# Analogue Filter Design

**Module:** SE2A2 Signals and Telecoms

**Lecturer:** James Grimbleby
**URL:** http://www.personal.rdg.ac.uk/~stsgrimb/
**email:** j.b.grimbleby@reading.ac.uk

**Number of Lectures:** 5

**Reference text:**
Design with Operational Amplifiers and Analog Integrated Circuits (3rd edition)
Sergio Franco
McGraw-Hill 2003 ISBN 0071207031

# Analogue Filter Design

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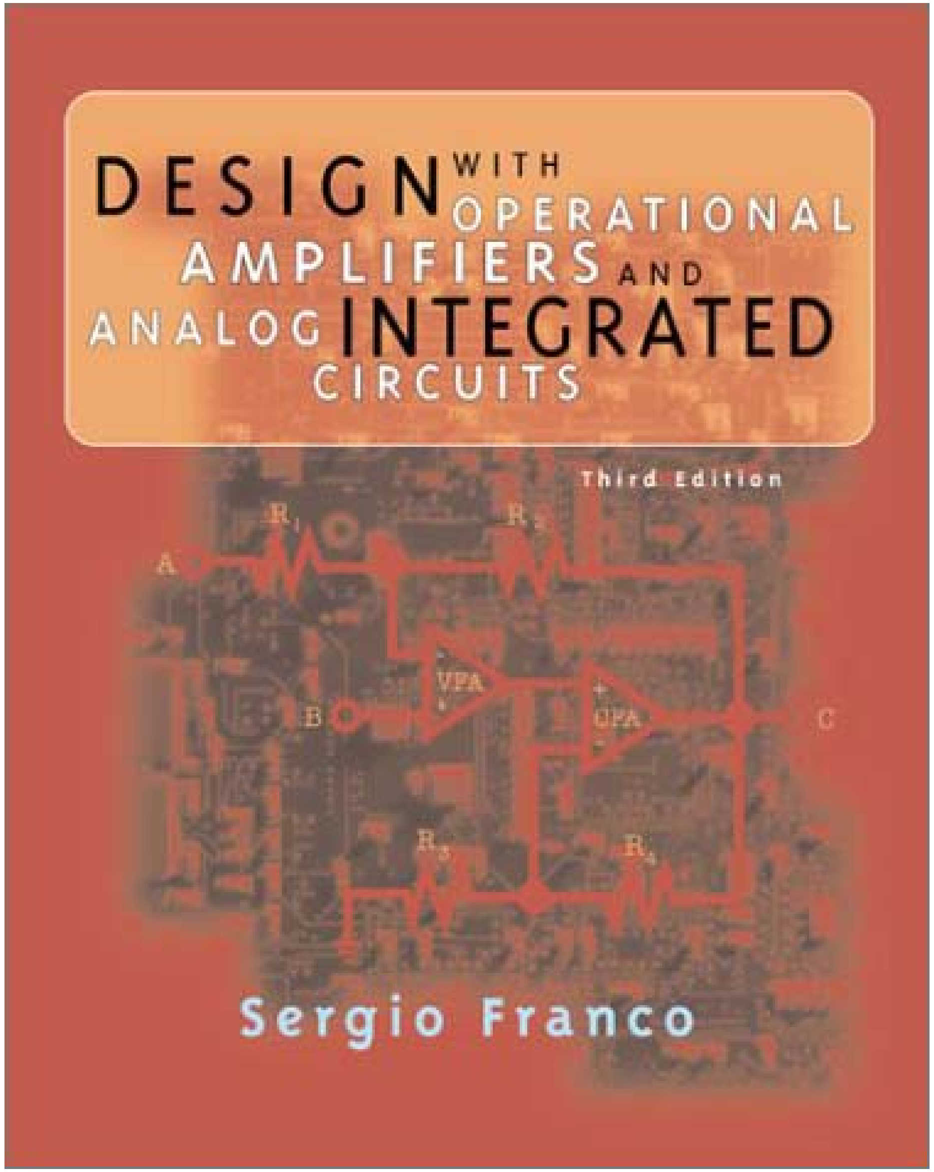
Approx £43



# Analogue Filter Design Syllabus

This course of lectures deals with the design of passive and active analogue filters

The topics that will be covered include:

- Frequency-domain filter approximations
- Filter transformations
- Passive equally-terminated ladder filters
- Passive ladder filters from filter design tables
- Active filters – cascade synthesis
- Component value sensitivity
- Copying methods
- Generalised immittance converter (GIC)
- Inductor simulation using GICs

# Analogue Filter Design Prerequities

You should be familiar with the following topics:

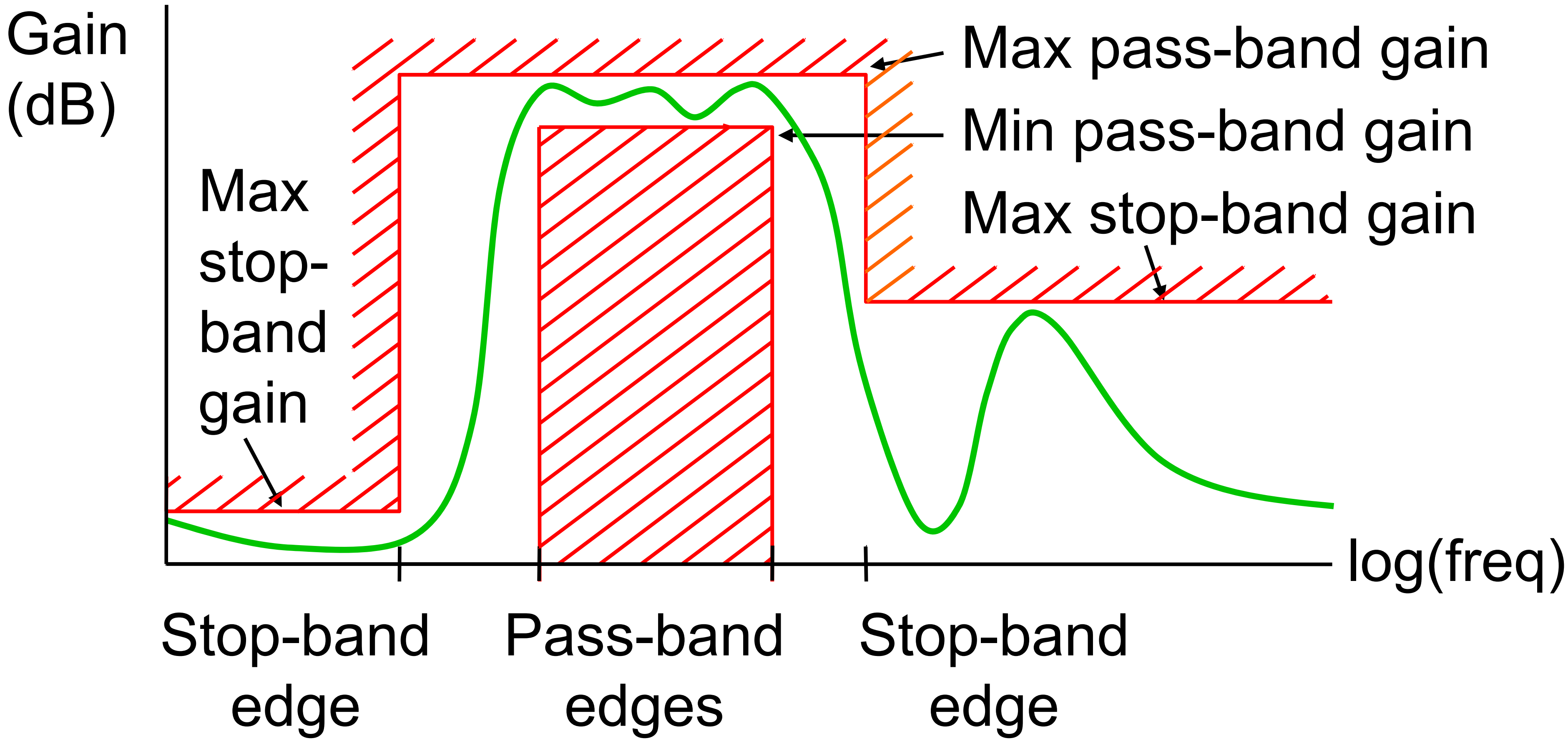
**SE1EA5: Electronic Circuits**

- Circuit analysis using Kirchhoff's Laws
- Thévenin and Norton's theorems
- The Superposition Theorem
- Semiconductor devices
- Complex impedances
- Frequency response function – gain and phase
- The infinite-gain approximation

**SE1EC5: Engineering Mathematics**

# Analogue Filter Design

Filters are normally used to modify the frequency spectrum of a signal and are therefore specified in the frequency domain:



# Analogue Filter Design

The design procedure for analogue filters has two distinct stages

In the first stage a frequency response function $H(j\omega)$ is derived which meets the specification

In the second stage an electronic circuit is designed to generate the required frequency response function

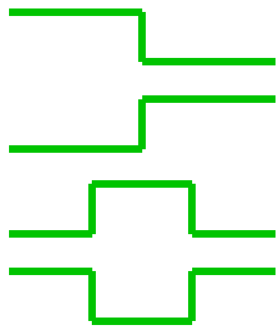
Filter circuits can be constructed entirely from passive components or can contain active elements such as operational amplifiers

# Frequency-Domain Filters

The design of frequency-domain filters usually starts by deriving a low-pass prototype filter normalised to a cut-off frequency of 1 rad/s

This prototype filter is then transformed to the required type and cut-off frequency:

Low-pass
High-pass
Band-pass
Band-stop

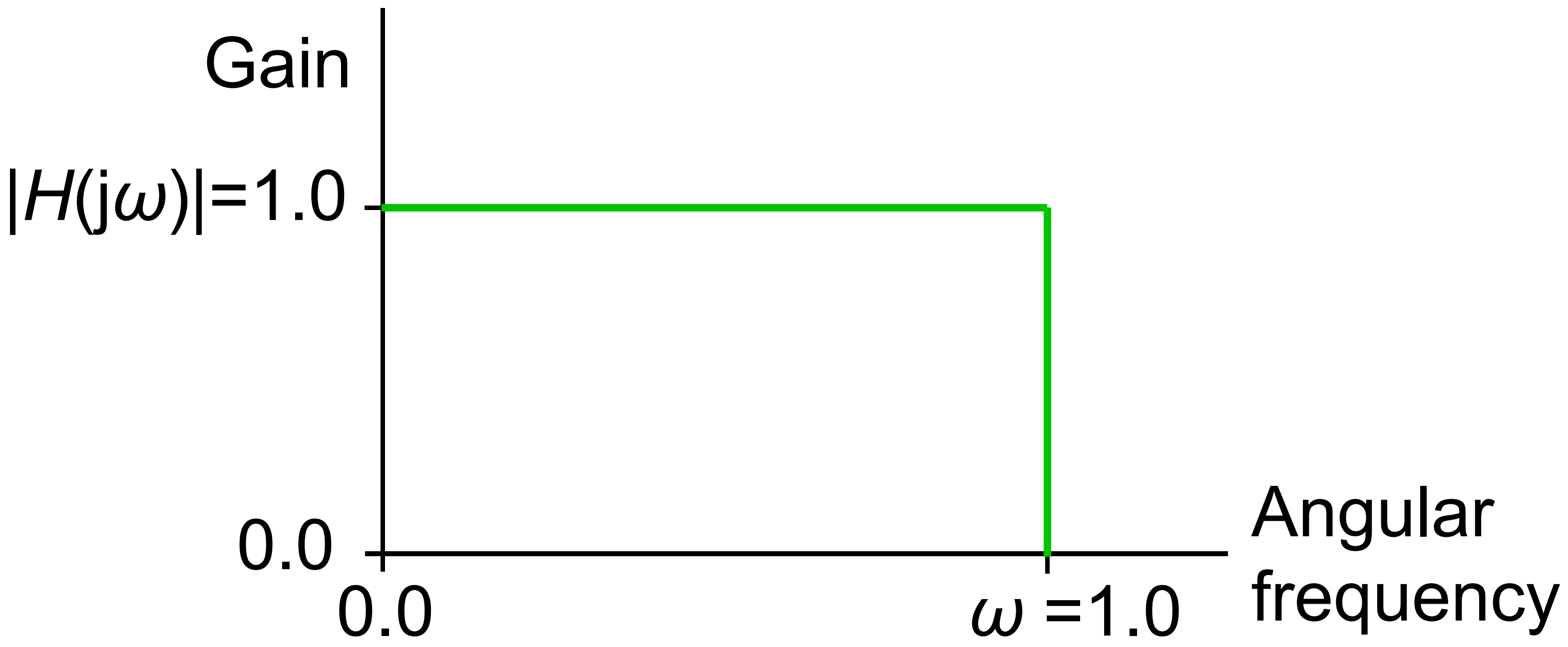


This procedure results in a frequency response function which meets the specification

# Frequency-Domain Filters



An ideal normalised sharp cut-off low-pass filter has a frequency response:




Unfortunately such a filter is not realisable and it is necessary to use approximations to the ideal response

# Frequency-Domain Filters

The frequency response function of a circuit containing no distributed elements is a rational function of j$\omega$ :

$$H(\mathrm{j}\omega) = \frac{a_0 + a_1(\mathrm{j}\omega) + a_2(\mathrm{j}\omega)^2 + .. + a_n(\mathrm{j}\omega)^n}{b_0 + b_1(\mathrm{j}\omega) + b_2(\mathrm{j}\omega)^2 + .. + b_n(\mathrm{j}\omega)^n}$$

The first stage in the design is to choose suitable values for the order $n$ and the coefficients $a_0..a_n$ and $b_0..b_n$

Five different approximations will be considered: Butterworth, Chebychev, Inverse Chebychev, Elliptic and Bessel

# Butterworth Approximation

The Butterworth approximation gain is maximally flat

That is to say the gain in the pass-band (below $\omega$=1) is as flat as possible

The gain falls off monotonically in both pass-band (below $\omega$=1) and stop-band (above $\omega$=1)

The gain of a Butterworth filter of order *n* is give by:

$$\left|H(\mathrm{j}\omega)\right| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

# Butterworth Approximation

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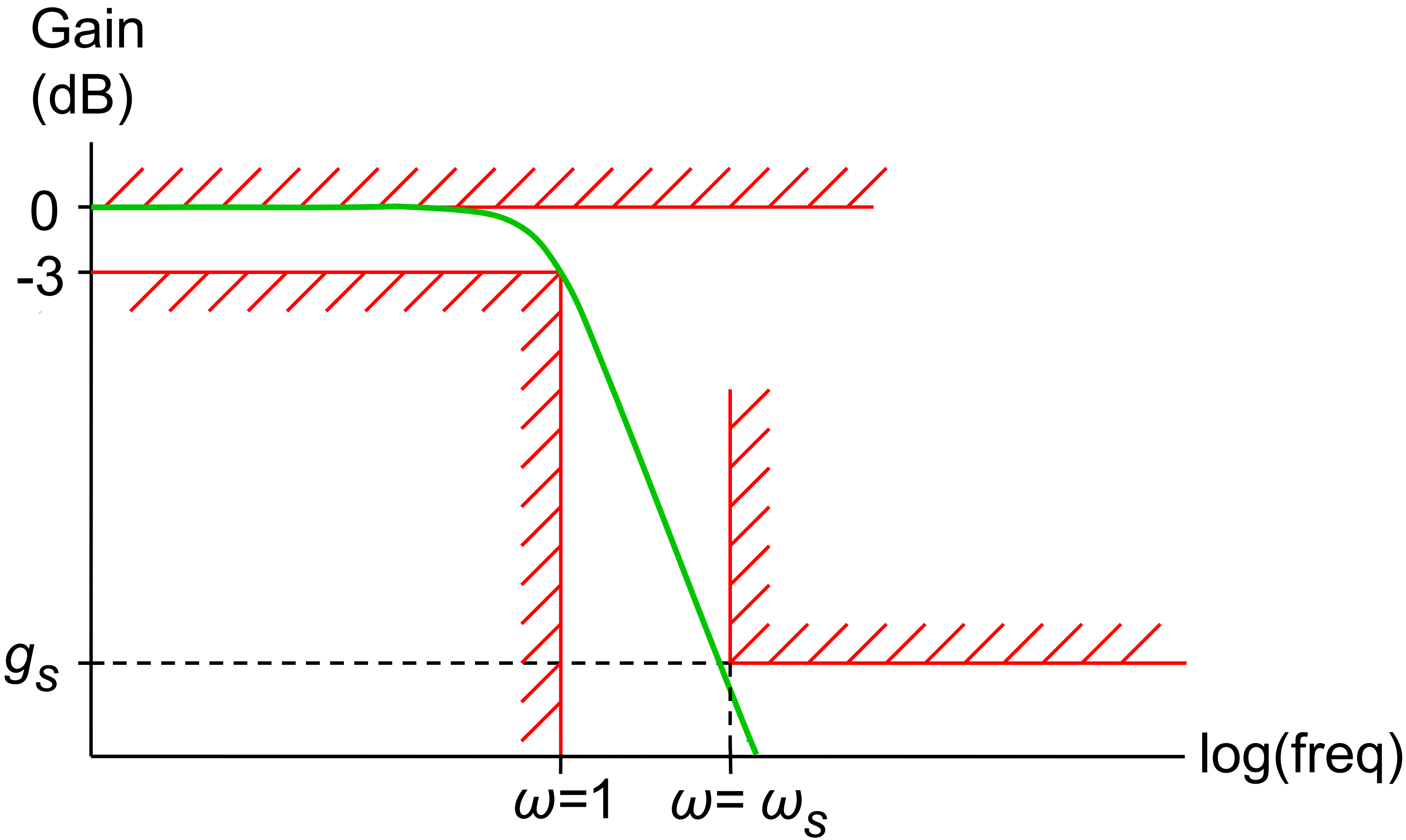

# Butterworth Approximation

The poles $p_k$ of a Butterworth frequency response function are given by:

$$p_k = x_k + \mathrm{j}y_k$$

where:

$$x_k = -\sin\frac{(2k-1)}{2n}\pi \qquad y_k = \cos\frac{(2k-1)}{2n}\pi$$

$$k = 1, 2, \ldots, n$$

Values of $k$ from 1 to $n$ are substituted into this formula giving the $n$ poles

# Butterworth Approximation

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The poles of the Butterworth response $p_1$, $p_2$, ... , $p_n$ are then combined to give the Butterworth frequency response function:

$$H(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega - p_1)(\mathrm{j}\omega - p_2)..(\mathrm{j}\omega - p_n)}$$

The Butterworth approximation is an all-pole response

That is, the numerator consists simply of a constant and there are no zeros

# Butterworth Approximation

3rd-order Butterworth approximation:

$$p_k = -\sin\frac{(2k-1)}{6}\pi + \mathrm{j}\cos\frac{(2k-1)}{6}\pi$$

| $k$ | $p_k$ |
|---|---|
| 1 | $-0.5$ +j 0.866 |
| 2 | $-1.0$ +j 0.0 |
| 3 | $-0.5$ $-$j 0.866 |

# Butterworth Approximation

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Poles of $H(\mathrm{j}\omega)$:

$-0.5 + \mathrm{j}\ 0.866 \qquad -0.5 - \mathrm{j}\ 0.866 \qquad -1.0 + \mathrm{j}\ 0.0$

Combining the poles:

