8

Continuous-Time Fourier Transform

In this lecture, we extend the Fourier series representation for continuous-time periodic signals to a representation of aperiodic signals. The basic approach is to construct a periodic signal from the aperiodic one by periodically replicating it, that is, by adding it to itself shifted by integer multiples of an assumed period T_0. As T_0 is increased indefinitely, the periodic signal then approaches or represents in some sense the aperiodic one. Correspondingly, since the periodic signal can be represented through a Fourier series, this Fourier series representation as T_0 goes to infinity can be considered to be a representation as a linear combination of complex exponentials of the aperiodic signal. The resulting synthesis equation and the corresponding analysis equation are referred to as the *inverse Fourier transform* and the *Fourier transform* respectively.

In understanding how the Fourier series coefficients behave as the period of a periodic signal is increased, it is particularly useful to express the Fourier series coefficients as samples of an envelope. The form of this envelope is dependent on the shape of the signal over a period, but it is not dependent on the value of the period. The Fourier series coefficients can then be expressed as samples of this envelope spaced in frequency by the fundamental frequency. Consequently, as the period increases, the envelope remains the same and the samples representing the Fourier series coefficients become increasingly dense, that is, their spacing in frequency becomes smaller. In the limit as the period approaches infinity, the envelope itself represents the signal. This envelope is defined as the Fourier transform of the aperiodic signal remaining when the period goes to infinity.

Although the Fourier transform is developed in this lecture beginning with the Fourier series, the Fourier transform in fact becomes a framework that can be used to encompass both aperiodic and periodic signals. Specifically, for periodic signals we can define the Fourier transform as an impulse train with the impulses occurring at integer multiples of the fundamental frequency and with amplitudes equal to 2π times the Fourier series coefficients. With this as the Fourier transform, the Fourier transform synthesis equation in fact reduces to the Fourier series synthesis equation.

As suggested by the above discussion, a number of relationships exist between the Fourier series and the Fourier transform, all of which are important to recognize. As stated in the last paragraph, the Fourier transform of a periodic signal is an impulse train with the areas of the impulses proportional to the Fourier series coefficients. An additional relationship is that the Fourier series coefficients of a periodic signal are *samples* of the Fourier transform of one period. Thus the Fourier transform of a period describes the envelope of the samples. Finally, the Fourier series of a periodic signal approaches the Fourier transform of the aperiodic signal represented by a single period as the period goes to infinity.

We now have a single framework, the Fourier transform, that incorporates both periodic and aperiodic signals. In the next lecture, we continue the discussion of the continuous-time Fourier transform in particular, focusing on some important and useful properties.

Suggested Reading

Section 4.4, Representation of Aperiodic Signals: The Continuous-Time Fourier Transform, pages 186–195

Section 4.5, Periodic Signals and the Continuous-Time Fourier Transform, pages 196–202

MARKERBOARD
8.1 (a)

Continuous-Time
Fourier Series

$x(t)$ periodic T_0

$x(t) = x(t+T_0)$

$\omega_0 = \frac{2\pi}{T_0}$

Synthesis

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Analysis

$$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

Continuous-Time
Fourier Transform

Synthesis

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Analysis

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

$$X(\omega) = \mathcal{R}e\{X(\omega)\} + j\,\mathcal{I}m\{X(\omega)\}$$

$$= |X(\omega)| e^{j\measuredangle X(\omega)}$$

Example (Text 4.7)

$$x(t) = e^{-at} u(t)$$

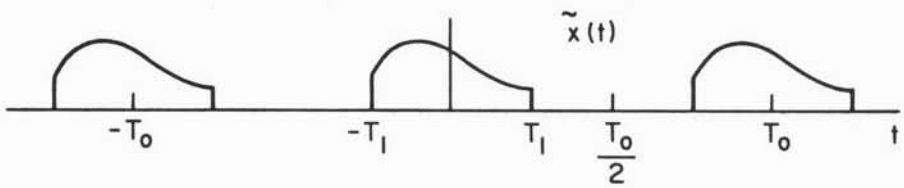
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} \underbrace{e^{-at} e^{-j\omega t}}_{e^{-t(a+j\omega)}} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^\infty$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$$

FOURIER REPRESENTATION OF APERIODIC SIGNALS



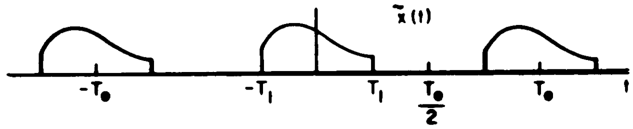
$$\tilde{x}(t) = x(t) \qquad |t| < \frac{T_0}{2}$$

As $T_0 \to \infty$ $\qquad \tilde{x}(t) \to x(t)$

- use Fourier series to represent $\tilde{x}(t)$
- let $T_0 \to \infty$ to represent $x(t)$

TRANSPARENCY
8.1
Representation of an aperiodic signal as a periodic signal with the period increasing to infinity.

