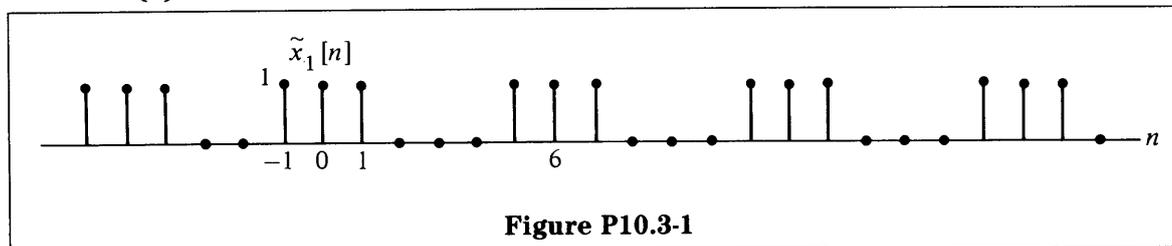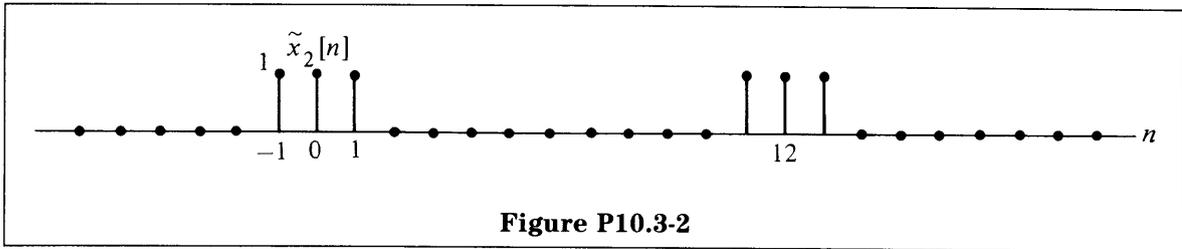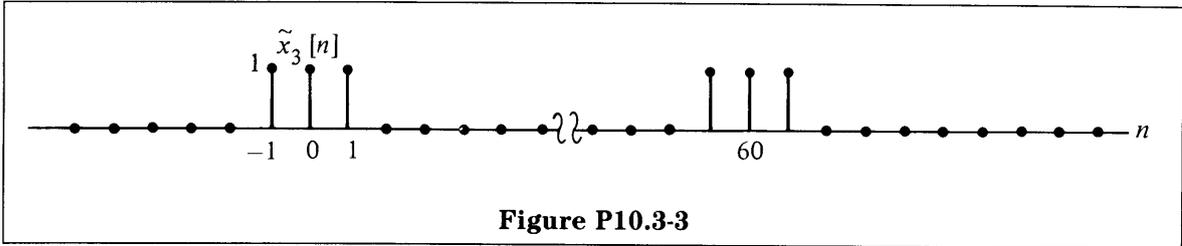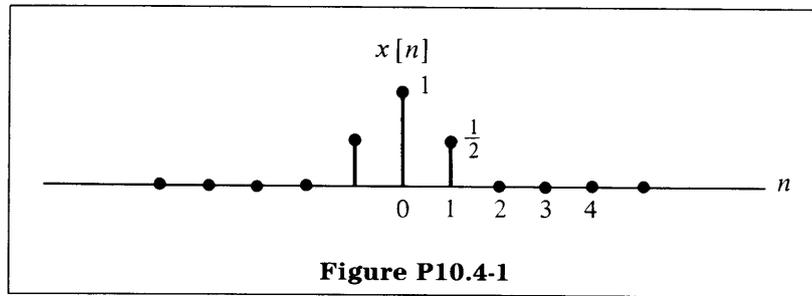**(a)**

Figure P10.3-1

(b)

Figure P10.3-2

(c)