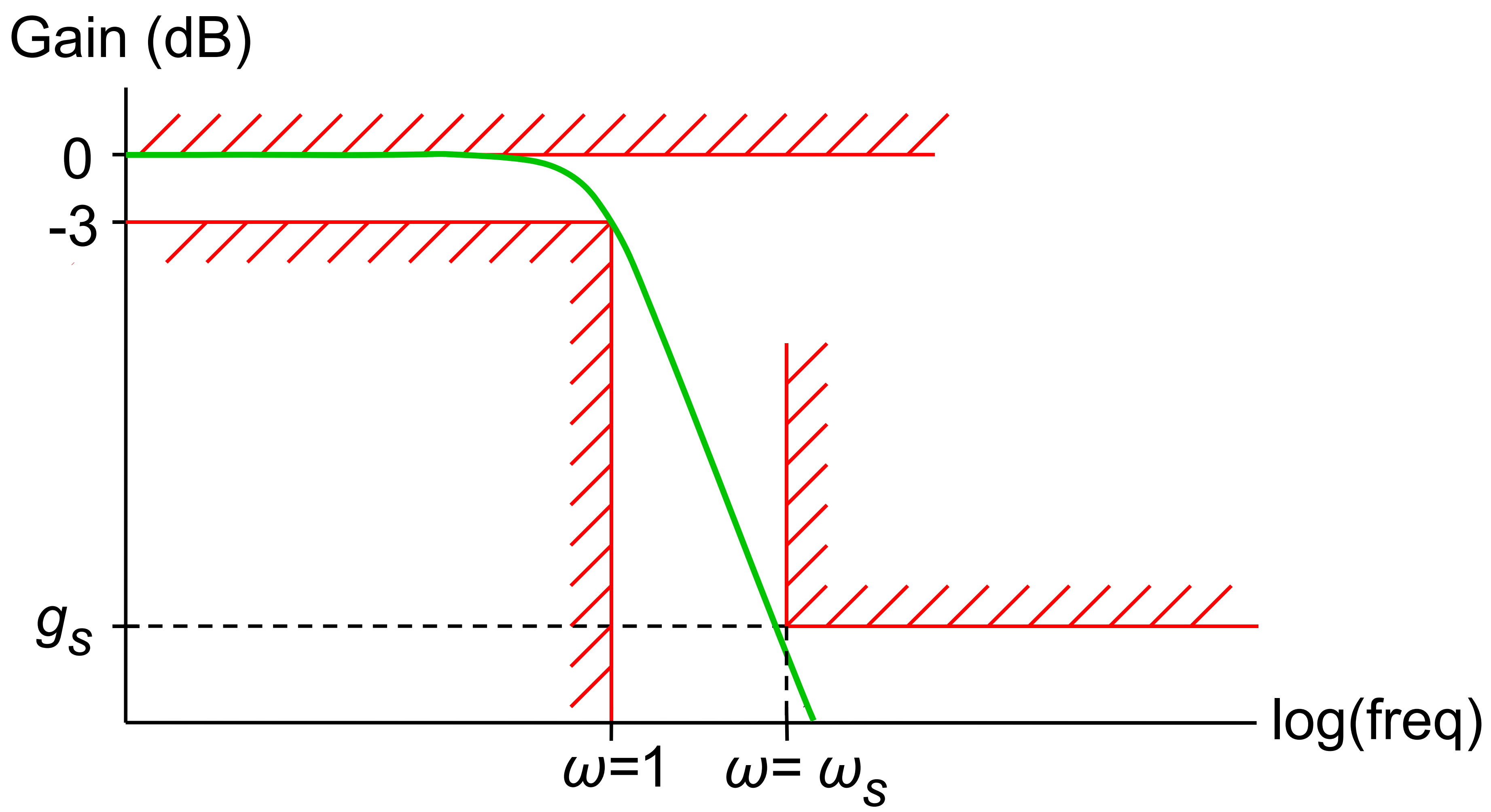
$$H(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega + 0.5 - \mathrm{j}0.866)(\mathrm{j}\omega + 0.5 + \mathrm{j}0.866)(\mathrm{j}\omega + 1.0)}$$

$$= \frac{1}{((\mathrm{j}\omega)^2 + \mathrm{j}\omega + 1.0)(\mathrm{j}\omega + 1.0)}$$

$$= \frac{1}{(\mathrm{j}\omega)^3 + 2.0(\mathrm{j}\omega)^2 + 2.0\mathrm{j}\omega + 1.0}$$

# Butterworth Approximation

What order *n* is required to meet specification?



# Butterworth Approximation

The gain $g$ of a Butterworth filter, expressed in dB, is given by:

$$g = 20\log_{10}\left|H(\mathrm{j}\omega)\right|$$

$$= 20\log_{10}(1+\omega^{2n})^{-1/2}$$

$$= -10\log_{10}(1+\omega^{2n})$$

The stop-band edge is at $\omega_s$, and the stop-band gain is required to be less than $g_s$ dB:

$$g_s \geq -10\log_{10}(1+\omega_s{}^{2n}) \approx -10\log_{10}\omega_s{}^{2n}$$

$$g_s \geq -20n\log_{10}\omega_s$$

$$n \geq \frac{-g_s}{20\log_{10}\omega_s}$$

# Butterworth Approximation

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Example:

Pass-band gain: $g_p \geq -3$ dB
Stop-band gain: $g_s \leq -40$ dB
Pass-band edge: $\omega_p = 1.0$
Stop-band edge: $\omega_s = 1.5$

Using the formula for the filter order:

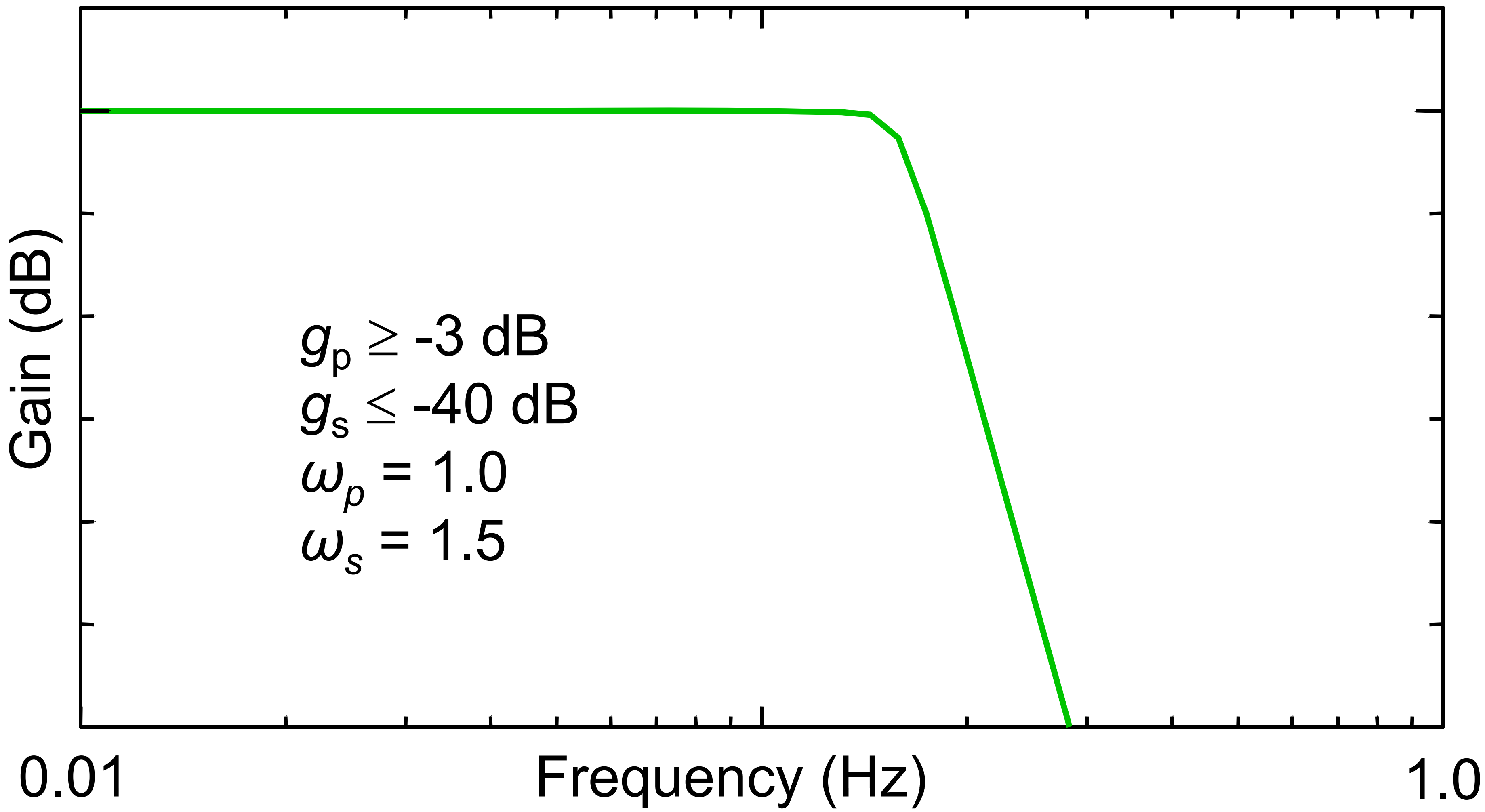
$$n \geq \frac{-g_s}{20\log_{10}\omega_s} = \frac{40.0}{20\log_{10}1.5}$$

$$\geq 11.36$$

Thus a Butterworth approximation of order 12 is required to meet the specification

# Butterworth Approximation

Response of 12th-order Butterworth approximation:



# Chebychev Approximation

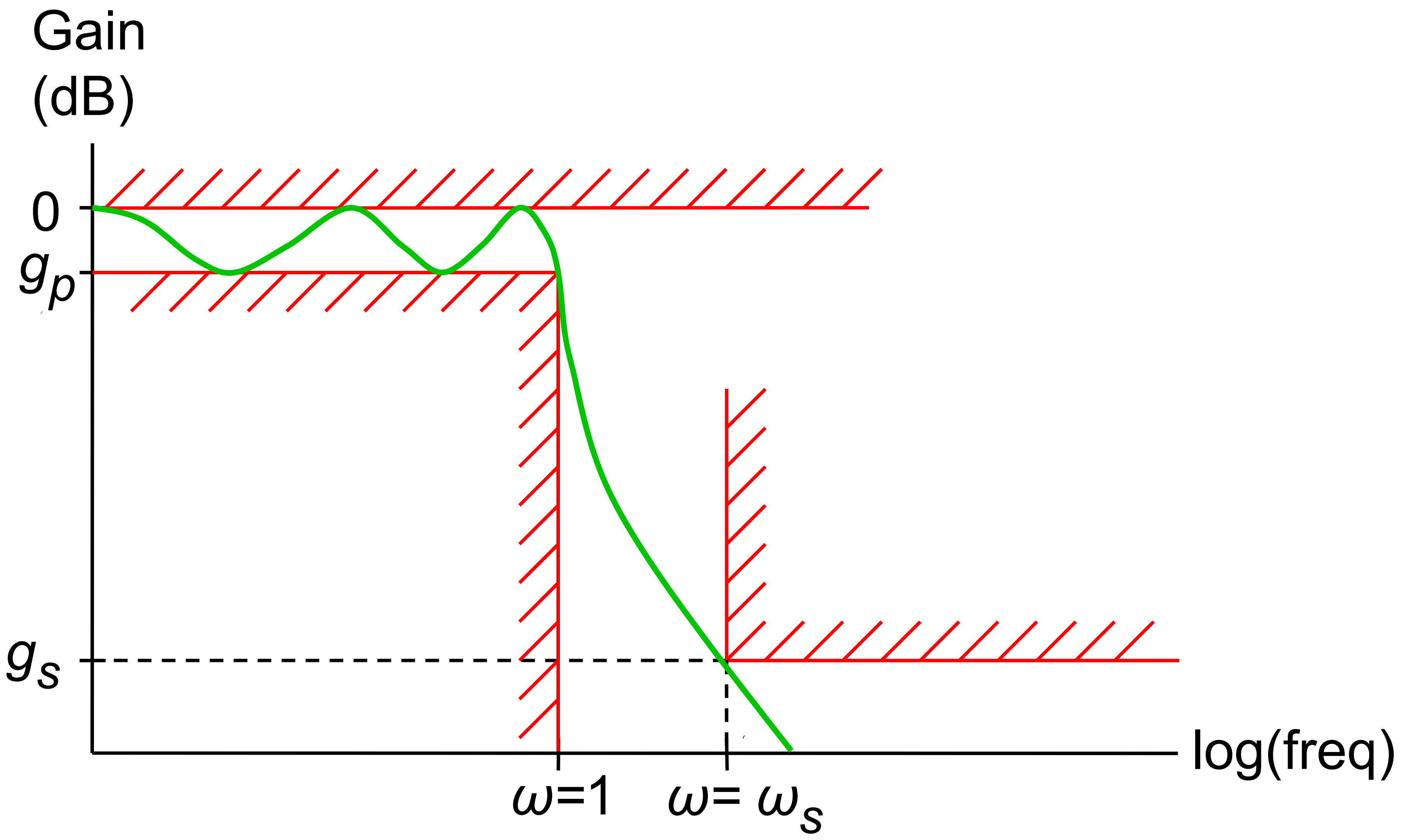
The Chebychev approximation gain oscillates between 0 dB and $g_p$ dB in the pass-band (below $\omega$=1)

In the stop-band (above $\omega$=1) the gain falls off monotonically

The Chebychev approximation is not a single approximation for each $n$, but a group of approximations with different values of the pass-band ripple $g_p$

Like the Butterworth approximation, the Chebychev approximation is an all-pole response

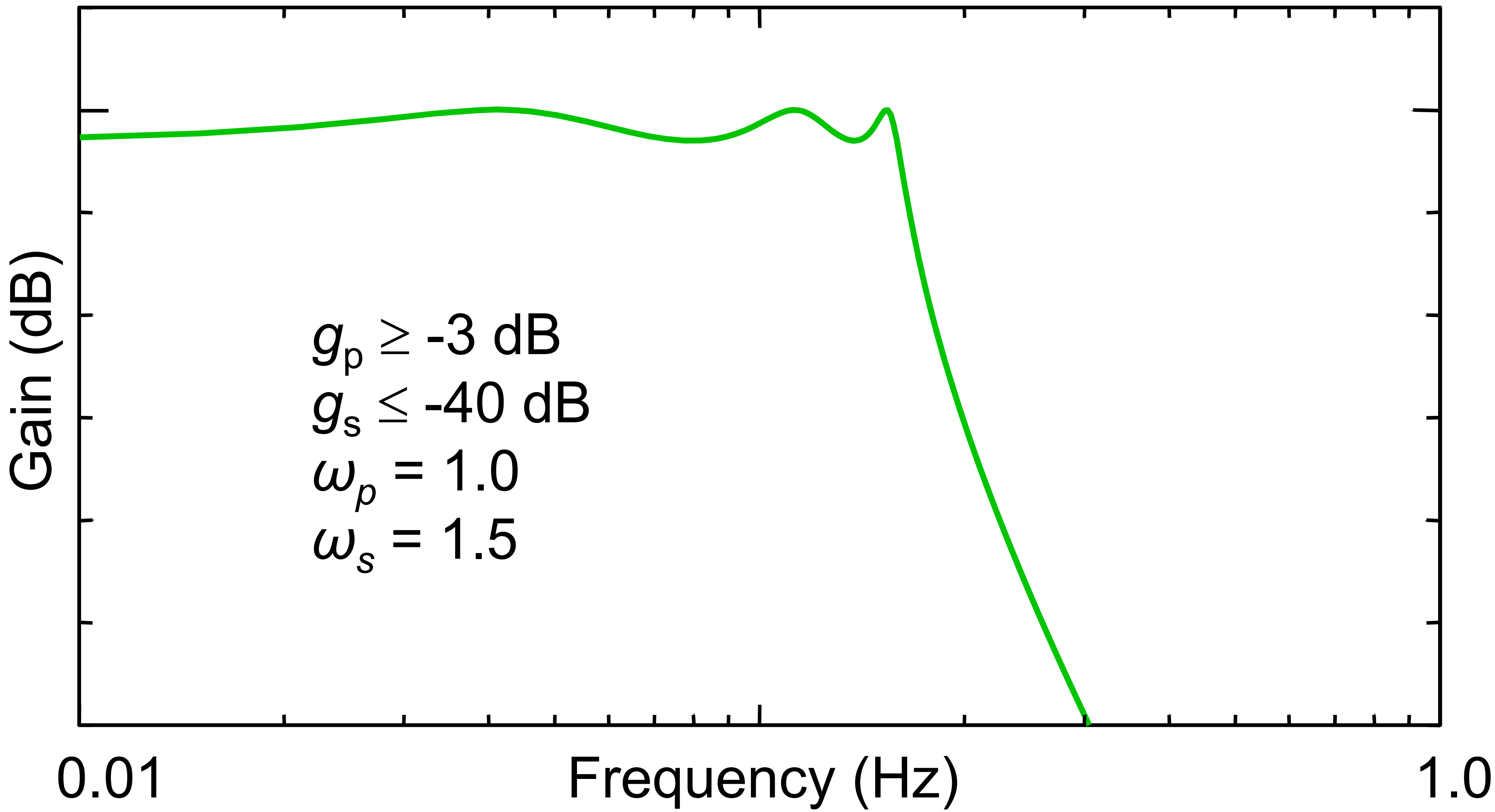
# Chebychev Approximation



# Chebychev Approximation



Response of 6th-order Chebychev approximation:



# Inverse Chebychev Approx

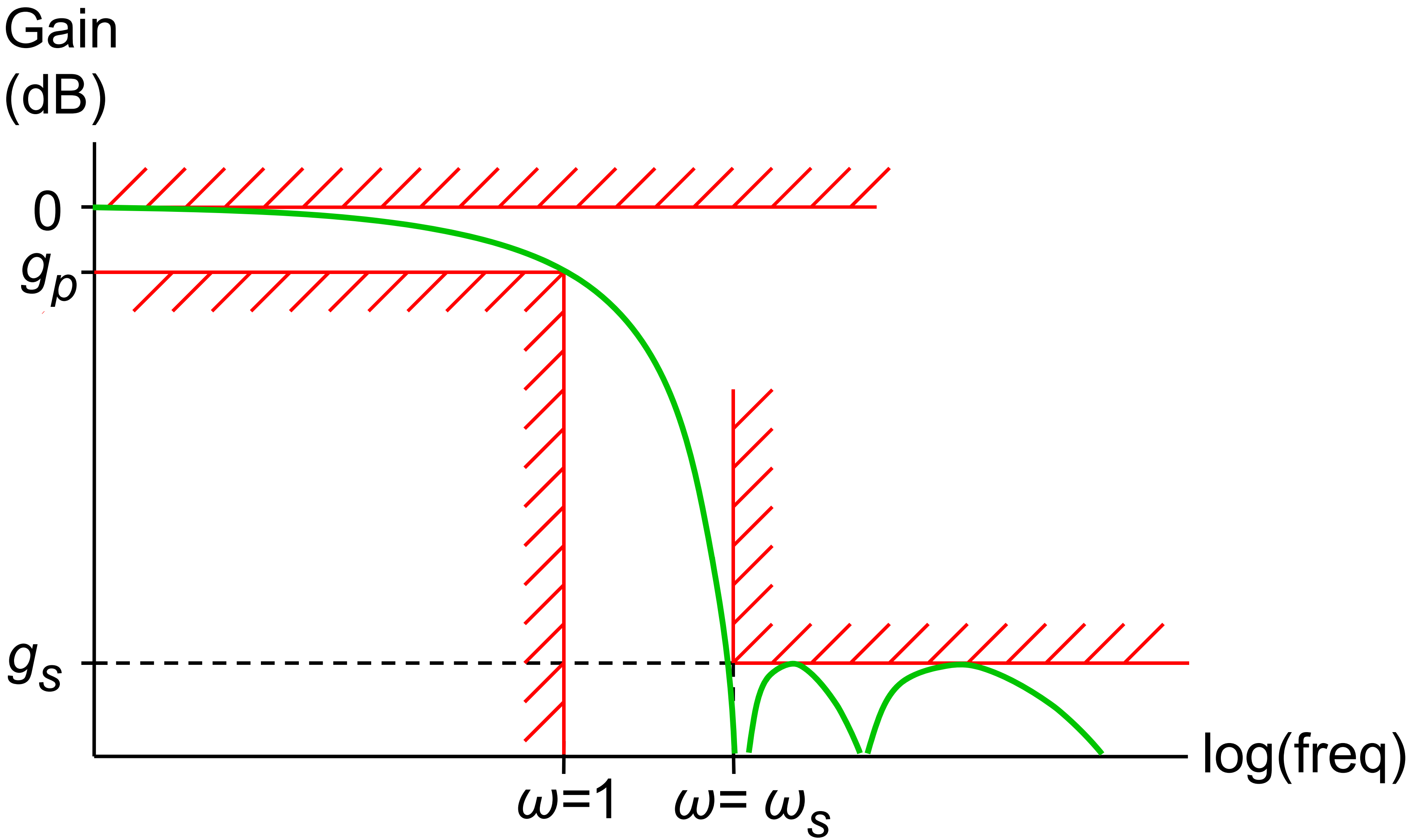
The Inverse Chebychev approximation gain falls off monotonically in the pass-band (below $\omega$=1)