TRANSPARENCY
8.2
Fourier series representation of the aperiodic signal $\tilde{x}(t)$.



$$\tilde{x}(t) = x(t) \qquad |t| < \frac{T_o}{2}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \qquad \omega_o = \frac{2\pi}{T_o}$$

$$a_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} \tilde{x}(t)\, e^{-jk\omega_o t}\, dt$$

$$a_k = \frac{1}{T_o} \int_{-\infty}^{+\infty} x(t)\, e^{-jk\omega_o t}\, dt$$

TRANSPARENCY
8.3
Representation of the Fourier series coefficients as samples of the envelope $X(\omega)$.

$$a_k = \frac{1}{T_o} \int_{-\infty}^{+\infty} x(t)\, e^{-jk\omega_o t}\, dt$$

Define: $$X(\omega) \triangleq \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Then: $$T_o a_k = X(\omega) \Big|_{\omega = k\omega_o}$$

$X(\omega)$ is the <u>envelope</u> of $T_o a_k$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T_o} X(k\omega_o)\, e^{jk\omega_o t}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(k\omega_o)\, e^{jk\omega_o t}\, \omega_o$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier transform
- analysis

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(k\omega_o)\, e^{jk\omega_o t}\, \omega_o$$

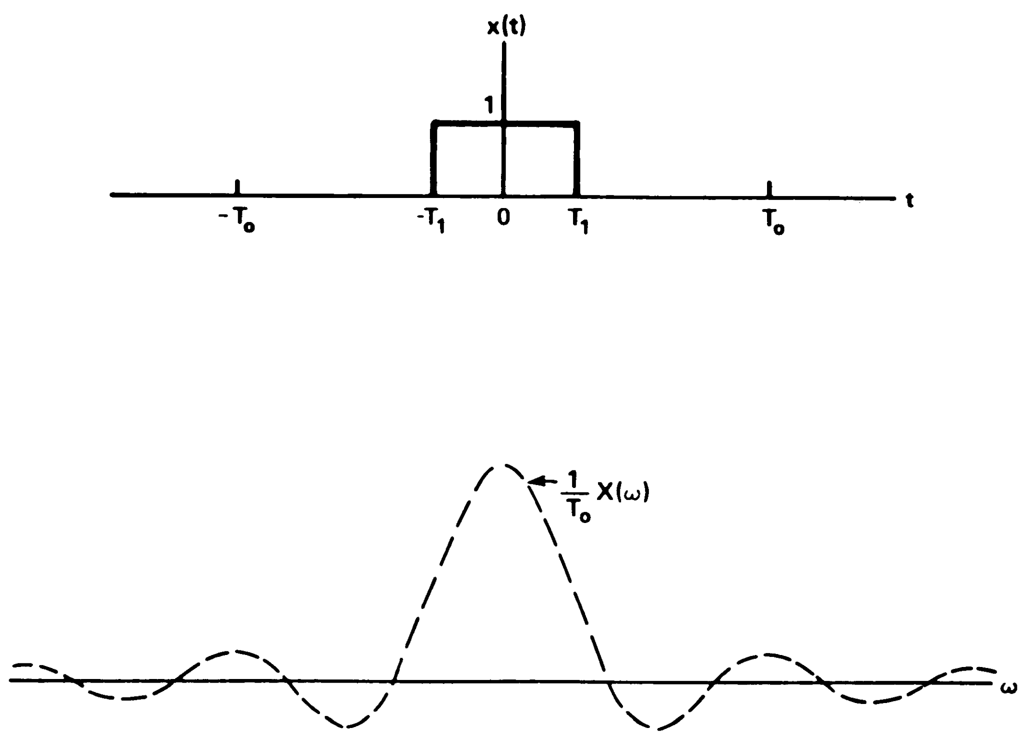
As $T_o \rightarrow \infty$,

$$\omega_o \rightarrow 0,\ \tilde{x}(t) \rightarrow x(t),\ \omega_o \rightarrow d\omega,\ \sum \rightarrow \int$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