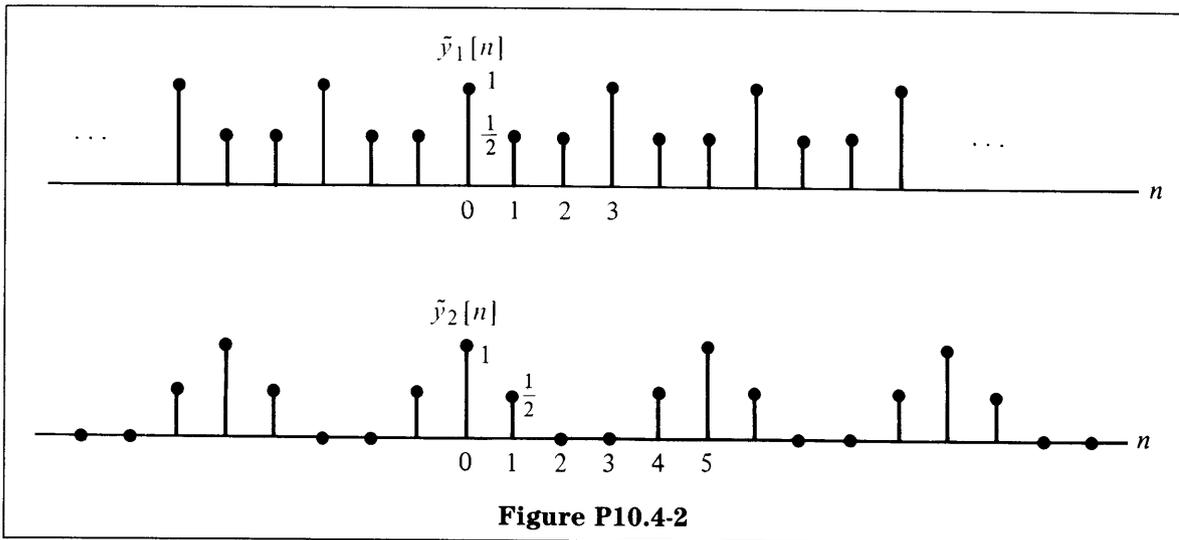**Figure P10.3-3**

P10.4

(a) Determine and sketch the discrete-time Fourier transform of the sequence in Figure P10.4-1.

Figure P10.4-1

(b) Using your result in part (a), determine the discrete-time Fourier series of the two periodic sequences in Figure P10.4-2.

Figure P10.4-2

P10.5

Consider the signal $x[n]$ depicted in Figure P10.5. This signal is periodic with period $N = 4$. The signal $x[n]$ can be expressed in terms of a discrete-time Fourier series:

$$x[n] = \sum_{k=0}^{3} a_k e^{jk(2\pi/4)n} \tag{P10.5-1}$$

As mentioned in the text, one way to determine the Fourier series coefficients is to treat eq. (P10.5-1) as a set of four linear equations [eq. (P10.5-1) for $n = 0, 1, 2, 3$] in the four unknowns (a_0, a_1, a_2, and a_3).
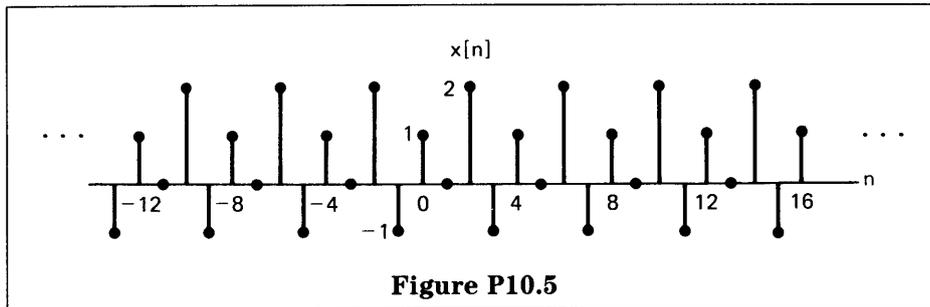

Figure P10.5

(a) Explicitly write out the four equations and solve them directly using any standard technique for solving four equations in four unknowns. (Be sure to first reduce the complex exponentials to the simplest form.)

(b) Check your answer by calculating the coefficients a_k directly, using the Fourier series analysis equation

$$a_k = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk(2\pi/4)n}$$

P10.6

Figure P10.6 shows a real periodic signal $\tilde{x}[n]$. Using the properties of the Fourier series and without explicitly evaluating the Fourier series coefficients, determine whether the following are true for the Fourier series coefficients a_k.

(a) $a_k = a_{k+10}$ for all k

(b) $a_k = a_{-k}$ for all k

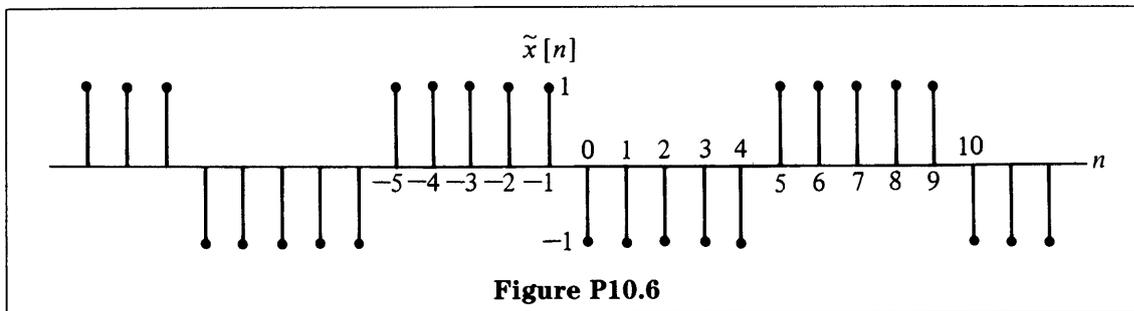**(c)** $a_k e^{jk(2\pi/5)}$ is real for all k

(d) $a_0 = 0$

Figure P10.6

Optional Problems

In parts (a)–(d) we specify the Fourier series coefficients of a signal that is periodic with period 8. Determine the signal $x[n]$ in each case.