In the stop-band (above $\omega$=1) the gain at first falls to -∞ dB, and then oscillates between -∞ dB and $g_s$ dB

The Inverse Chebychev approximation is not a single approximation for each $n$, but a group of approximations with different values of the stop-band ripple $g_s$

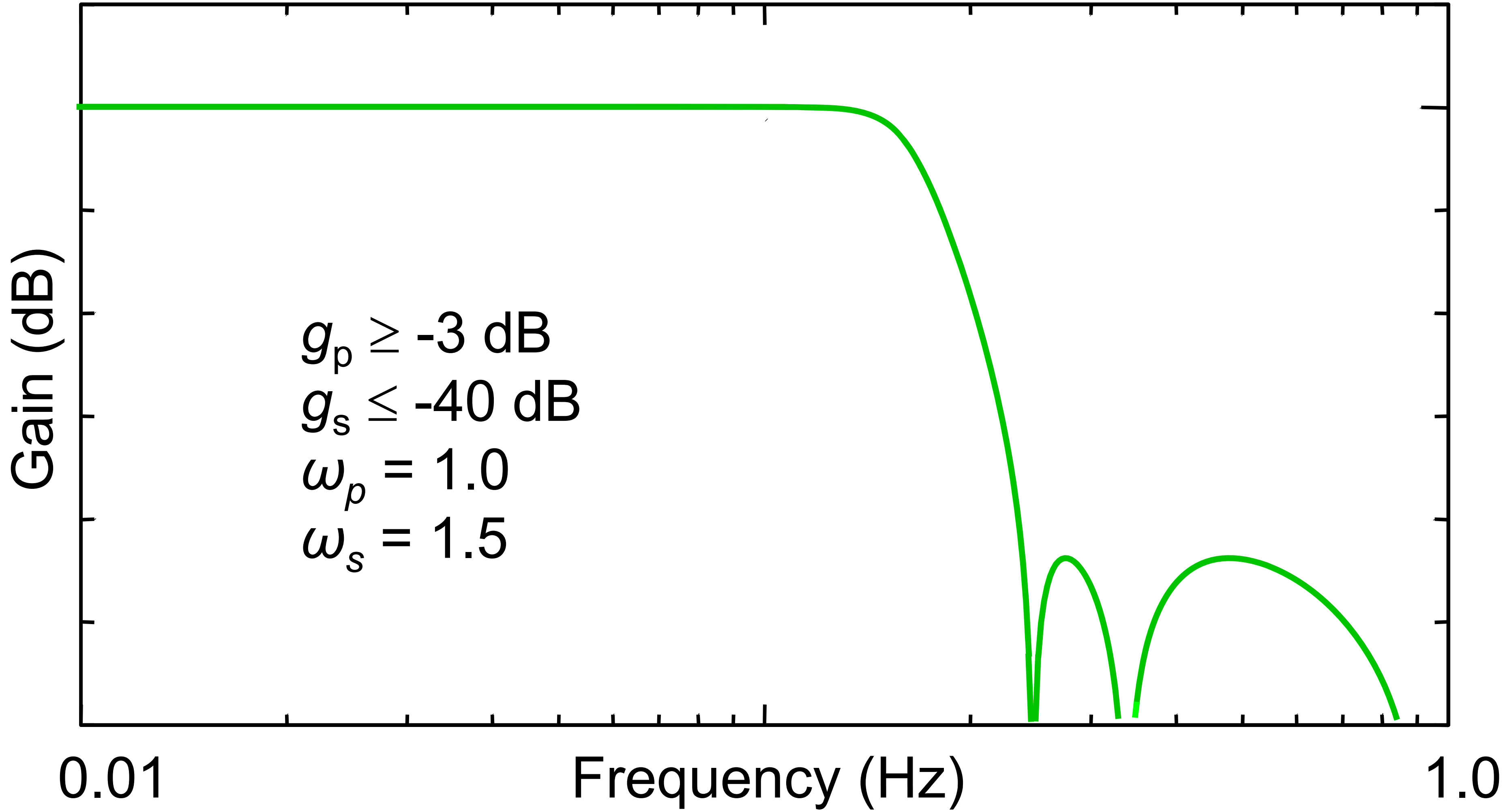
The Inverse Chebychev approximation has imaginary zeros

# Inverse Chebychev Approx



# Inverse Chebychev Approx

Response of 6th-order Inverse Chebychev approximation:



# Elliptic Approximation

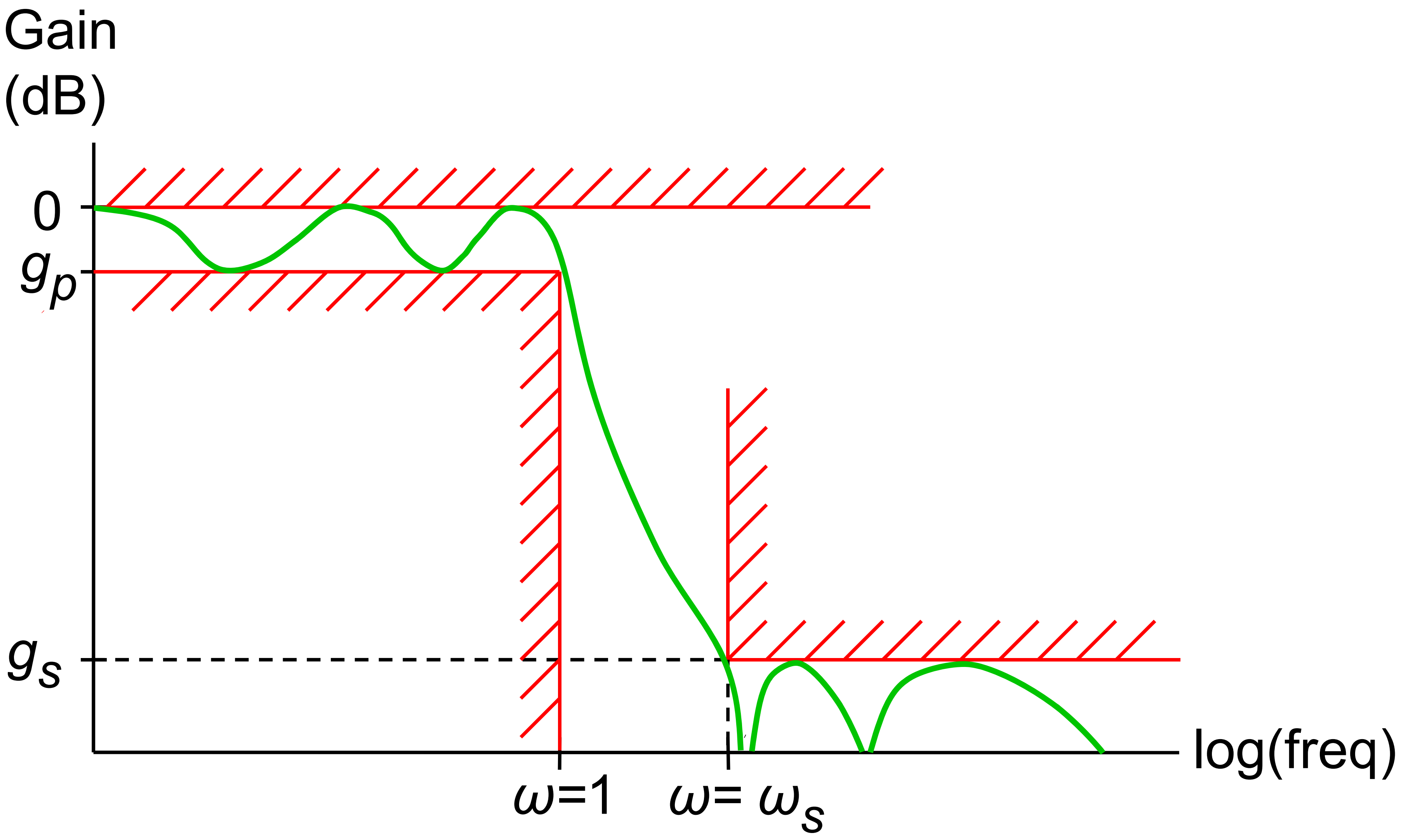
The elliptic response gain oscillates between 0 and $g_p$ dB in the pass-band (below $\omega$=1)

The gain in the stop-band (above $\omega$=1) oscillates between $g_s$ dB and -∞ dB

The elliptic approximation has imaginary zeros

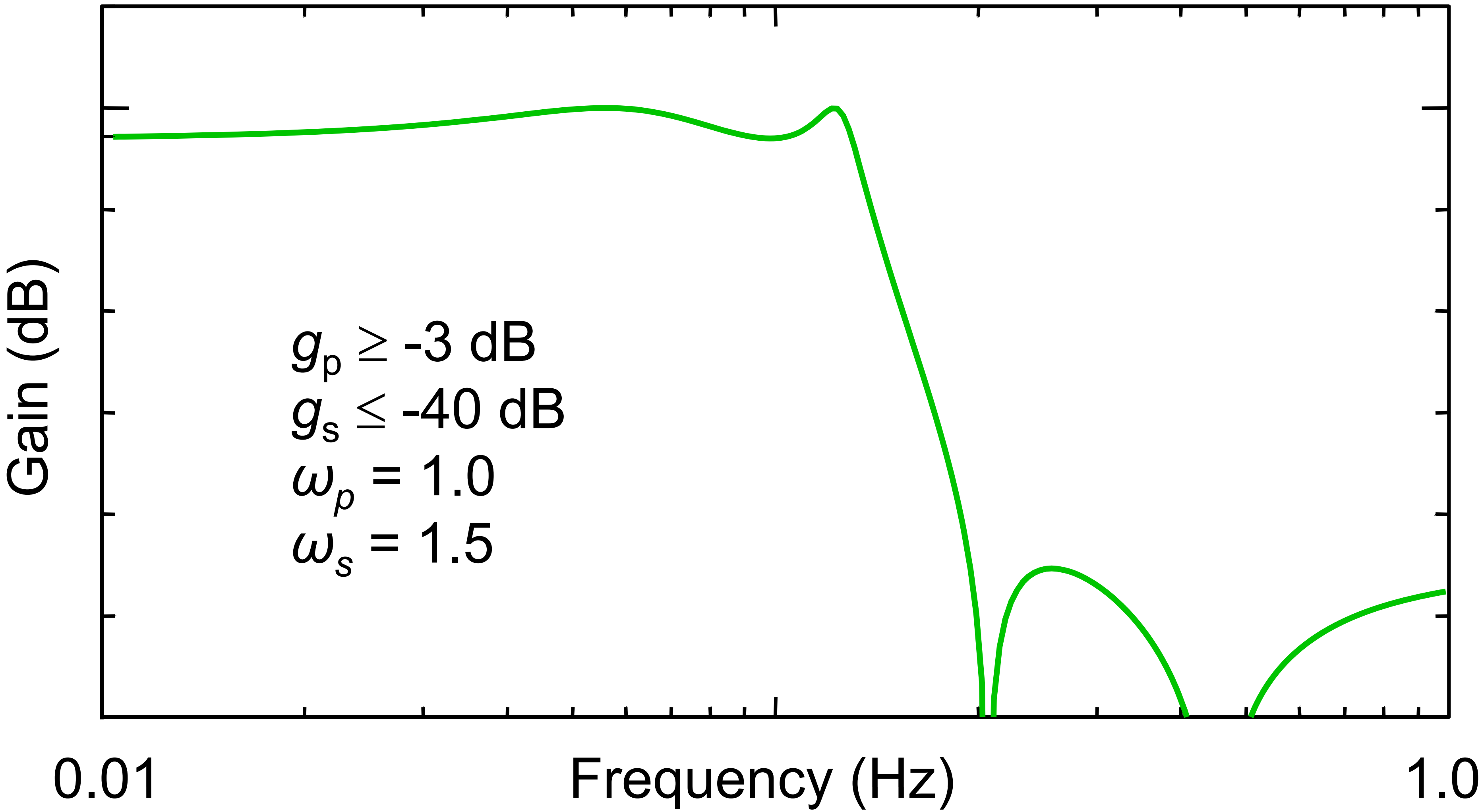
For a given filter specification the elliptic approximation gives the lowest order frequency response function

# Elliptic Approximation



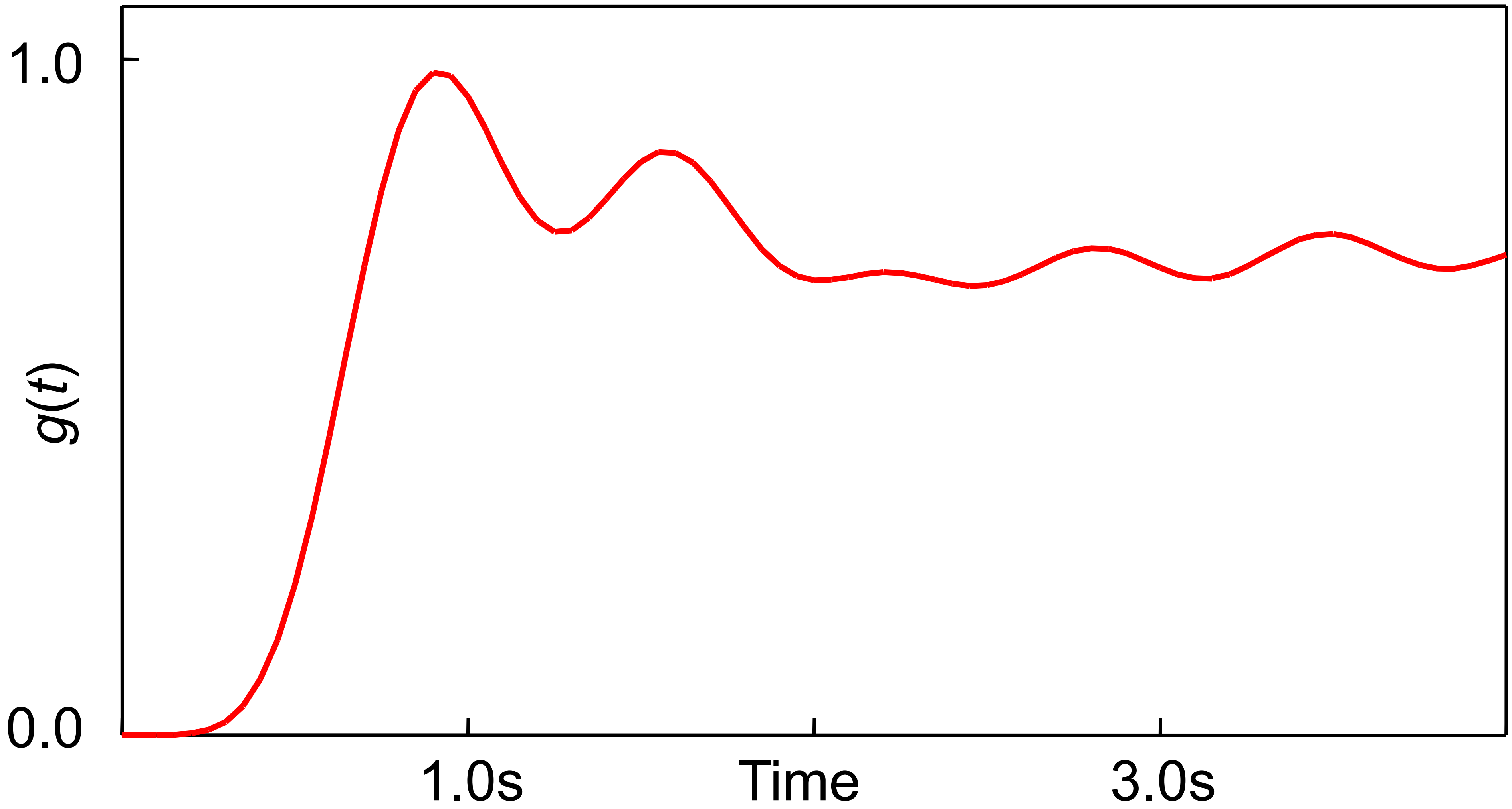
# Elliptic Approximation

Response of 4th-order elliptic approximation:



# Time-Domain Response

Unit-step response of 6th-order Chebychev:



# Bessel Approximation

The Bessel approximation is used where a frequency-domain filter is required which also has a good time-domain behaviour

All filters generate a frequency-dependent phase shift

In a Bessel filter the phase varies approximately linearly with frequency and the different frequency components are delayed by the same amount

A time-domain waveform is therefore delayed, but is not seriously distorted

# Bessel Approximation

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The Bessel approximation has an all-pole frequency response function:

$$H(\mathrm{j}\omega) = \frac{a_0}{b_0 + b_1(\mathrm{j}\omega) + b_2(\mathrm{j}\omega)^2 + .. + b_n(\mathrm{j}\omega)^n}$$