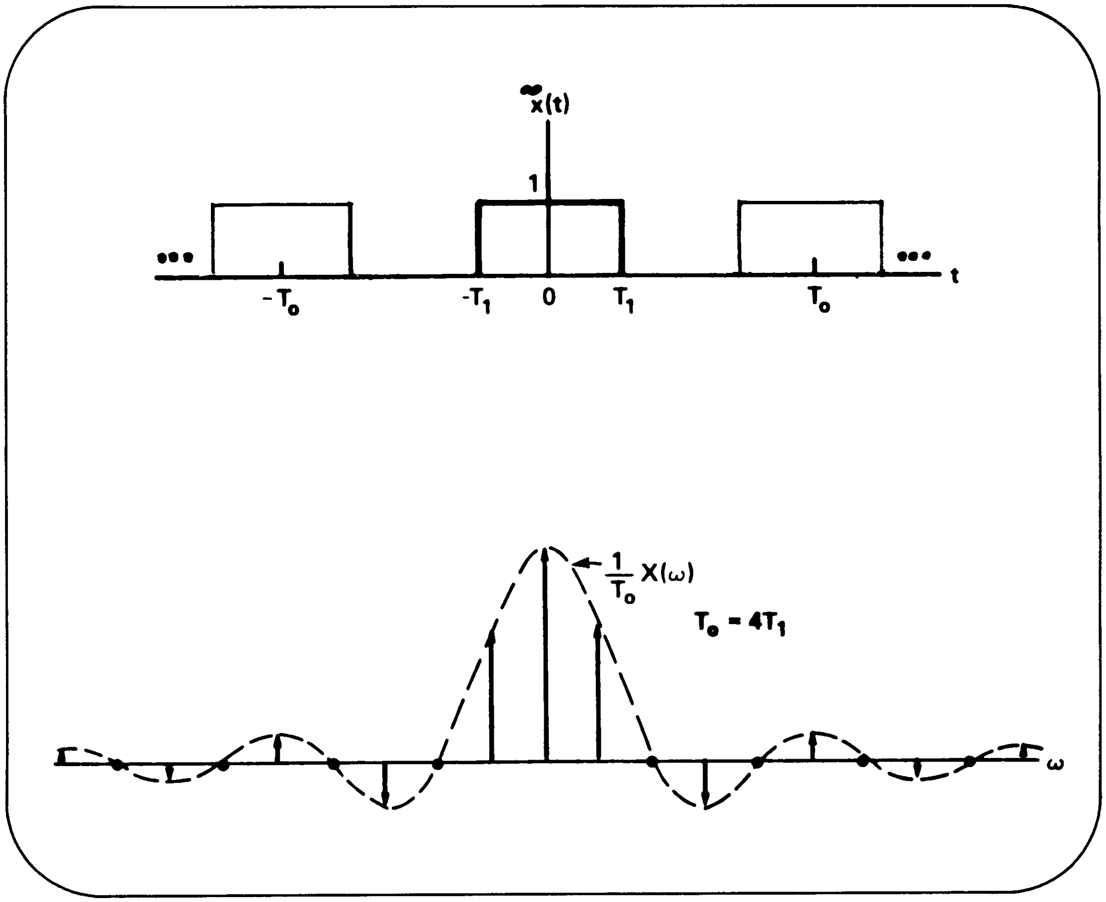
Inverse
Fourier transform
- synthesis

TRANSPARENCY 8.4
The analysis and synthesis equations associated with the Fourier transform.