(a) $a_k = \cos\left(k\dfrac{\pi}{4}\right) + \sin\left(3k\dfrac{\pi}{4}\right)$

(b) $a_k = \begin{cases} \sin\left(\dfrac{k\pi}{3}\right), & 0 \leq k \leq 6 \\ 0, & k = 7 \end{cases}$

(c) a_k as in Figure P10.7(a)

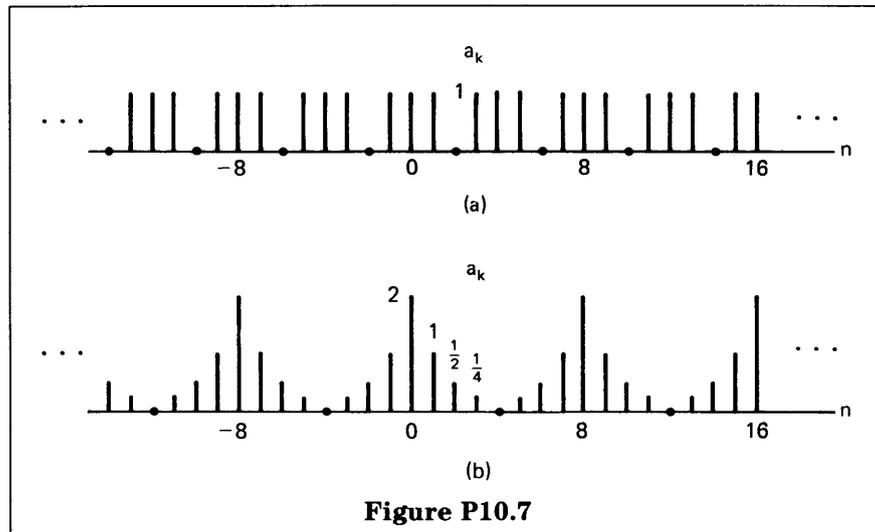**(d)** a_k as in Figure P10.7(b)

Figure P10.7

(a) Consider a linear, time-invariant system with impulse response

$$h[n] = \left(\tfrac{1}{2}\right)^{|n|}$$

Find the Fourier series representation of the output $\tilde{y}[n]$ for each of the following inputs.

(i) $\tilde{x}[n] = \sin\left(\dfrac{3\pi n}{4}\right)$

(ii) $\tilde{x}[n] = \displaystyle\sum_{k=-\infty}^{\infty} \delta[n - 4k]$

(iii) $\tilde{x}[n]$ is periodic with period 6, and

$$\tilde{x}[n] = \begin{cases} 1, & n = 0, \pm 1 \\ 0, & n = \pm 2, \pm 3, \pm 4 \end{cases}$$

(iv) $\tilde{x}[n] = j^n + (-1)^n$

(b) Repeat part (a) for

$$h[n] = \begin{cases} 1, & 0 \leq n \leq 2 \\ -1, & -2 \leq n \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

P10.9

Let $\tilde{x}[n]$ be a periodic sequence with period N and Fourier series representation

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \qquad \text{(P10.9-1)}$$

The Fourier series coefficients for each of the following signals can be expressed in terms of the coefficients a_k in eq. (P10.9-1). Derive these expressions.

(a) $\tilde{x}[n - n_0]$

(b) $\tilde{x}[n] - \tilde{x}[n - 1]$

(c) $\tilde{x}[n] - \tilde{x}\left[n - \dfrac{N}{2}\right]$ (assume that N is even)

(d) $\tilde{x}[n] + x\left[n + \dfrac{N}{2}\right]$ (assume that N is even; note that this signal is periodic with period $N/2$)

(e) $\tilde{x}^*[-n]$

P10.10

Consider two specific periodic sequences $\tilde{x}[n]$ and $\tilde{y}[n]$. $\tilde{x}[n]$ has period N and $\tilde{y}[n]$ has period M. The sequence $\tilde{w}[n]$ is defined as $\tilde{w}[n] = \tilde{x}[n] + \tilde{y}[n]$.

(a) Show that $\tilde{w}[n]$ is periodic with period MN.

(b) Since $\tilde{x}[n]$ has period N, its discrete Fourier series coefficients a_k also have period N. Similarly, since $\tilde{y}[n]$ has period M, its discrete Fourier series coefficients b_k also have period M. The discrete Fourier series coefficients of $\tilde{w}[n]$, c_k, have period MN. Determine c_k in terms of a_k and b_k.

P10.11

Determine the Fourier series coefficients for each of the following periodic discrete-time signals. Plot the magnitude and phase of each set of coefficients a_k.

(a) $x[n] = \sin\left[\dfrac{\pi(n - 1)}{4}\right]$

(b) $x[n] = \cos\left(\dfrac{2\pi n}{3}\right) + \sin\left(\dfrac{2\pi n}{7}\right)$

(c) $x[n] = \cos\left(\dfrac{11\pi n}{4} - \dfrac{\pi}{3}\right)$

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