In the high-frequency limit $\omega \to \infty$:

$$\left|H(\mathrm{j}\omega)\right| \approx \frac{a_0}{b_n \omega^n}$$

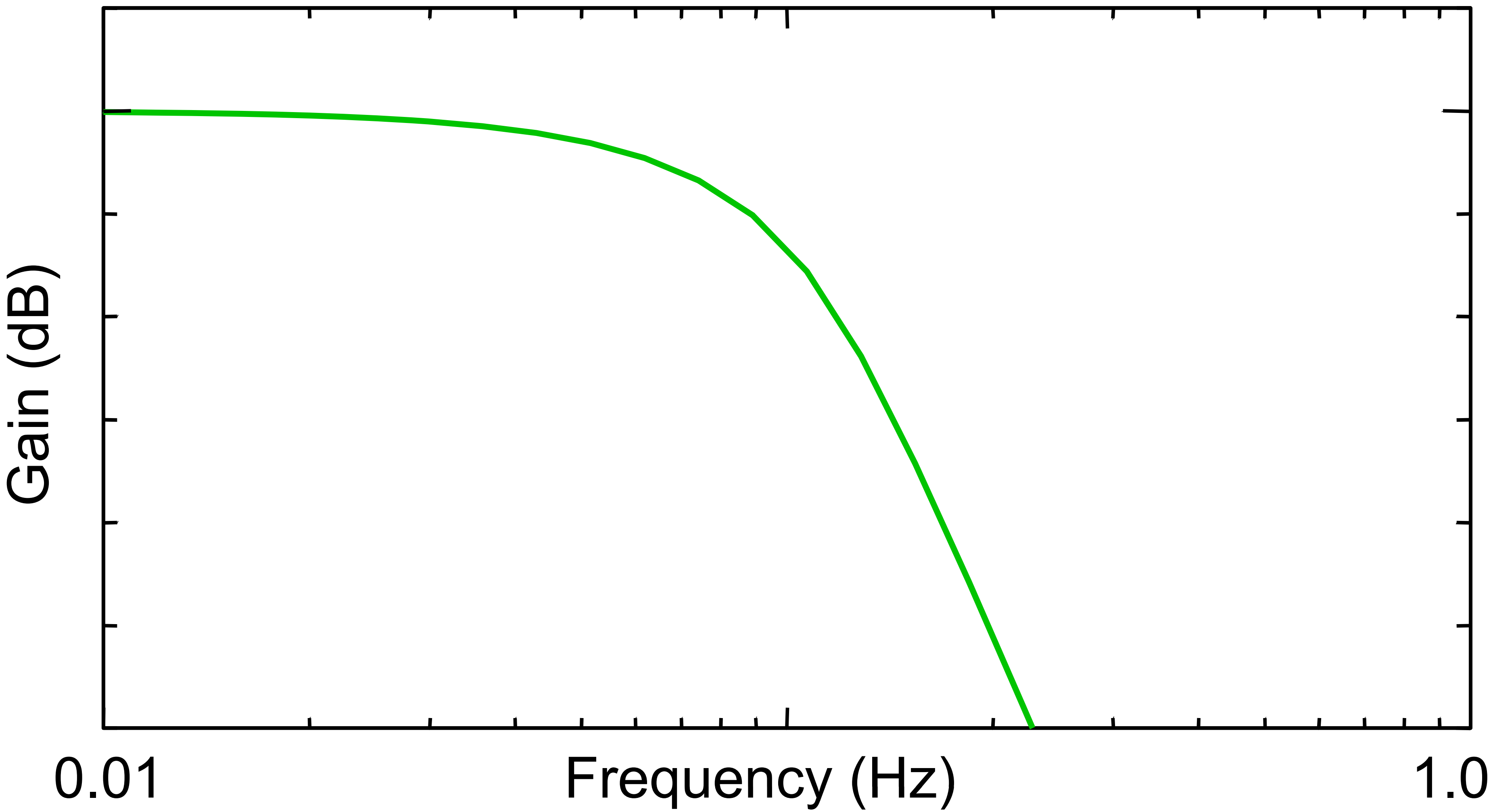
The coefficients are related to Bessel functions:

$$a_0 = b_0$$

$$b_i = \frac{(2n-i)!}{2^{n-i}\, i!(n-i)!}$$

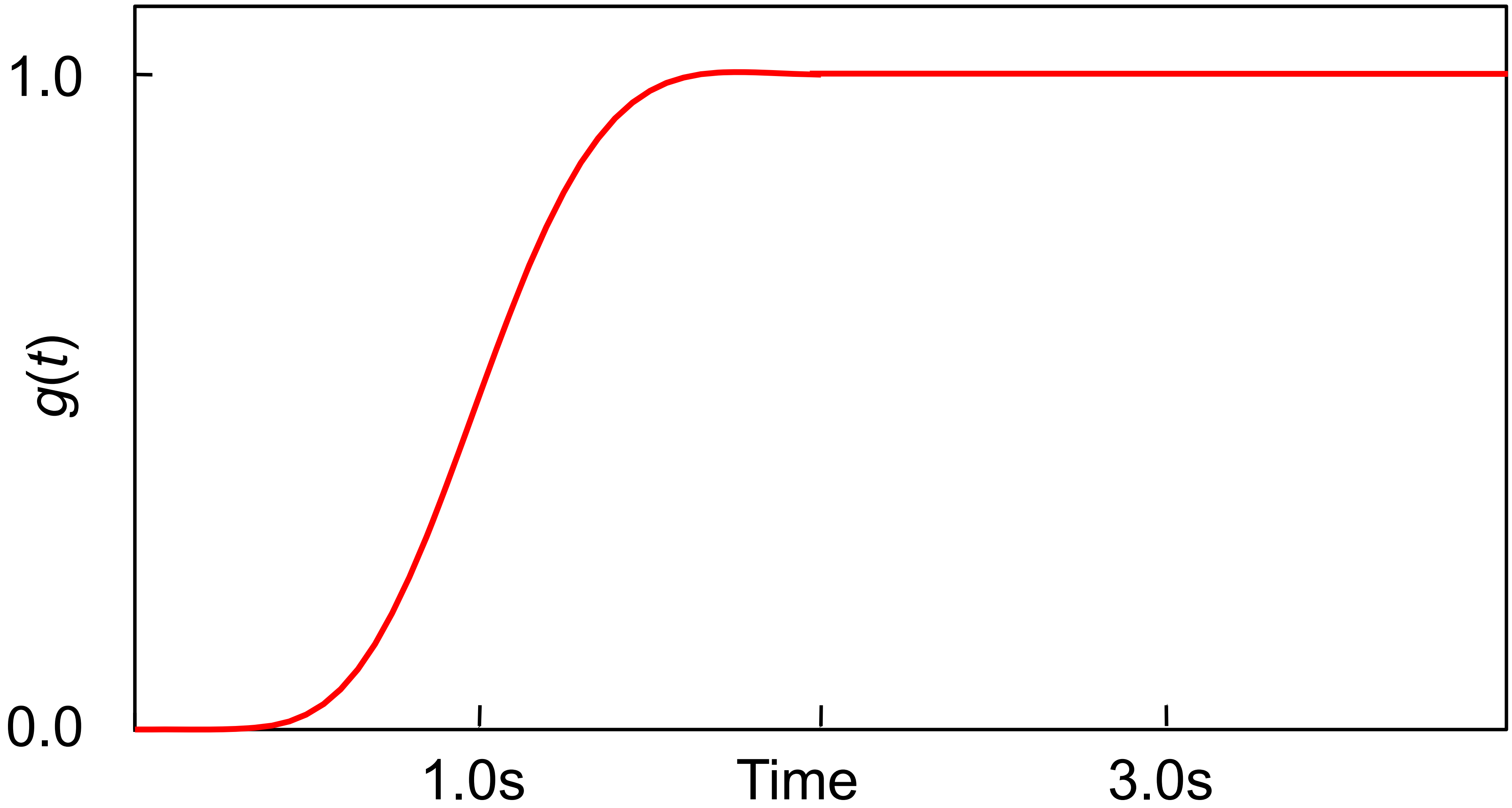
# Bessel Approximation

Response of 8th-order Bessel approximation:



# Bessel Approximation

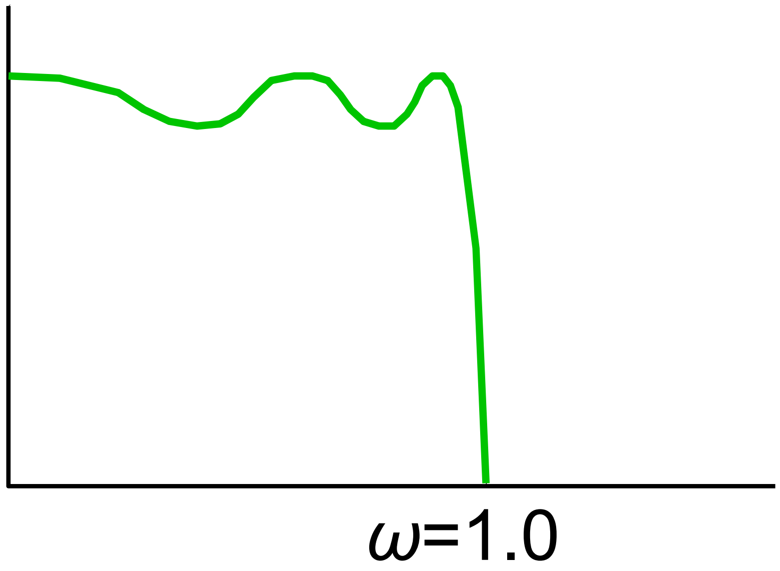
Unit-step response of 8th-order Bessel:

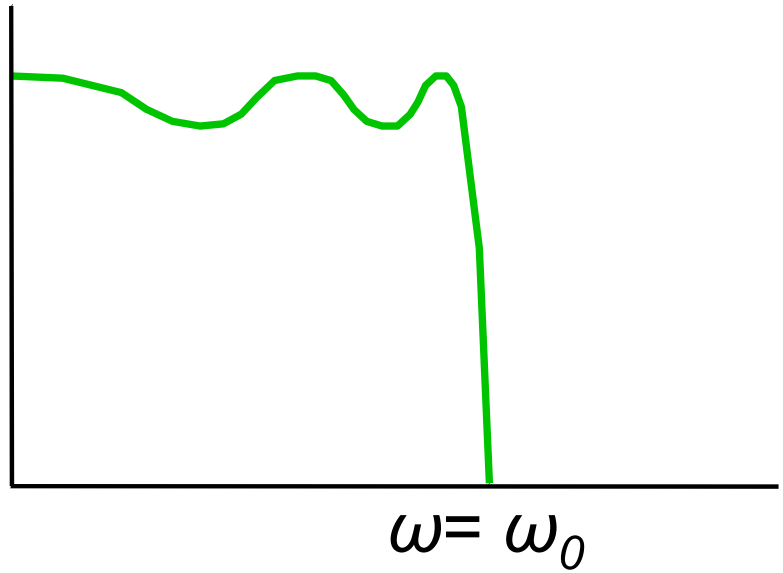


# Low-Pass to Low-Pass Transformation

This transformation shifts the cut-off frequency to $\omega_0$:

$$j\omega \rightarrow \frac{j\omega}{\omega_0}$$



# Low-Pass to Low-Pass Transformation

Design example: A Chebychev low-pass filter is required with the following specification:

| | |
|---|---|
| Pass-band gain: | $g_p \geq -3$ dB |
| Stop-band gain: | $g_s \leq -50$ dB |
| Pass-band edge: | $f_p = 1000$ Hz |
| Stop-band edge: | $f_s = 2000$ Hz |

This corresponds to a normalised low-pass filter with $\omega_p$=1.0, $\omega_s$=2.0

# Low-Pass to Low-Pass Transformation

Normalised low-pass filter:

| | |
|---|---|
| Pass-band gain: | $g_p \geq -3$ dB |
| Stop-band gain: | $g_s \leq -50$ dB |
| Pass-band edge: | $\omega_p=1.0$ |
| Stop-band edge: | $\omega_s=2.0$ |

This specification can be met by a 5th-order Chebychev approximation with 3 dB pass-band ripple:

$$H(j\omega) = \frac{0.06265}{(j\omega)^5 + 0.5745(j\omega)^4 + 1.415(j\omega)^3 + 0.5489(j\omega)^2 + 0.4080(j\omega) + 0.06265}$$

# Low-Pass to Low-Pass Transformation

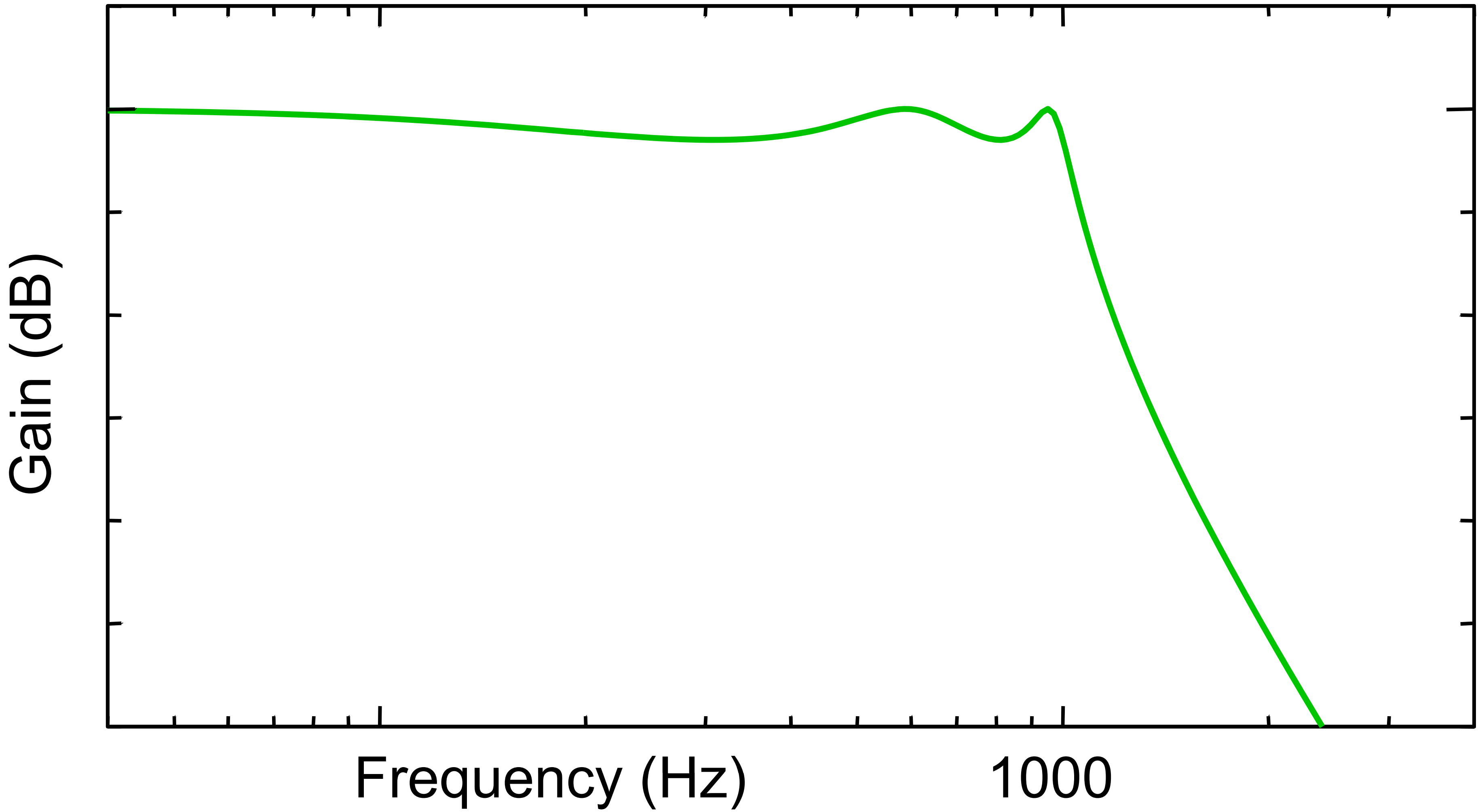
Applying the low-pass to low-pass transformation with $\omega_0$=2π×1000=6283:

$$H(j\omega)=\frac{0.06265}{\left\{\frac{j\omega}{6283}\right\}^5+0.5745\left\{\frac{j\omega}{6283}\right\}^4+1.415\left\{\frac{j\omega}{6283}\right\}^3+0.5489\left\{\frac{j\omega}{6283}\right\}^2+0.4080\left\{\frac{j\omega}{6283}\right\}+0.06265}$$

$$=\frac{0.06265}{1.021\times10^{-19}(j\omega)^5+3.686\times10^{-16}(j\omega)^4+5.705\times10^{-12}(j\omega)^3+1.390\times10^{-8}(j\omega)^2+6.493\times10^{-5}(j\omega)+0.06265}$$

# Low-Pass to Low-Pass Transformation

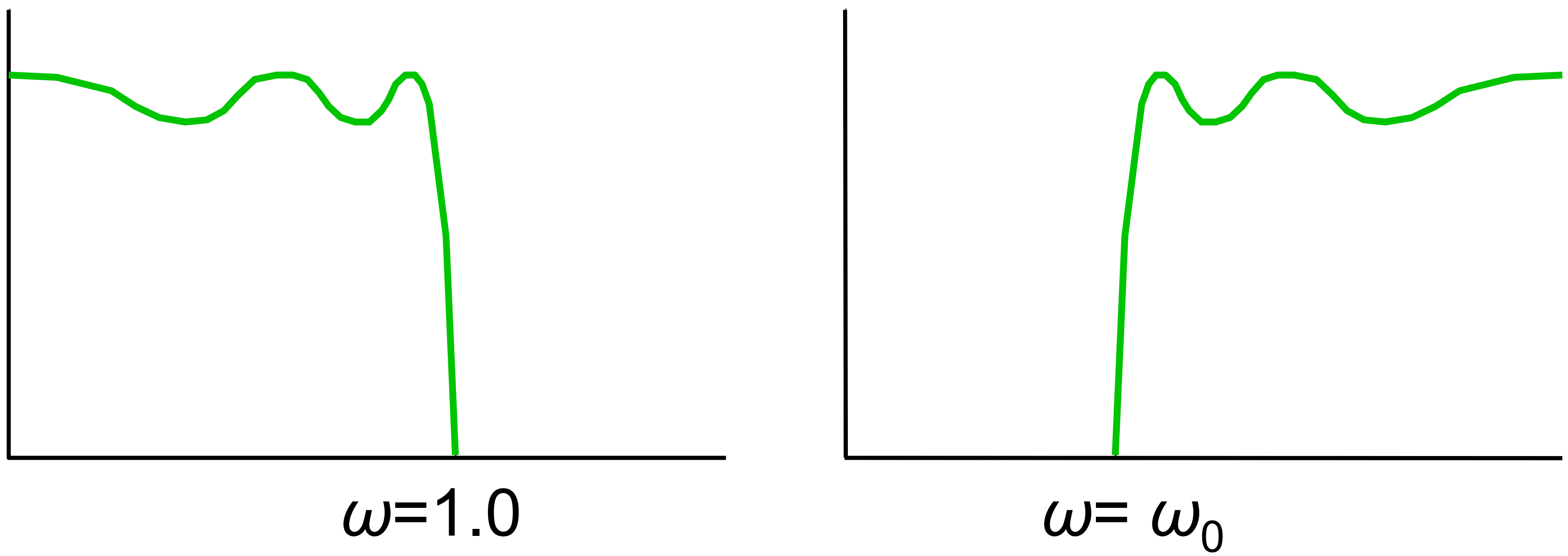
5th-order Chebychev low-pass with cut-off frequency 1000 Hz :



# Low-Pass to High-Pass Transformation

This transformation converts to a high-pass response and shifts the cut-off frequency to $\omega_0$:

$$j\omega \rightarrow \frac{\omega_0}{j\omega}$$



# Low-Pass to High-Pass Transformation

Design example: A Chebychev high-pass filter is required with the following specification:

| | |
|---|---|
| Pass-band gain: | $g_p \geq -3$ dB |
| Stop-band gain: | $g_s \leq -25$ dB |
| Pass-band edge: | $f_p = 2000$ Hz |
| Stop-band edge: | $f_s = 1000$ Hz |

This corresponds to a normalised low-pass filter with $\omega_p=1.0$, $\omega_s=2.0$

# Low-Pass to High-Pass Transformation

Normalised low-pass filter:

Pass-band gain: $g_p \geq$ -3 dB
Stop-band gain: $g_s \leq$ -25 dB
Pass-band edge: $\omega_p$=1.0
Stop-band edge: $\omega_s$=2.0

This specification can be met by a 3rd-order Chebychev approximation with 3 dB pass-band ripple:

$$H(j\omega) = \frac{0.2506}{(j\omega)^3 + 0.5972(j\omega)^2 + 0.9284(j\omega) + 0.2506}$$

# Low-Pass to High-Pass Transformation

$$H(j\omega) = \frac{0.2506}{(j\omega)^3 + 0.5972(j\omega)^2 + 0.9284(j\omega) + 0.2506}$$