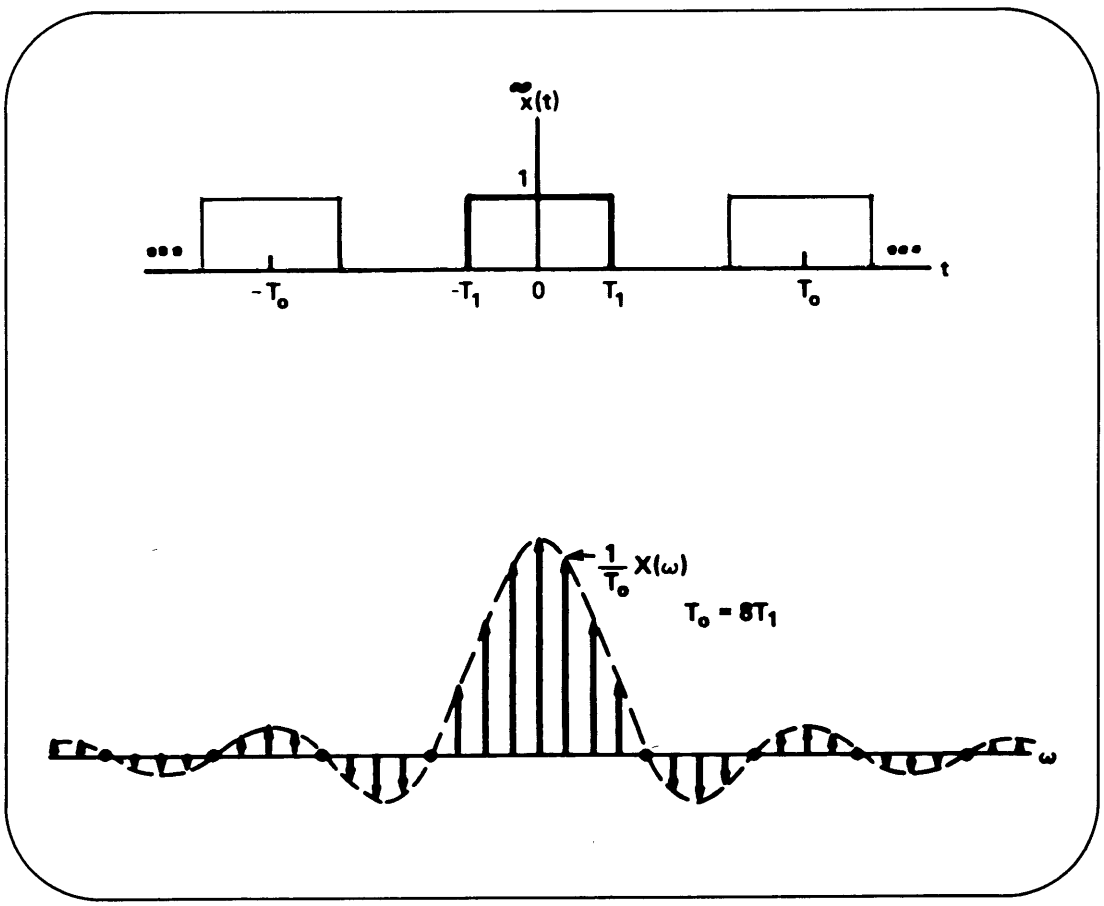


TRANSPARENCY 8.5
Transparencies 8.5–8.7 illustrate that the Fourier series coefficients of $\tilde{x}(t)$ are samples of the Fourier transform of one period. As the period T_0 increases, the samples become more closely spaced. Here is shown $x(t)$ (i.e., one period of $\tilde{x}(t)$) and its Fourier transform.

TRANSPARENCY
8.6
$\tilde{x}(t)$ and its Fourier series coefficients with $T_0 = 4T_1$. [As the figure is drawn, T_0 and T_1 are not to scale.]



TRANSPARENCY
8.7
$\tilde{x}(t)$ and its Fourier series coefficients with $T_0 = 8T_1$. [As the figure is drawn, T_0 and T_1 are not to scale.]



MARKERBOARD
8.1 (b)

Continuous-Time Fourier Series

$x(t)$ periodic T_0

$x(t) = x(t+T_0)$

$\omega_0 = \frac{2\pi}{T_0}$

Synthesis

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Analysis

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

Continuous-Time Fourier Transform

Synthesis

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Analysis

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$X(\omega) = \mathrm{Re}\{X(\omega)\} + j\,\mathrm{Im}\{X(\omega)\}$$

$$= |X(\omega)| e^{j\sphericalangle X(\omega)}$$

Example (Text 4.7)