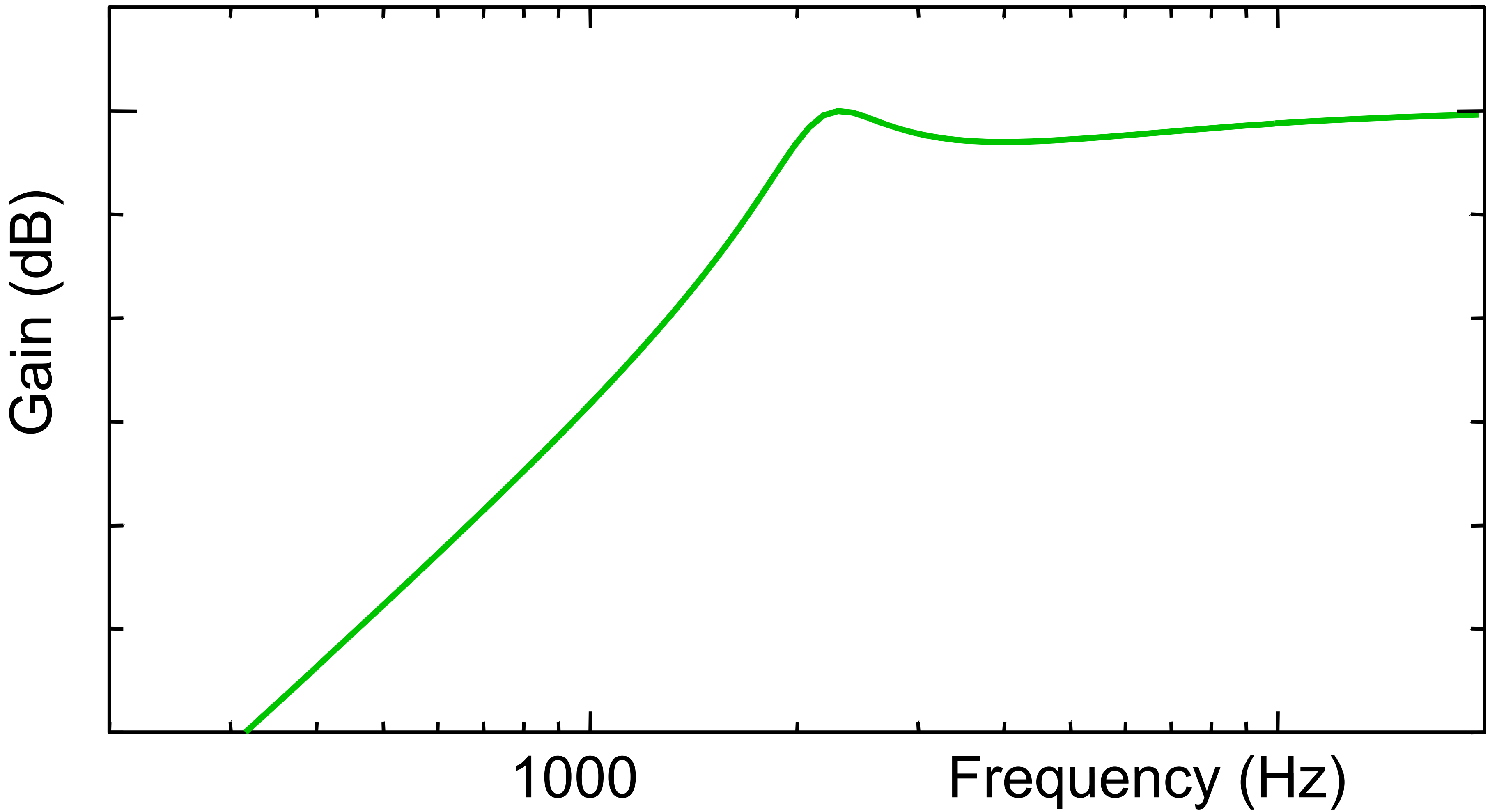
Applying the low-pass to high-pass transformation with $\omega_0 = 2\pi \times 2000$:

$$H(j\omega) = \frac{0.2506}{\left\{\frac{12566}{j\omega}\right\}^3 + 0.5972\left\{\frac{12566}{j\omega}\right\}^2 + 0.9284\left\{\frac{12566}{j\omega}\right\} + 0.2506}$$

$$= \frac{0.2506(j\omega)^3}{1.984 \times 10^{12} + 9.431 \times 10^7 (j\omega) + 1.167 \times 10^4 (j\omega)^2 + 0.2506(j\omega)^3}$$

# Low-Pass to High-Pass Transformation

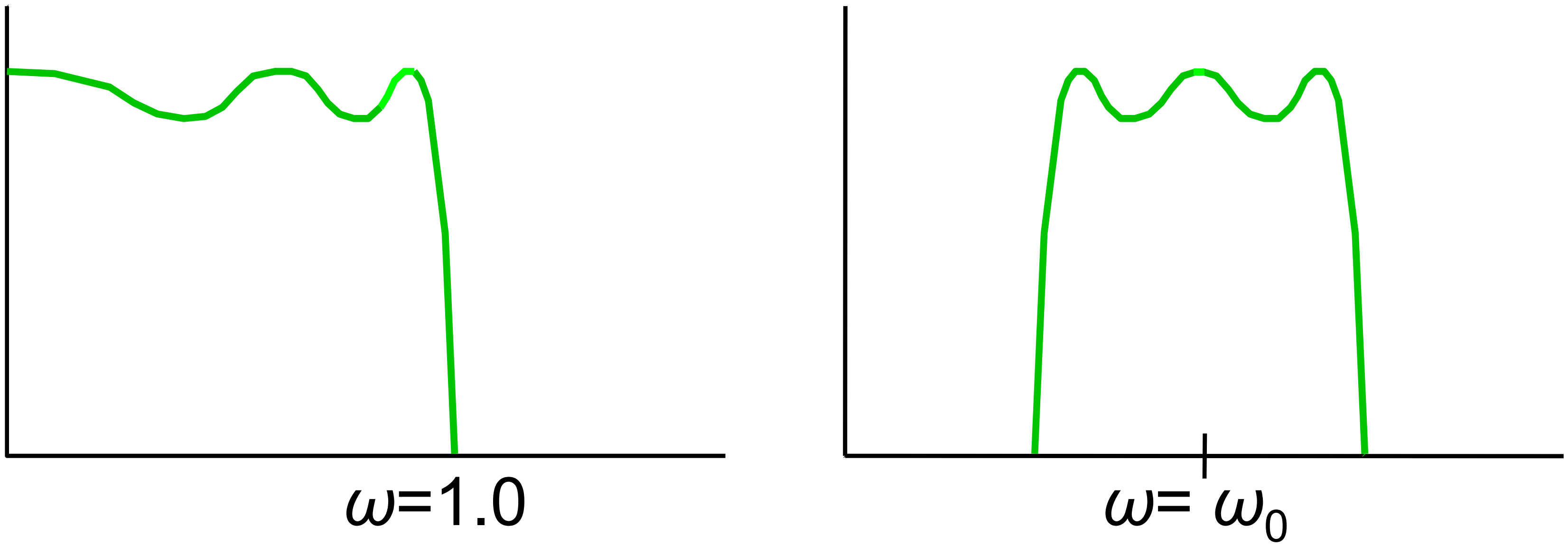
3rd-order Chebychev high-pass with cut-off frequency 2000 Hz :



# Low-Pass to Band-Pass Transformation

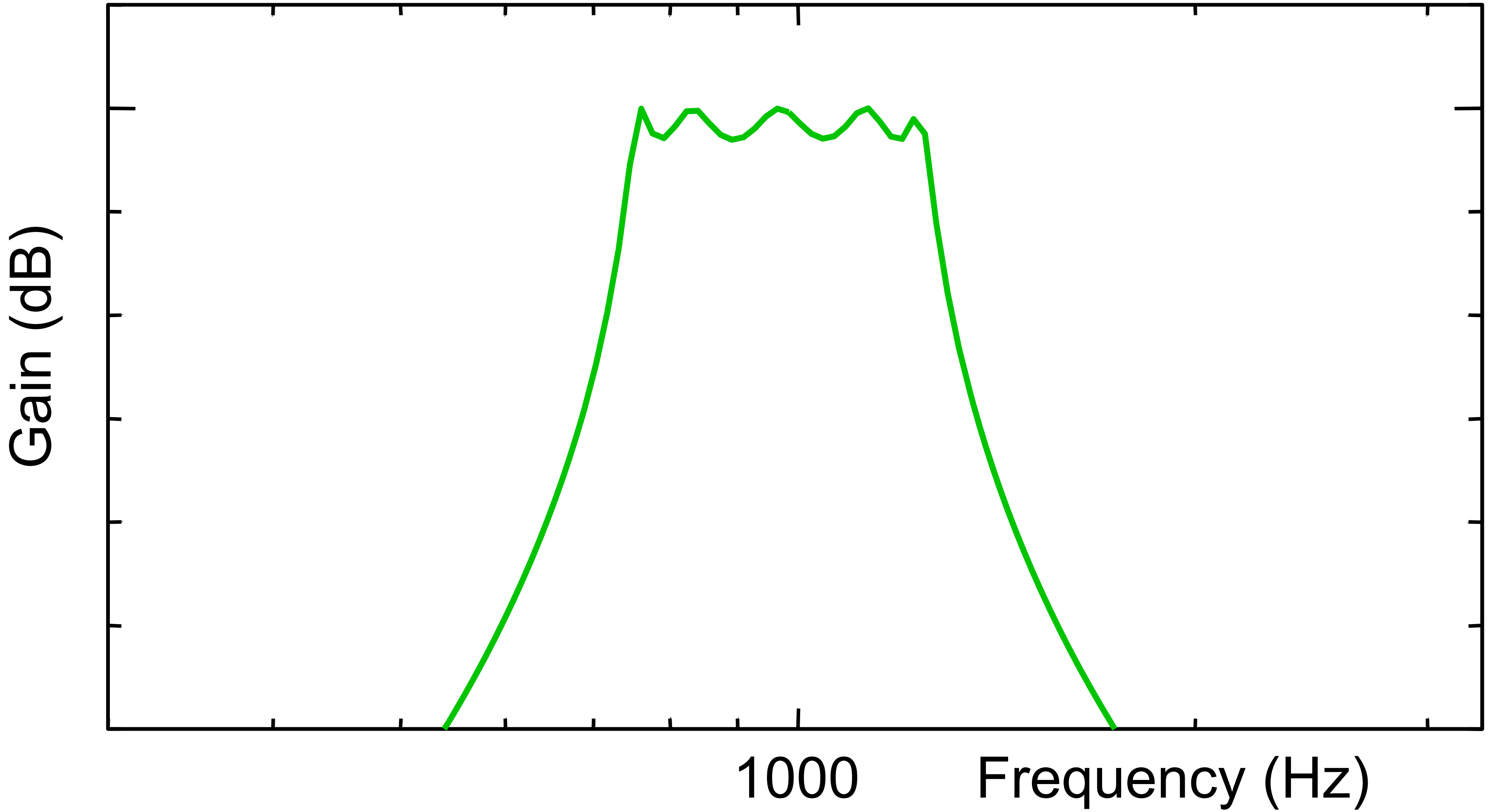
This transformation converts to a band-pass response centred on $\omega_0$ with relative bandwidth $k^2$:

$$j\omega \rightarrow \frac{\left\{\dfrac{\omega_0}{j\omega} + \dfrac{j\omega}{\omega_0}\right\}}{k}$$



# Low-Pass to Band-Pass Transformation

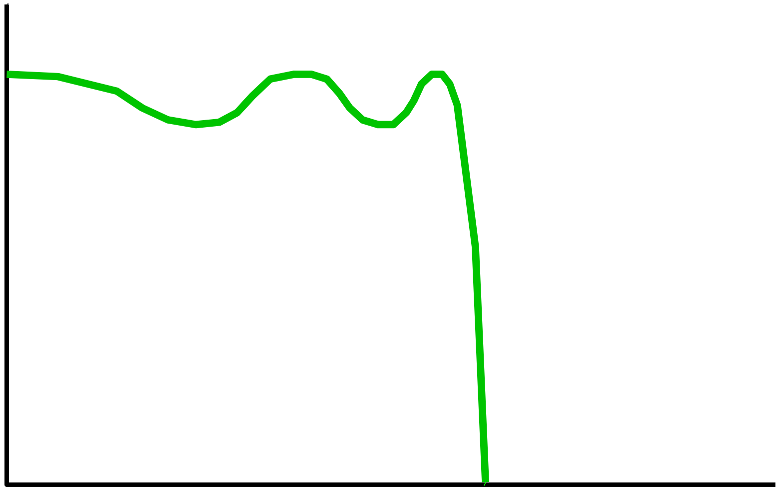
10th-order Chebychev band-pass centred on $f$=1000 Hz with relative bandwidth $k^2$=4:

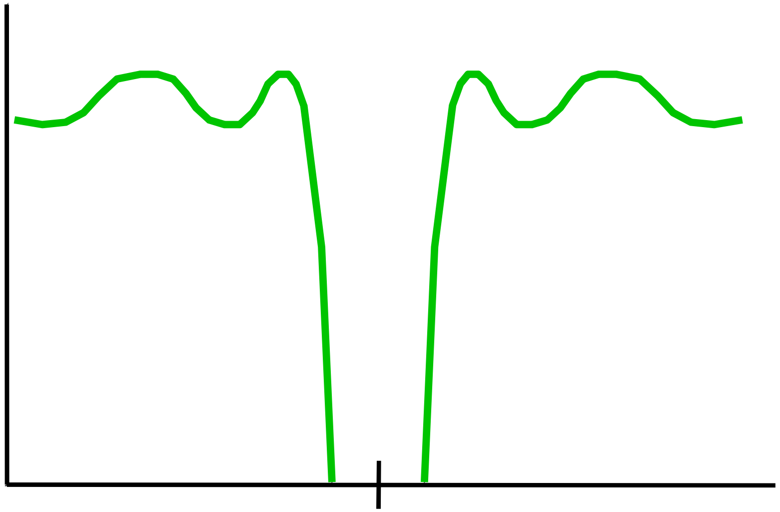


# Low-Pass to Band-Stop Transformation

This transformation converts to a band-stop response centred on $\omega_0$ with relative bandwidth $k^2$:

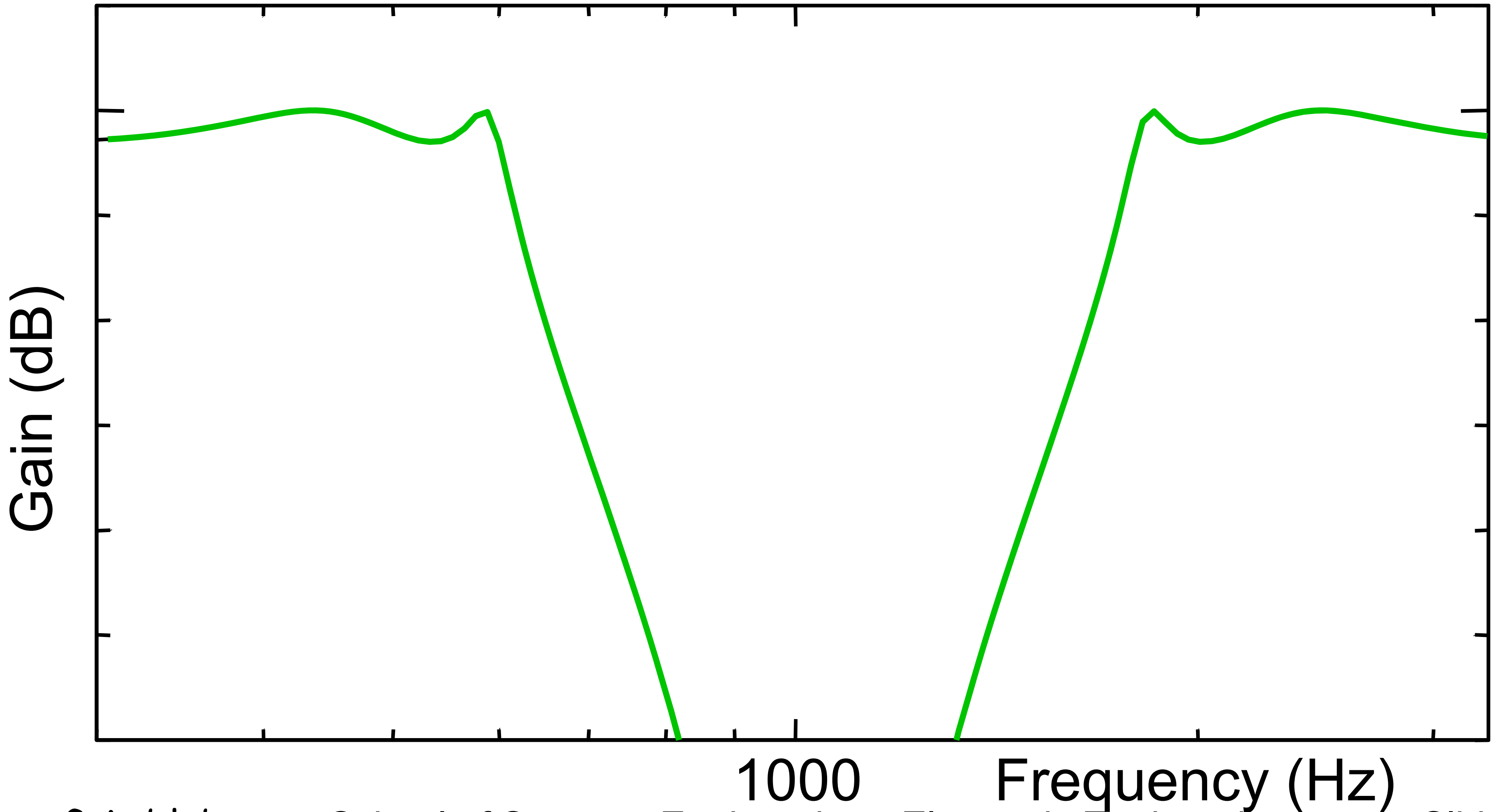
$$\mathrm{j}\omega \;\rightarrow\; \frac{k}{\left\{\dfrac{\omega_0}{\mathrm{j}\omega} + \dfrac{\mathrm{j}\omega}{\omega_0}\right\}}$$



# Low-Pass to Band-Stop Transformation

10th-order Chebychev band-stop centred on $f$=1000 Hz with relative bandwidth $k^2$=4:



# Passive Filter Realisation

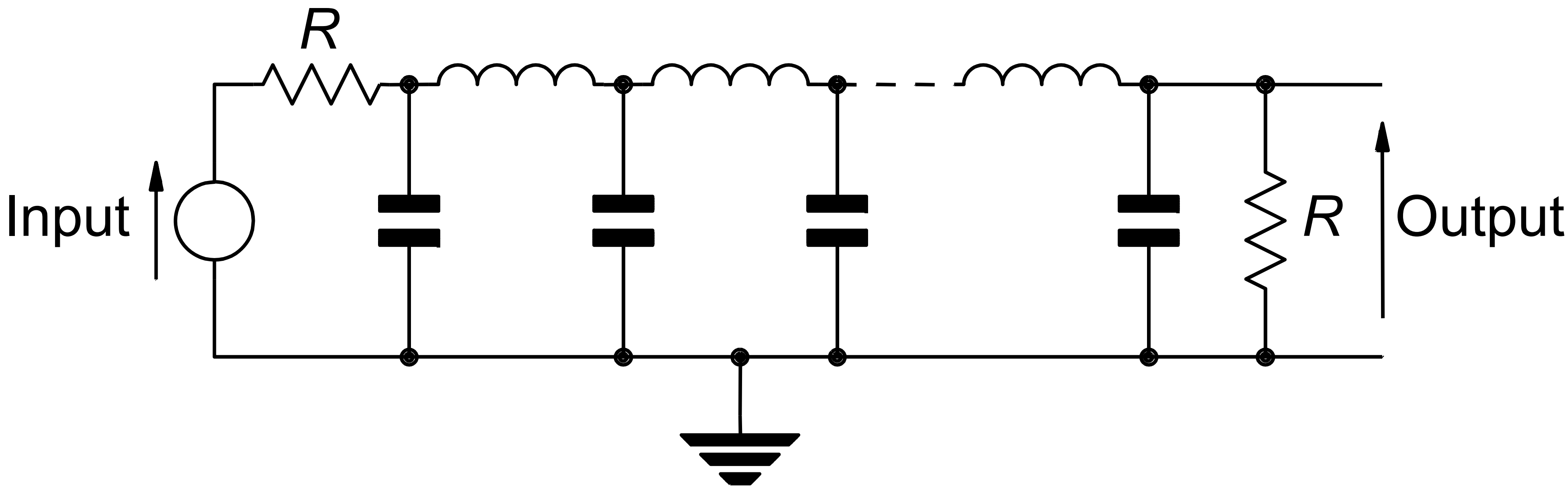
Passive filters are usually realised as equally-terminated ladder filters

This type of filter has a resistor in series with the input and a resistor of nominally the same value in parallel with the output; all other components are reactive (that is inductors or capacitors)

Passive equally-terminated ladder filters are not normally used at frequencies below about 10 kHz because they contain inductors

# Passive Filter Realisation

Equally-terminated low-pass all-pole ladder filter:

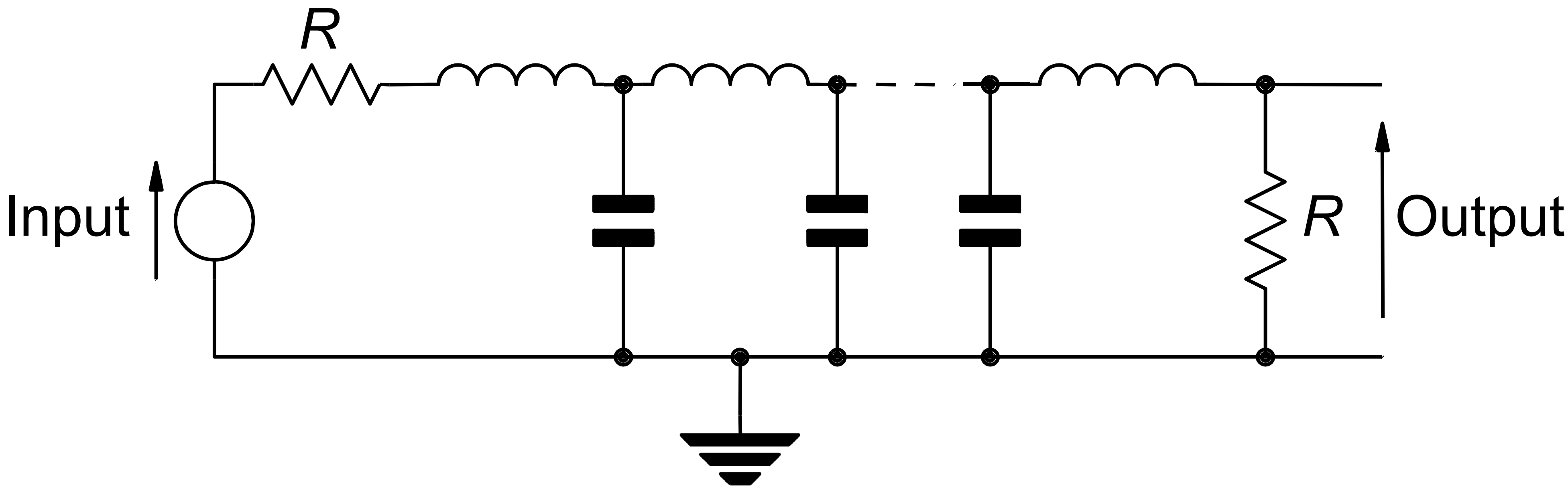



This type of filter is suitable for implementing all-pole designs such as Butterworth, Chebychev and Bessel

Order = number of capacitors + number of inductors

# Passive Filter Realisation

Alternative equally-terminated low-pass all-pole ladder filter:

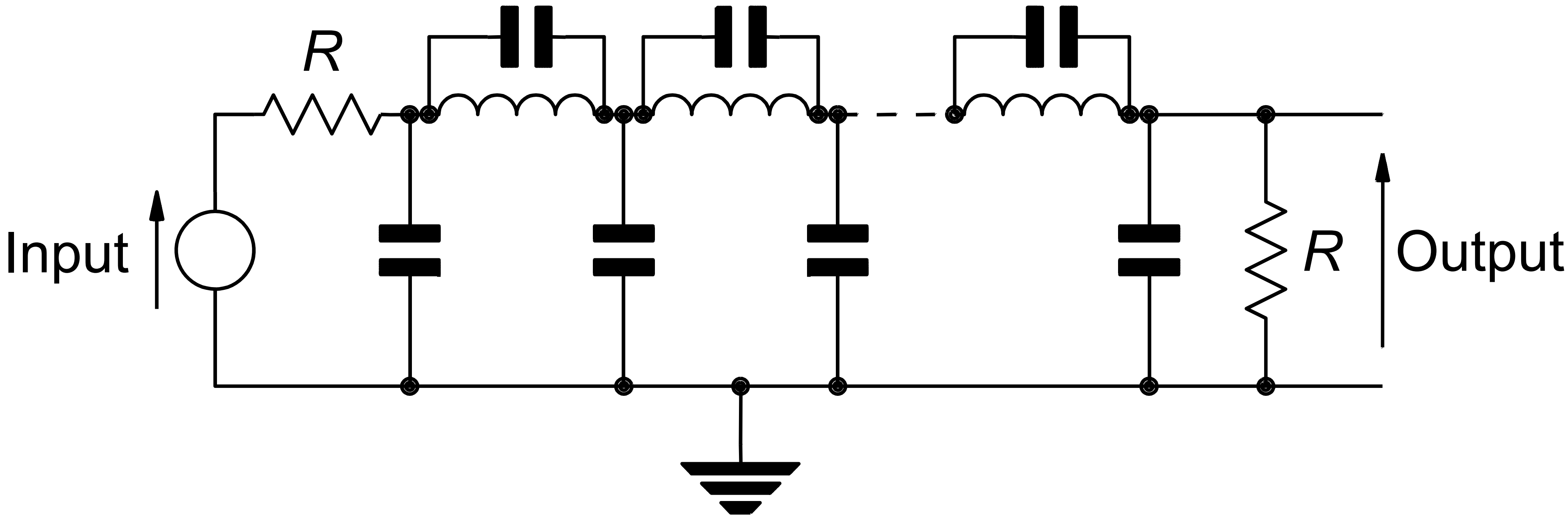



This type of filter is suitable for implementing all-pole designs such as Butterworth, Chebychev and Bessel

Order = number of capacitors + number of inductors

# Passive Filter Realisation

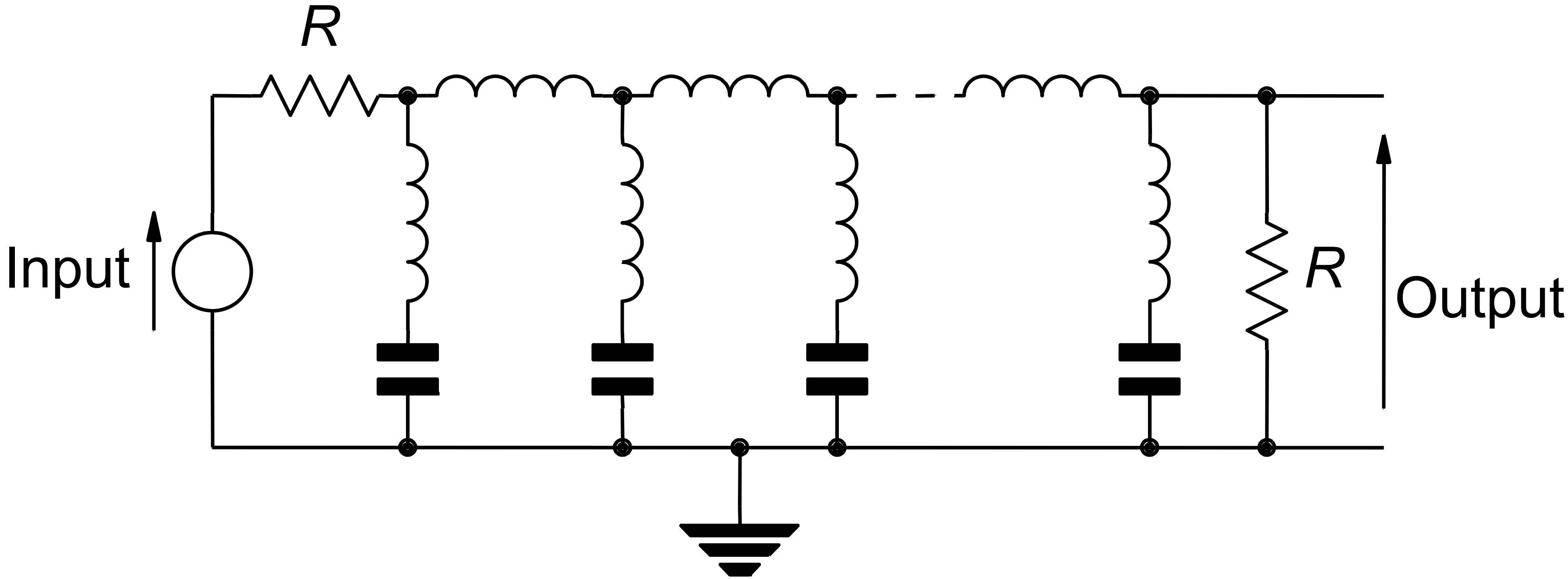
Equally-terminated low-pass ladder filter with imaginary zeros in response:




This type of filter is suitable for implementing an Inverse Chebychev or Elliptic response

# Passive Filter Realisation

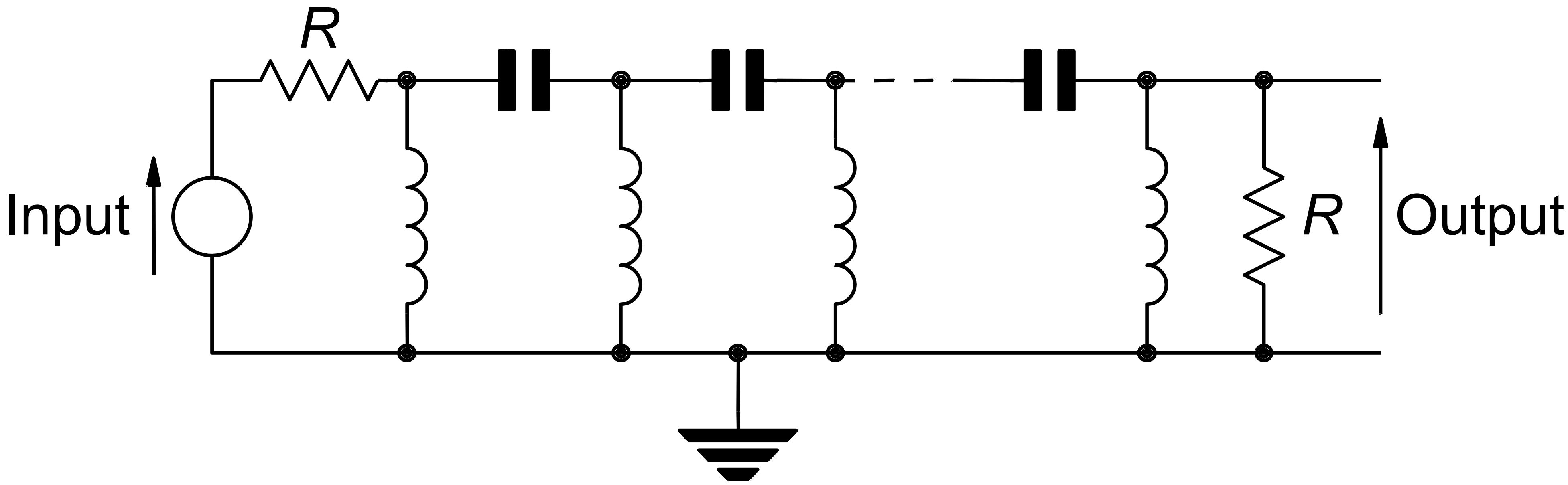
Alternative equally-terminated low-pass ladder filter with imaginary zeros in response:




This type of filter is suitable for implementing an Inverse Chebychev or Elliptic response

# Passive Filter Realisation

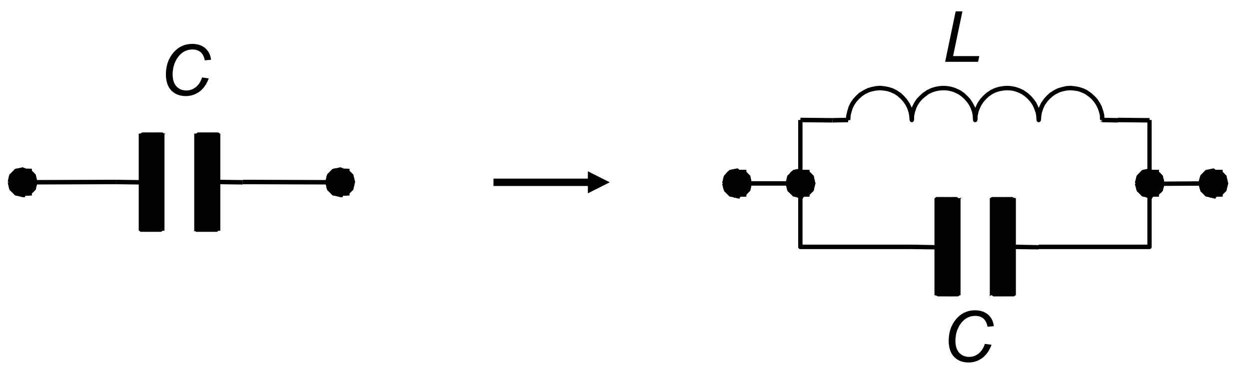
Low-pass filters can be changed to high-pass filters by replacing the inductors by capacitors, and the capacitors by inductors:



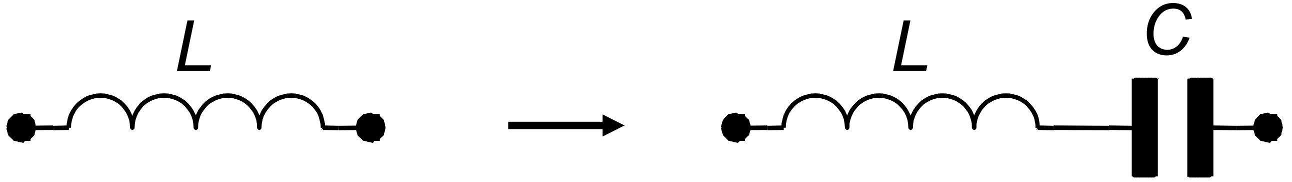

This type of filter is suitable for implementing all-pole designs such as Butterworth, Chebychev and Bessel

# Passive Filter Realisation

Low-pass filters can be changed to band-pass filters by replacing the inductors and capacitors by LC combinations




$$L = \frac{1}{\omega_0^2 C}$$

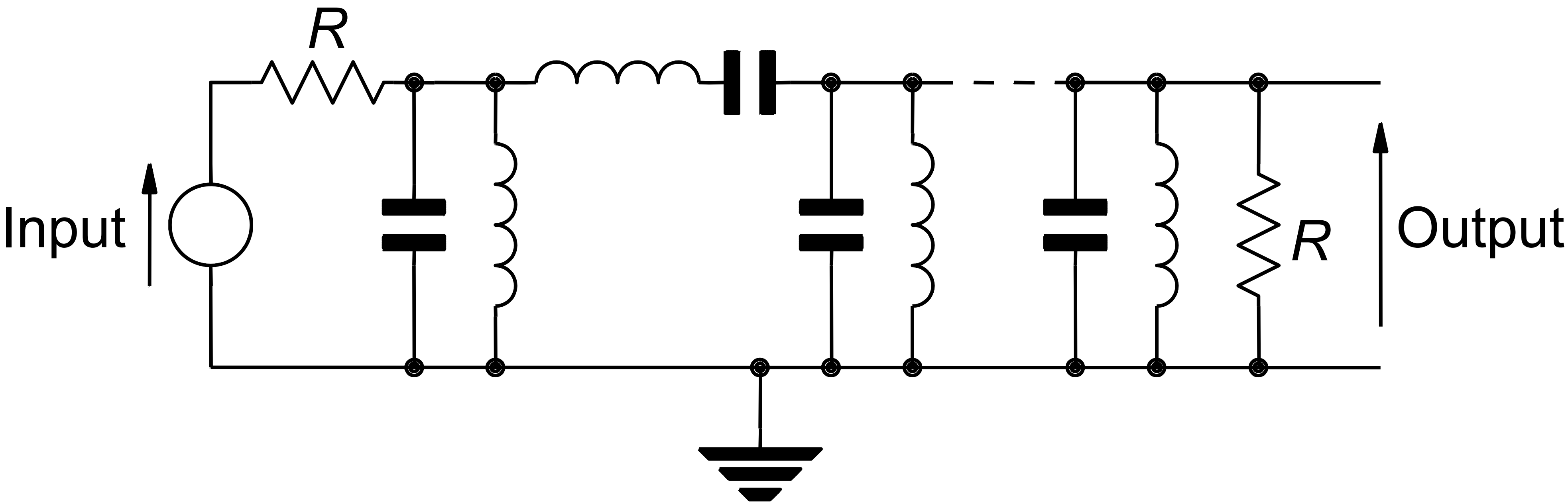



$$C = \frac{1}{\omega_0^2 L}$$

where $\omega_0$ is the geometric centre frequency of the passband

# Passive Filter Realisation

Low-pass filters can be changed to band-pass filters by replacing the inductors and capacitors by LC combinations




This type of filter is suitable for implementing all-pole designs such as Butterworth, Chebychev and Bessel

# Component Value Determination

Suitable component values can be determined by the following procedure:

1. Select a suitable filter circuit

2. Obtain its frequency-response function in symbolic form

3. Equate coefficients of the symbolic frequency-response function and the required response

4. Solve the simultaneous non-linear equations to obtain the component values

# Component Value Determination

University of Reading

A simpler procedure is to use filter design tables

Normalised low-pass Butterworth
($R$ = 1.0 Ω, $\omega_0$ = 1.0 rad/s):

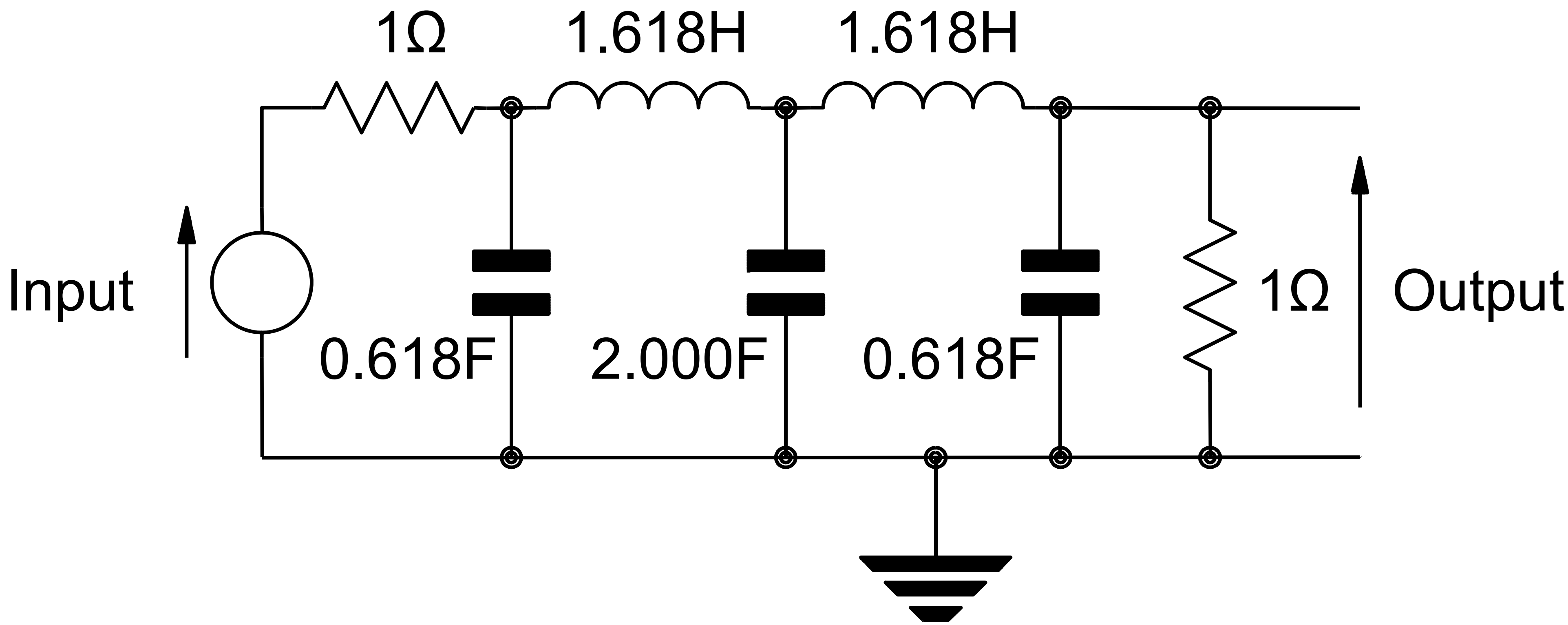
| *n* | *C1* | *L2* | *C3* | *L4* | *C5* | *L6* | *C7* |
|---|---|---|---|---|---|---|---|
| 2 | 1.414 | 1.414 | | | | | |
| 3 | 1.000 | 2.000 | 1.000 | | | | |
| 4 | 0.765 | 1.848 | 1.848 | 0.765 | | | |
| 5 | 0.618 | 1.618 | 2.000 | 1.618 | 0.618 | | |
| 6 | 0.518 | 1.414 | 1.932 | 1.932 | 1.414 | 0.518 | |
| 7 | 0.445 | 1.247 | 1.802 | 2.000 | 1.802 | 1.247 | 0.445 |