$$x(t) = e^{-at} u(t)$$

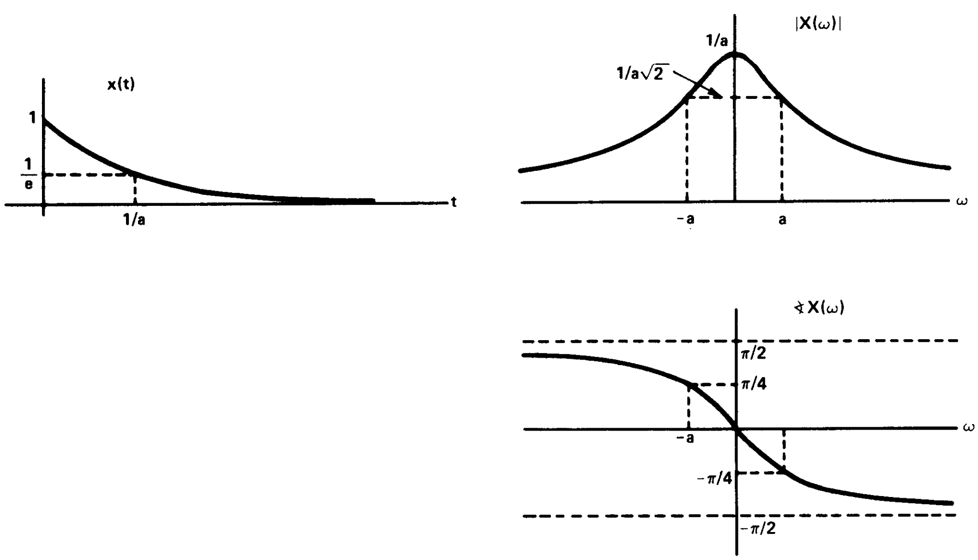
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} \underbrace{e^{-at} e^{-j\omega t}}_{e^{-t(a+j\omega)}} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

$$e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

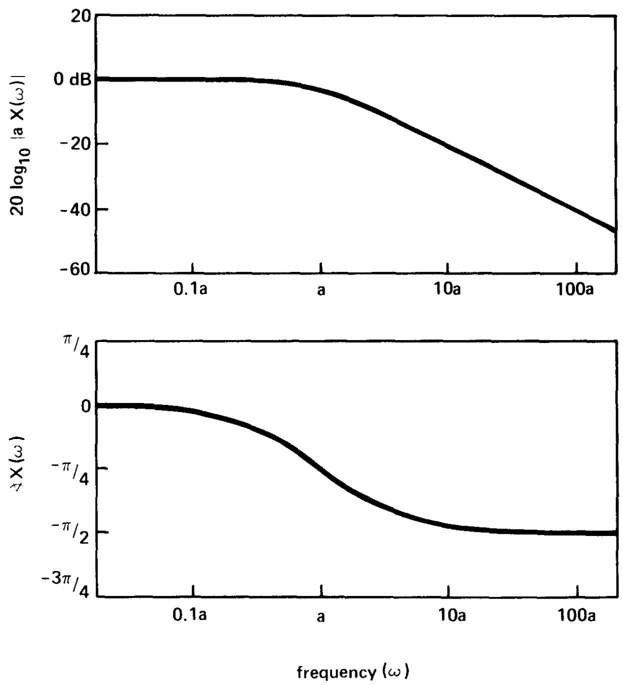
Example 4.7: $e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} \quad a > 0$



TRANSPARENCY
8.8
An exponential time function and its Fourier transform. [Example 4.7 from the text.]

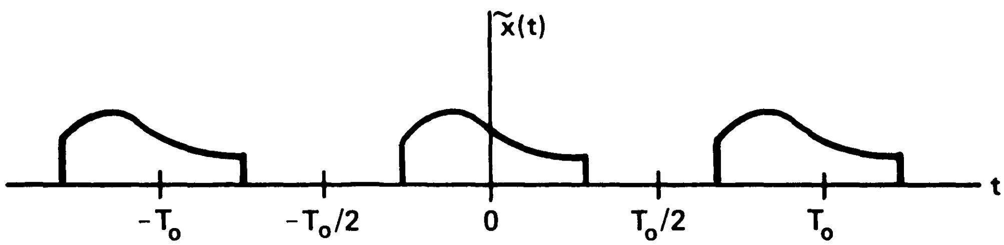
TRANSPARENCY 8.9
Bode plot representation of the Fourier transform of an exponential time function.

Bode Plot $\begin{cases} 20\log_{10}|X(\omega)| \triangleq \text{decibels (dB)} \\ \sphericalangle X(\omega) \end{cases}$



TRANSPARENCY 8.10
Relationship between the Fourier series coefficients of a periodic signal and the Fourier transform of one period.

Fourier Series coefficients equal $\frac{1}{T_o}$ times samples of Fourier transform of one period

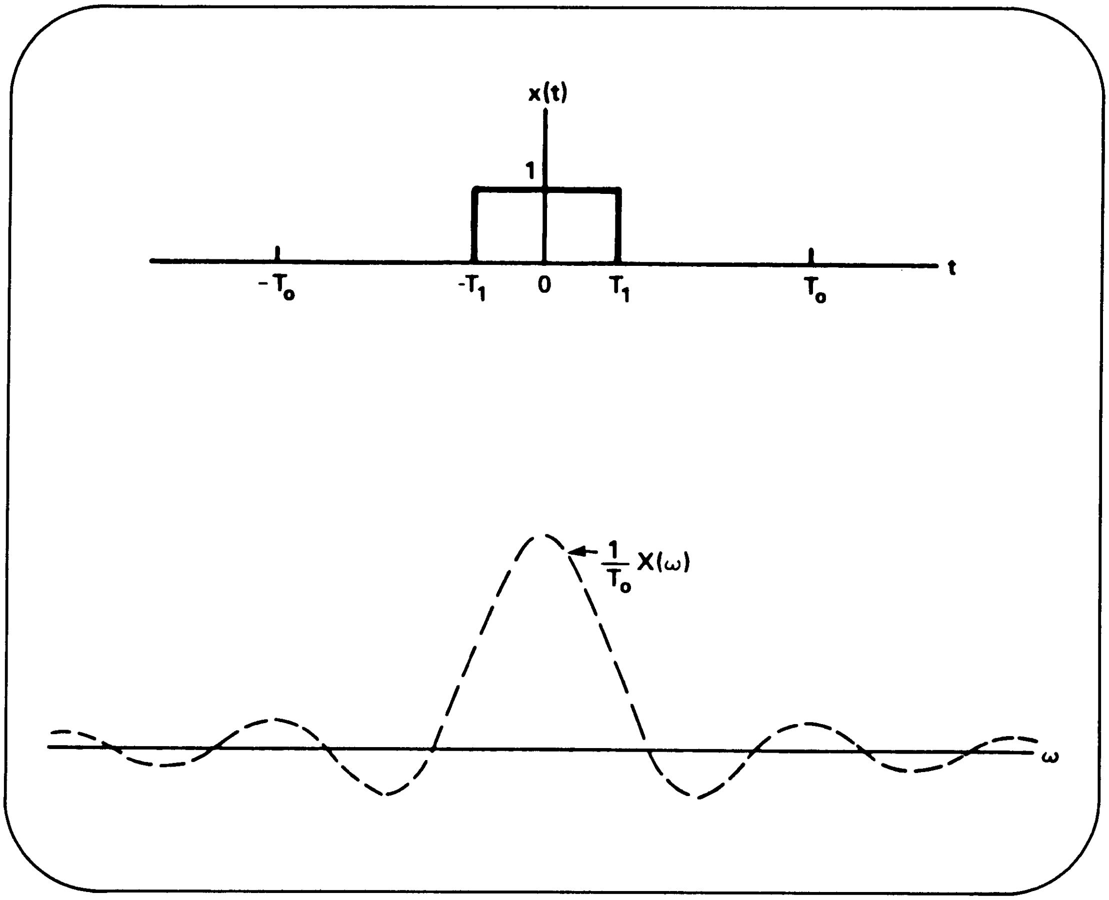


$x(t) \triangleq$ one period of $\tilde{x}(t)$

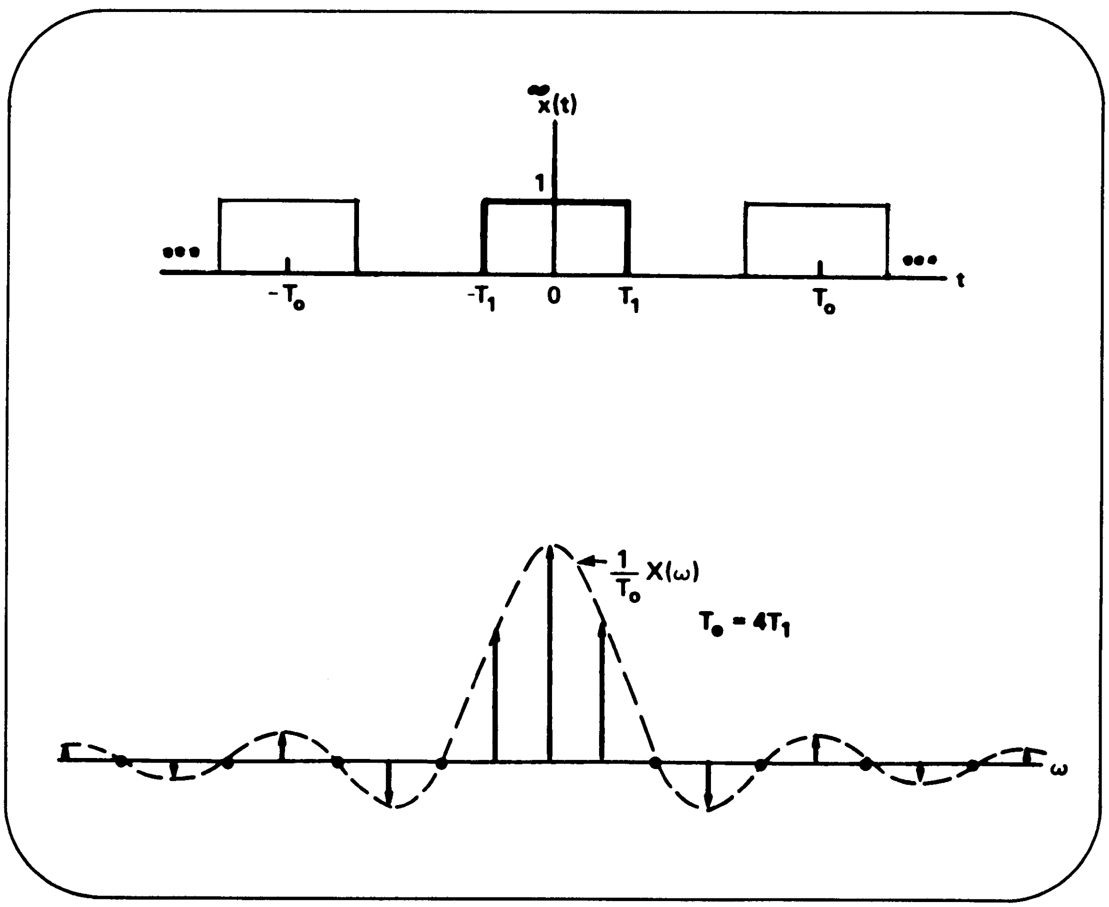
$\tilde{x}(t) \longleftrightarrow a_k$

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

$$\boxed{a_k = \frac{1}{T_o} X(\omega)\Big|_{\omega = k\omega_o}}$$

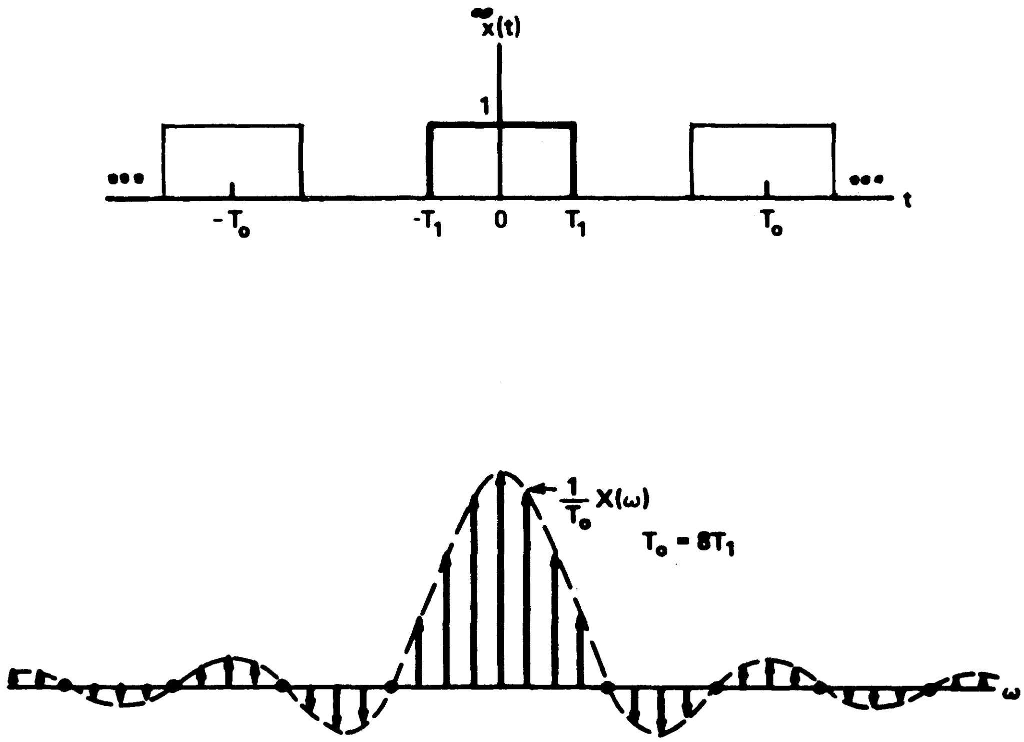


TRANSPARENCY 8.11
The Fourier transform of $x(t)$ as the envelope of the Fourier series coefficients of $\tilde{x}(t)$. As the period T_0 increases, the samples become more closely spaced. This transparency shows $x(t)$ and its Fourier transform. [Transparency 8.5 repeated]



TRANSPARENCY 8.12
$\tilde{x}(t)$ and its Fourier series coefficients with $T_0 = 4T_1$. [Transparency 8.6 repeated]

TRANSPARENCY 8.13
$\tilde{x}(t)$ and its Fourier series coefficients with $T_0 = 8T_1$. [Transparency 8.7 repeated]



TRANSPARENCY 8.14
Representation of the Fourier series in terms of the Fourier transform.

FOURIER TRANSFORM OF A PERIODIC SIGNAL $\tilde{x}(t)$

$\tilde{x}(t) \longleftrightarrow a_k$ **Fourier series coefficients**

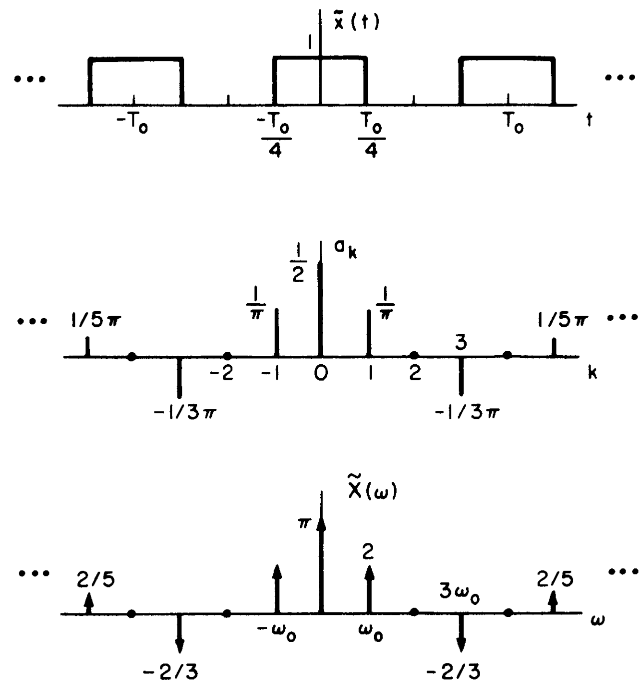
$\tilde{x}(t) \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}(\omega)$ **Fourier transform**

$$\boxed{\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0)}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} 2\pi\, a_k \underbrace{\int_{-\infty}^{+\infty} \delta(\omega - k\omega_0)\, e^{-j\omega t}\, d\omega}_{e^{-jk\omega_0}}$$

Symmetric square wave



TRANSPARENCY 8.15
Illustration of the Fourier series coefficients and the Fourier transform for a symmetric square wave.

1. x(t) APERIODIC

- construct periodic signal $\tilde{x}(t)$ for which one period is x(t)

- $\tilde{x}(t)$ has a Fourier series

- as period of $\tilde{x}(t)$ increases, $\tilde{x}(t) \rightarrow x(t)$ and Fourier series of $\tilde{x}(t) \rightarrow$ Fourier Transform of x(t)

TRANSPARENCY 8.16
Summary of the development of the Fourier transform from the Fourier series. [The periodic signal has been corrected here to read $\tilde{x}(t)$, not $x(t)$.]

TRANSPARENCY 8.17
Summary of the relationship between the Fourier transform and Fourier series for a periodic signal.

2. $\tilde{x}(t)$ PERIODIC, $x(t)$ REPRESENTS ONE PERIOD

- Fourier series coefficients of $\tilde{x}(t)$

 = $(1/T_0)$ times samples of Fourier transform of $x(t)$

3. $\tilde{x}(t)$ PERIODIC

- Fourier transform of $\tilde{x}(t)$ defined as impulse train:

$$\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$

TRANSPARENCY 8.18
Summary example illustrating some of the relationships between the Fourier series and Fourier transform.