# Component Value Determination

University of Reading

Design example: a 5th-order low-pass Butterworth filter with cut-off (-3 dB) frequency 2 kHz

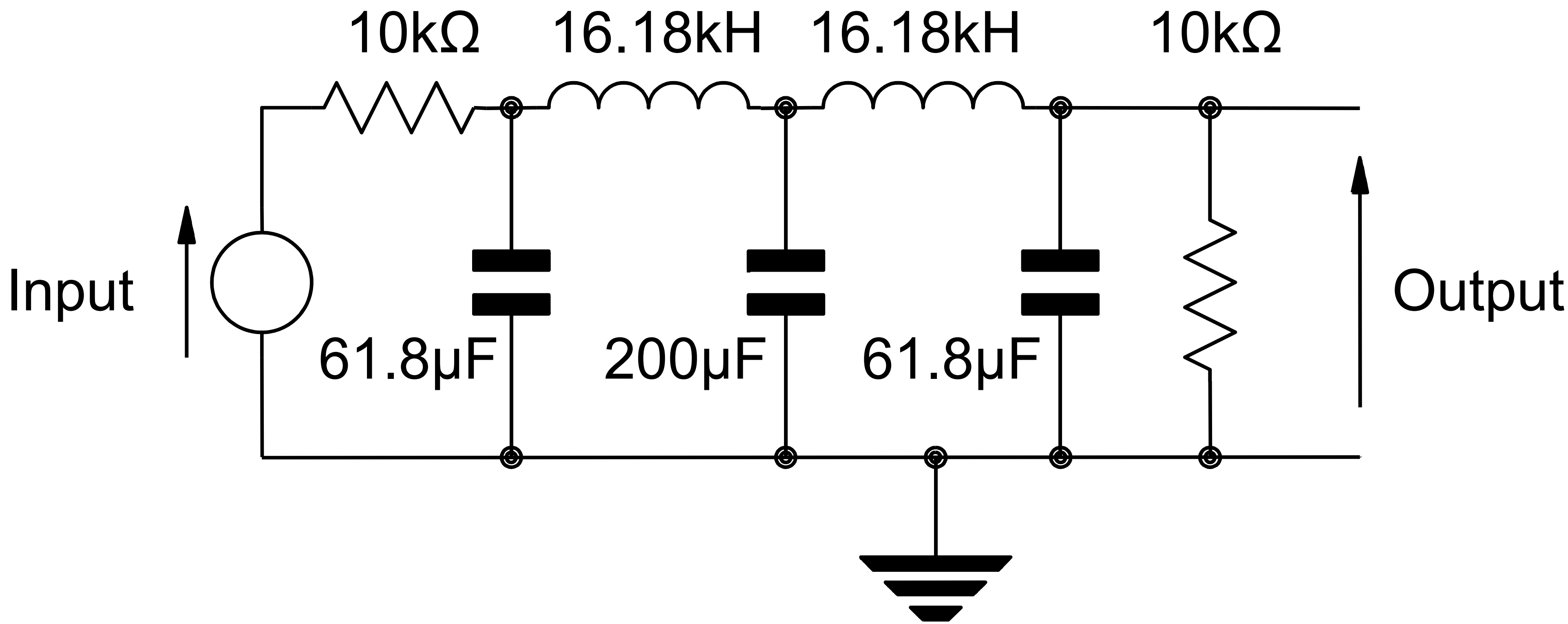
Normalised 5th-order Butterworth low-pass filter:



# Component Value Determination

Scale impedances: multiply resistor and inductor values by *k*, divide capacitor values by *k*

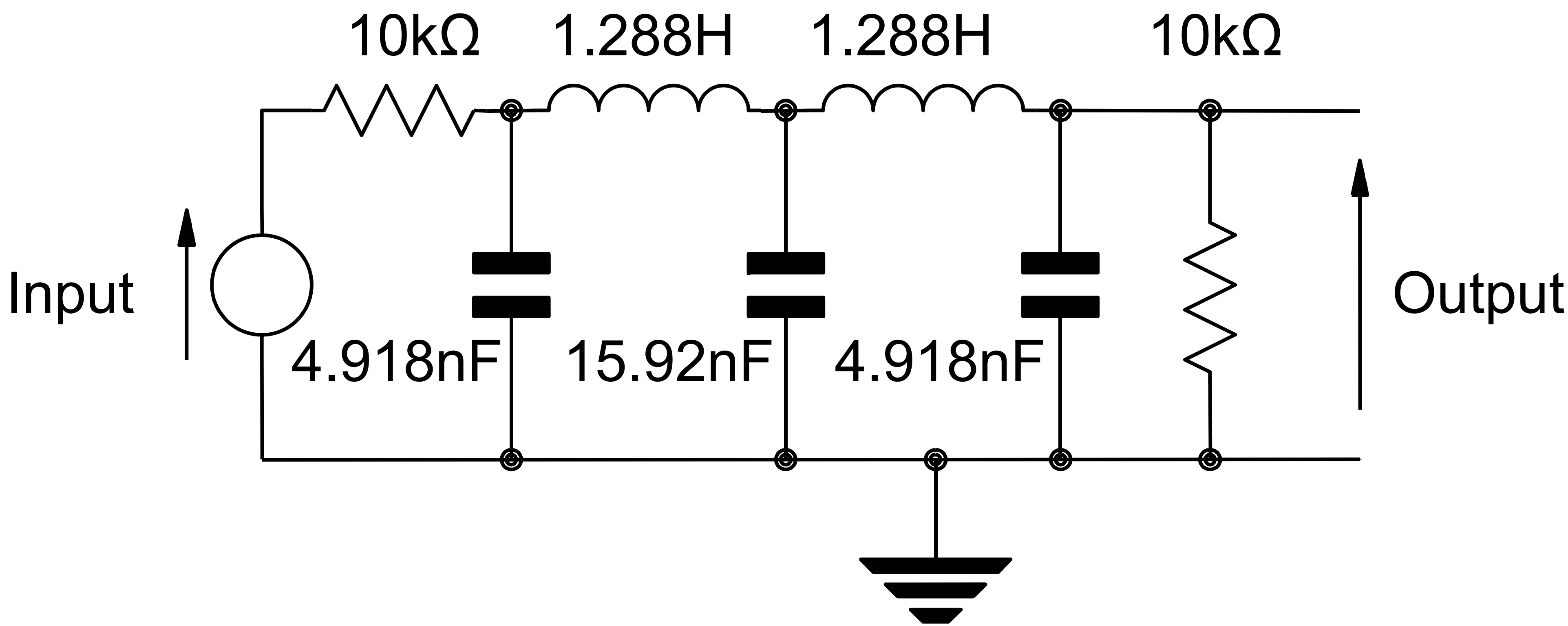
Choose *k* = 10000



# Component Value Determination

University of Reading

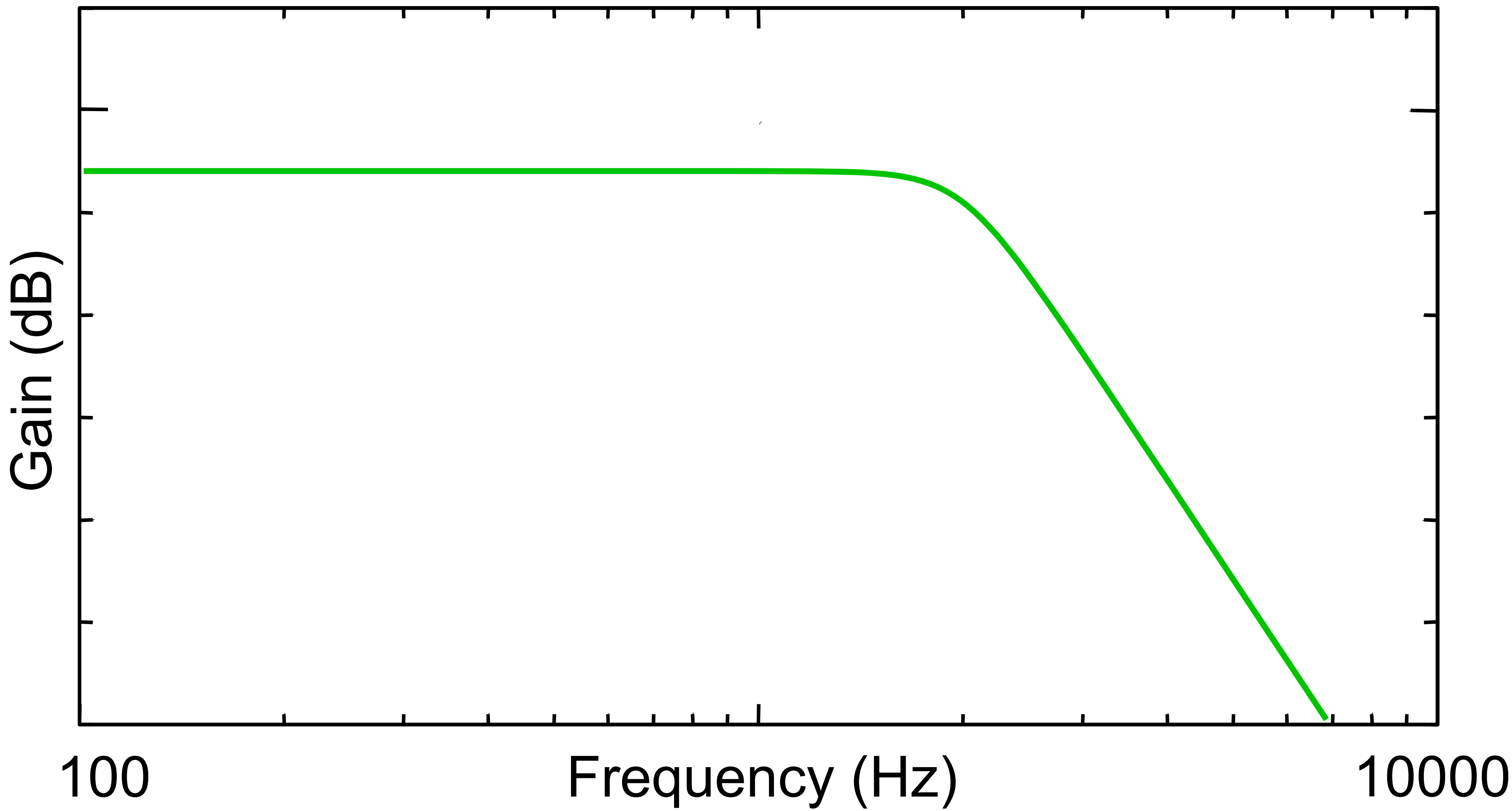
Scale frequency: divide inductor and capacitor values by $k$ where $k = 2\pi \times 2000 = 1.257 \times 10^4$



# Component Value Determination

University of Reading

Response of 5th-order low-pass Butterworth filter:



# Component Value Determination

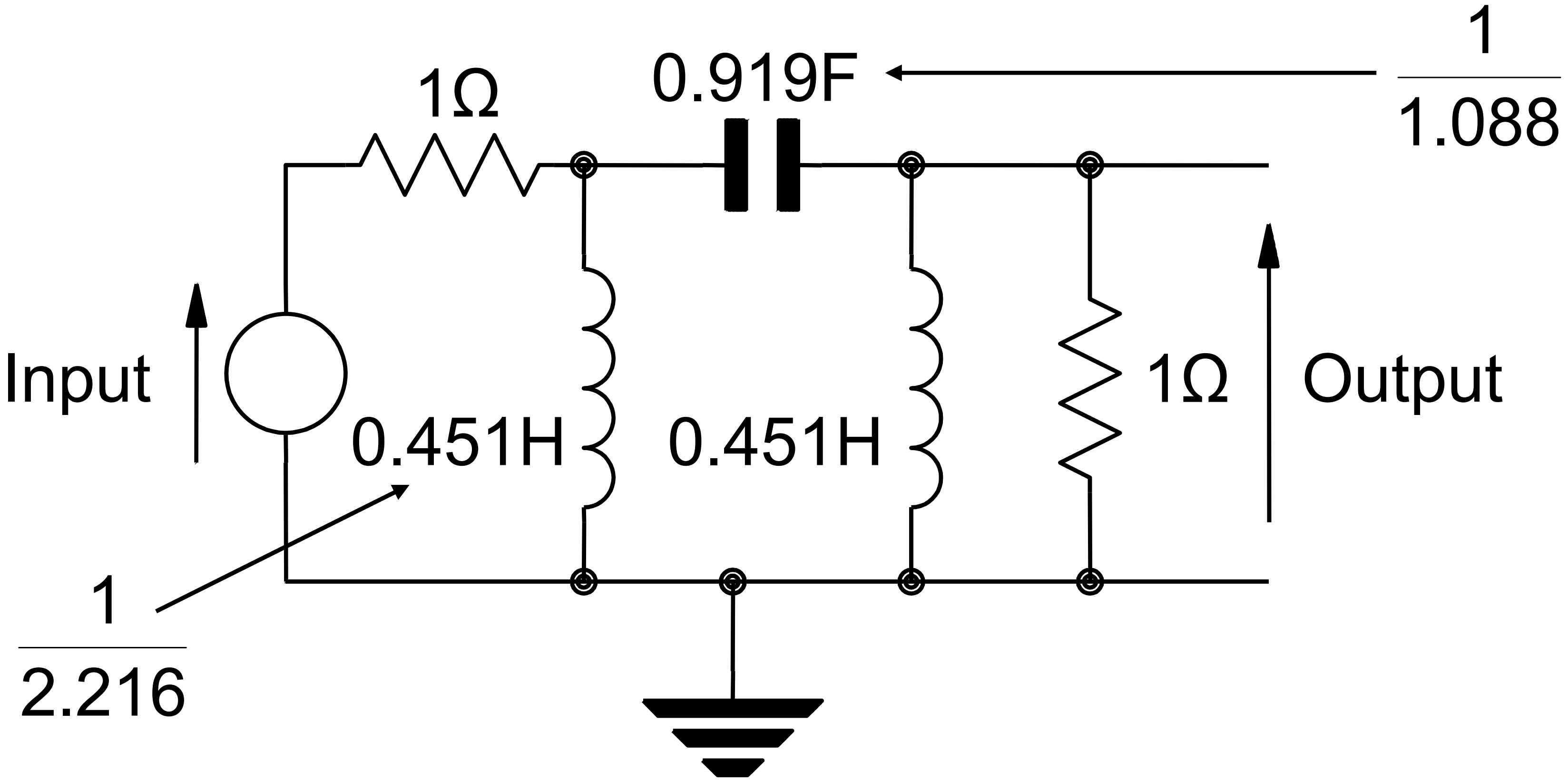
Normalised 1 dB ripple low-pass Chebychev
($R$ = 1.0 Ω, $\omega_0$ = 1.0 rad/s):

| *n* | *C1* | *L2* | *C3* | *L4* | *C5* | *L6* | *C7* |
|---|---|---|---|---|---|---|---|
| 2 | 0.572 | 3.132 | | | | | |
| 3 | 2.216 | 1.088 | 2.216 | | | | |
| 4 | 0.653 | 4.411 | 0.814 | 2.535 | | | |
| 5 | 2.207 | 1.128 | 3.103 | 1.128 | 2.207 | | |
| 6 | 0.679 | 3.873 | 0.771 | 4.711 | 0.969 | 2.406 | |
| 7 | 2.204 | 1.131 | 3.147 | 1.194 | 3.147 | 1.131 | 2.204 |

# Component Value Determination

Design example: a 3rd-order high-pass Chebychev filter with 1 dB ripple and a cut-off (-3 dB) frequency 5 kHz

Normalised 3rd-order Chebychev high-pass filter:

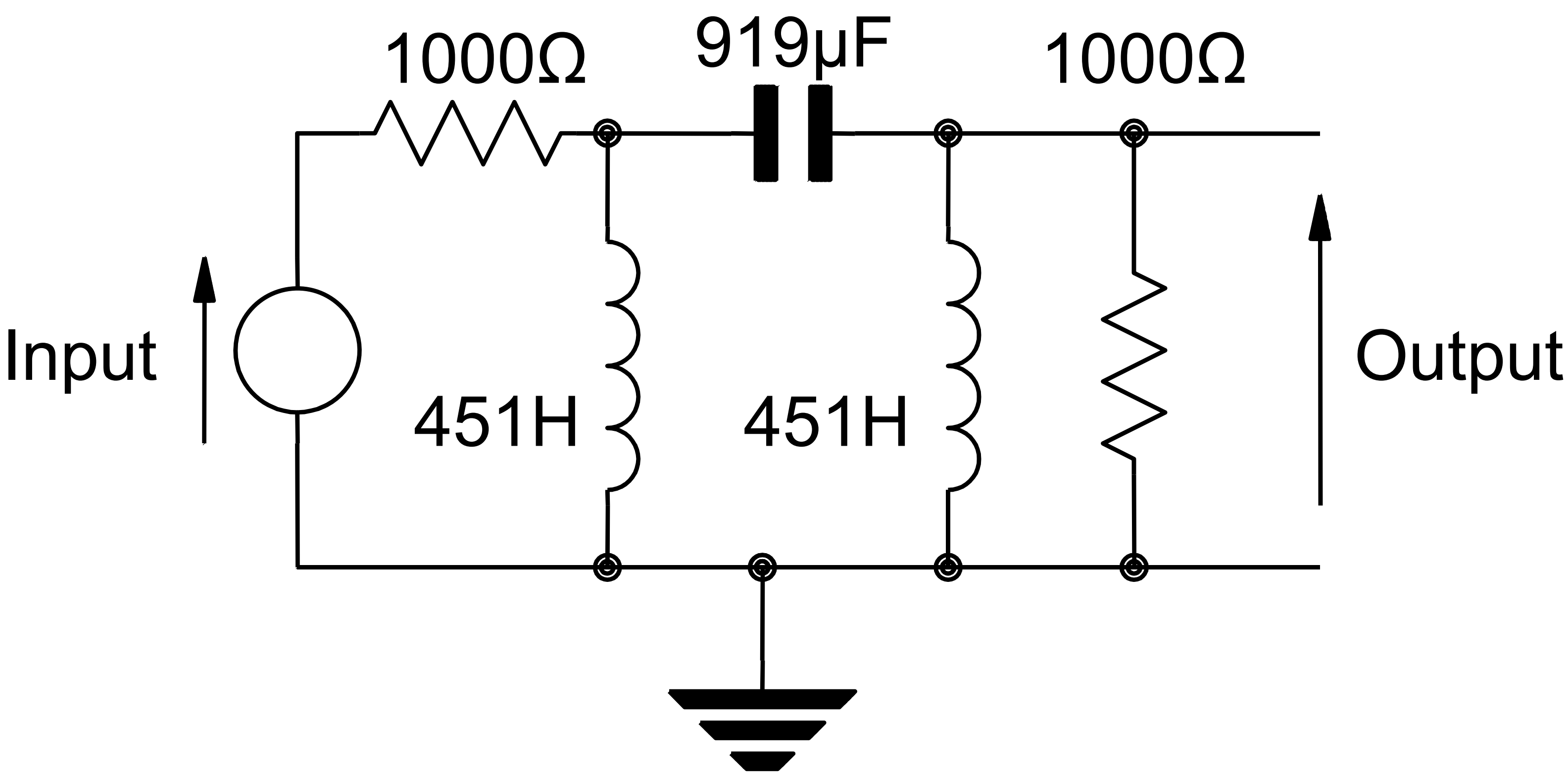


# Component Value Determination

University of Reading

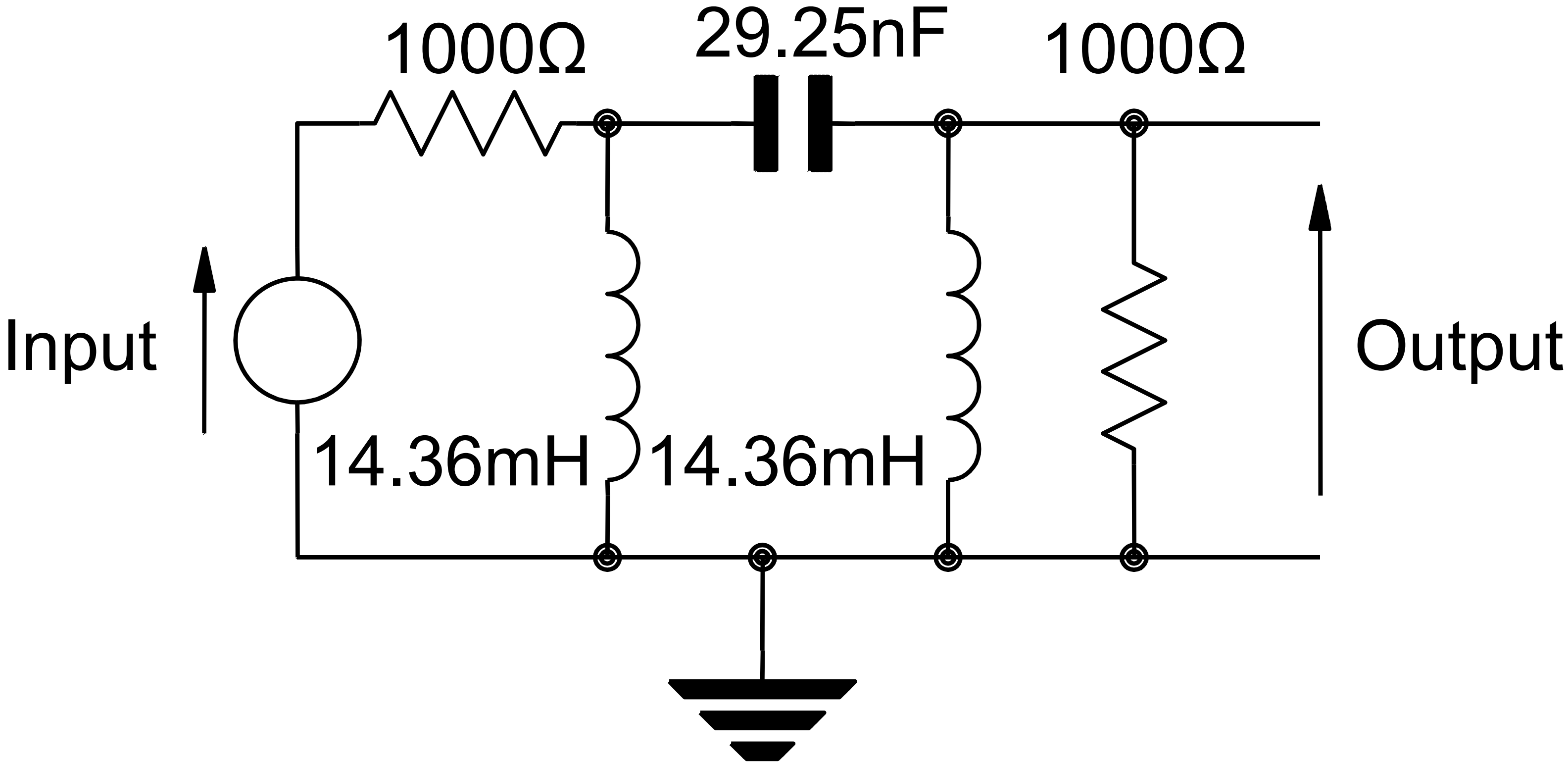
Scale impedances: multiply resistor and inductor values by *k*, divide capacitor values by *k*

Choose *k* = 1000



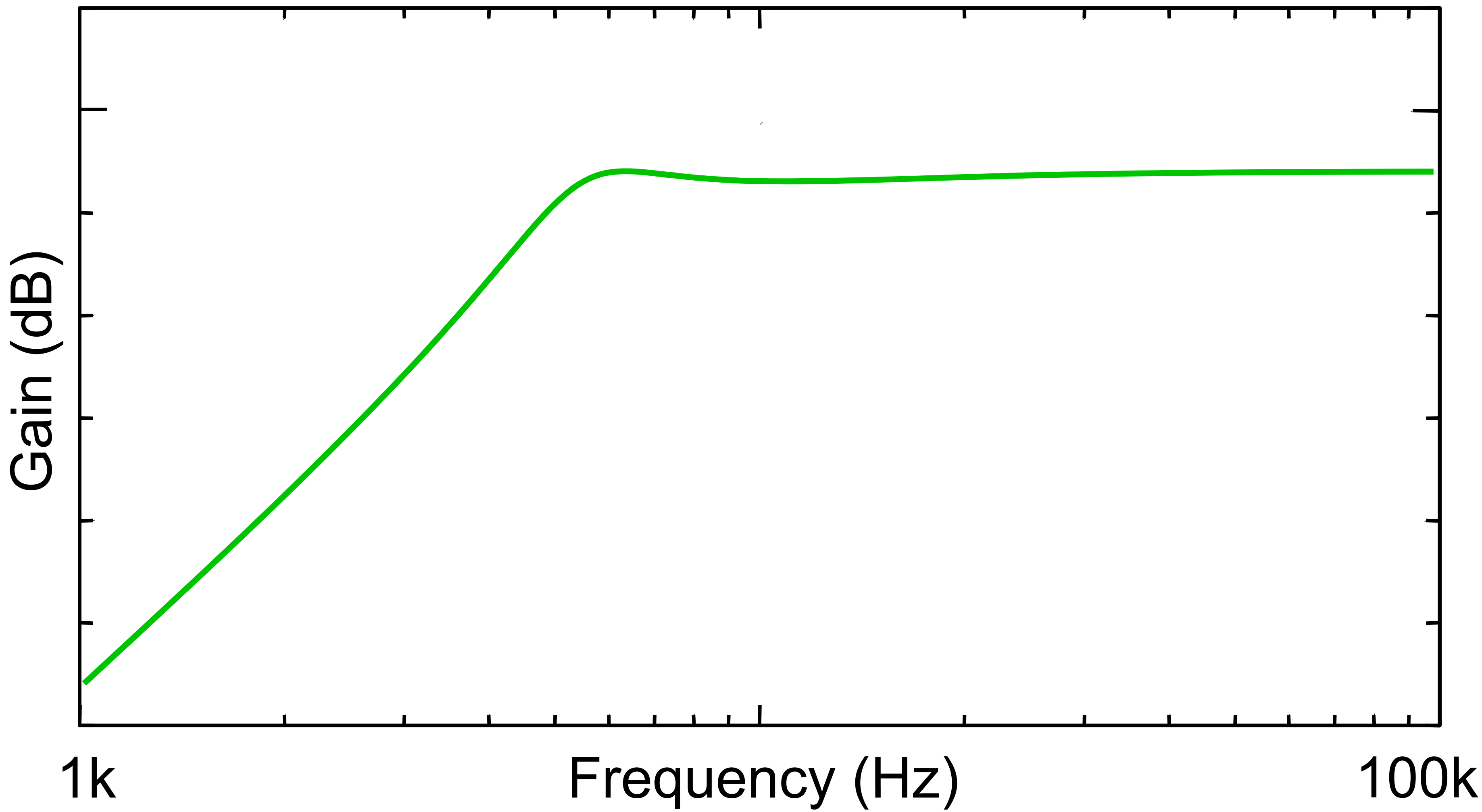
# Component Value Determination

Scale frequency: divide inductor and capacitor values by $k$ where $k = 2\pi \times 5000 = 3.142 \times 10^5$



# Component Value Determination

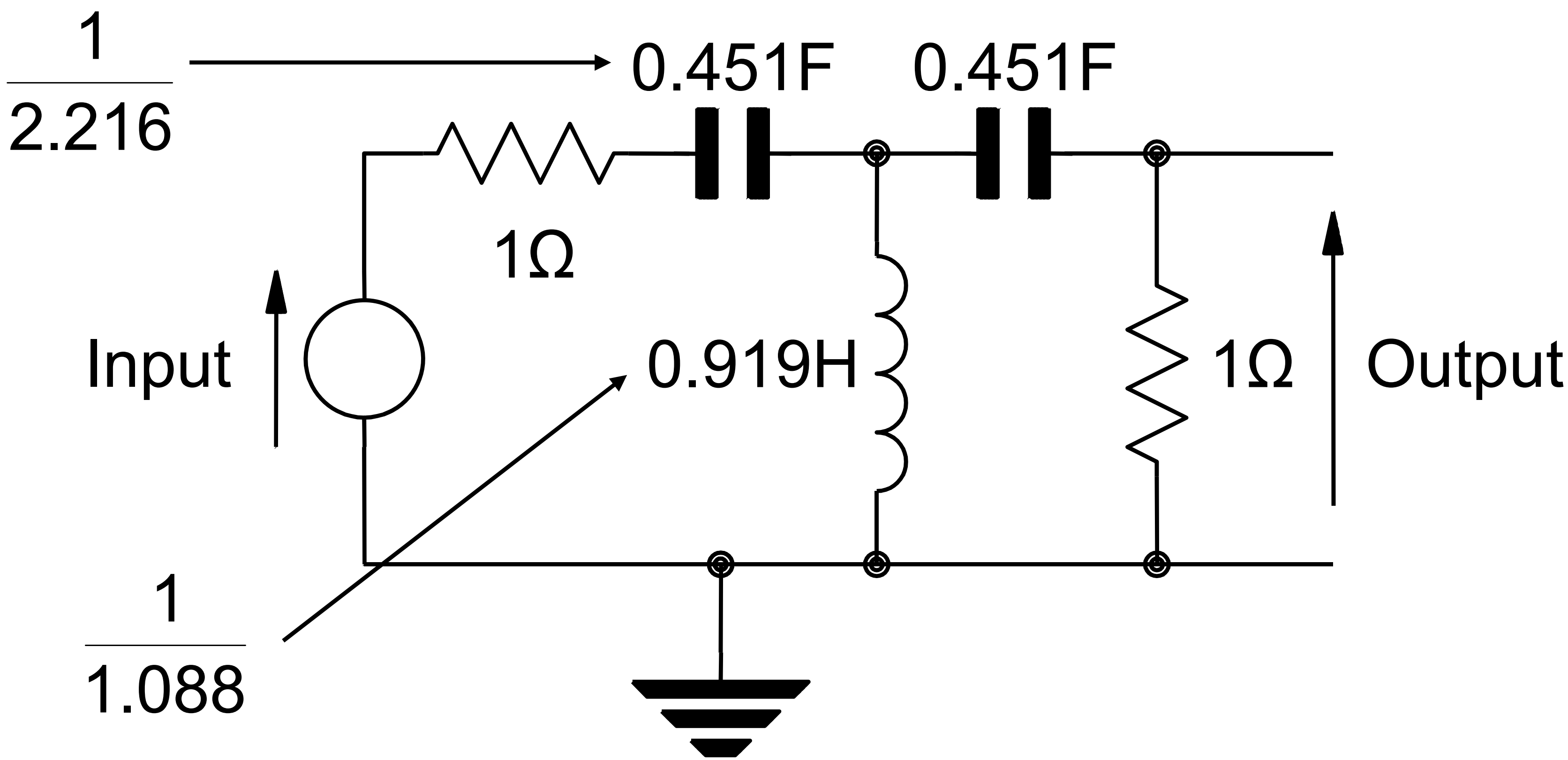
Response of 3rd-order high-pass Chebychev filter:



# Alternative Configuration

Design example: a 3rd-order high-pass Chebychev filter with 1 dB ripple and a cut-off (-3 dB) frequency 5 kHz

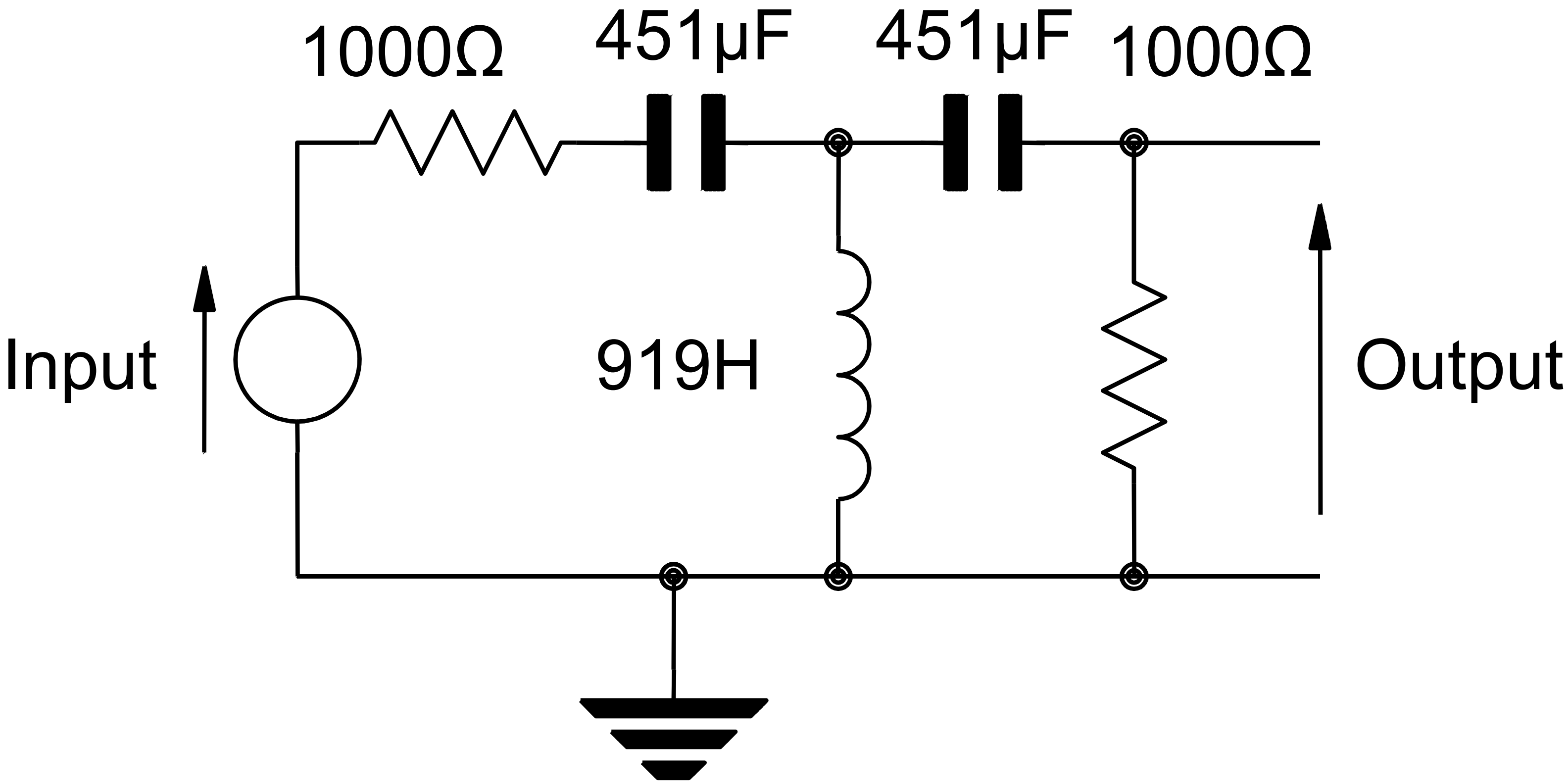
Normalised 3rd-order Chebychev high-pass filter:



# Alternative Configuration

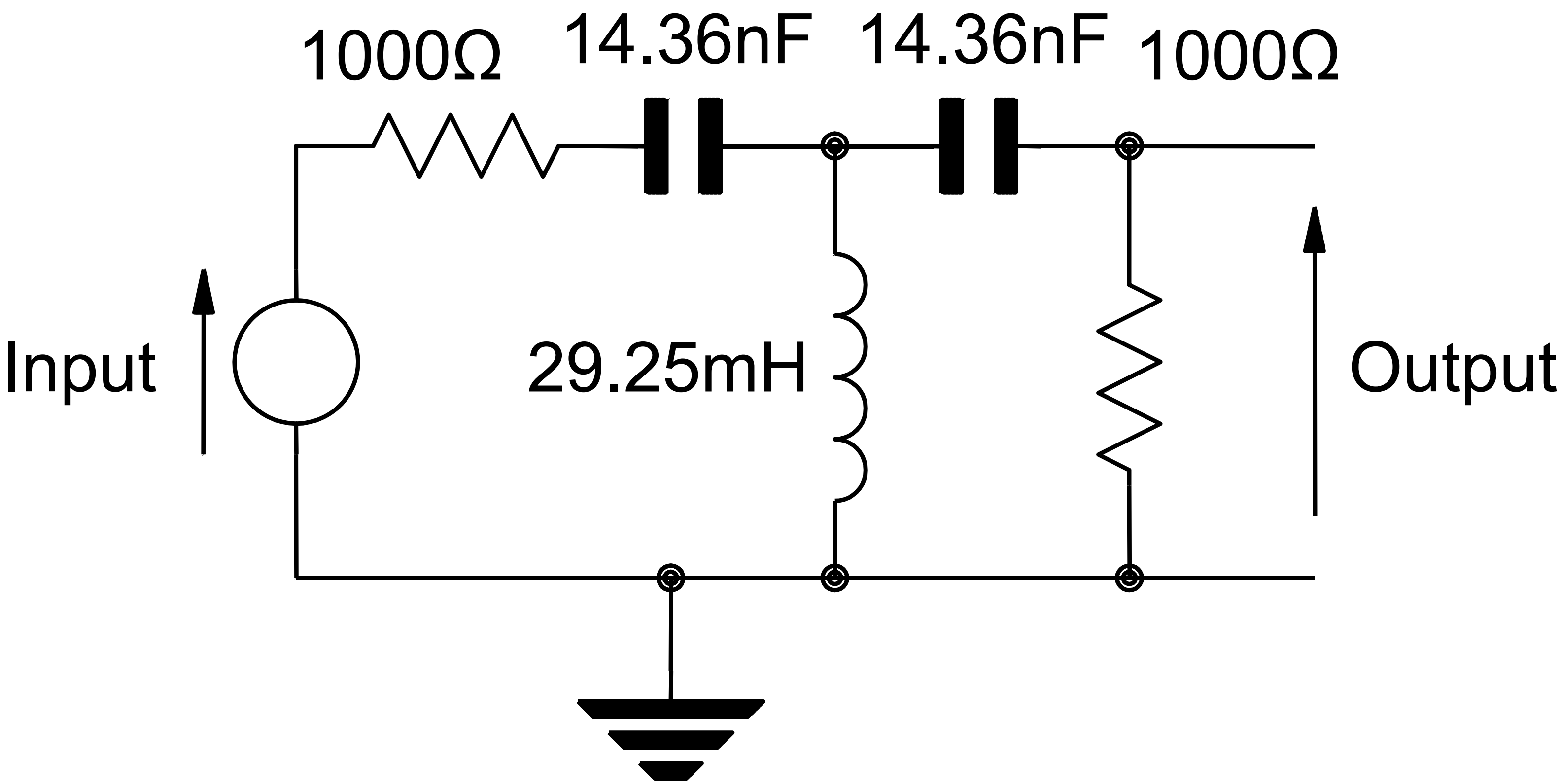
Scale impedances: multiply resistor and inductor values by $k$, divide capacitor values by $k$

Choose $k$ = 1000



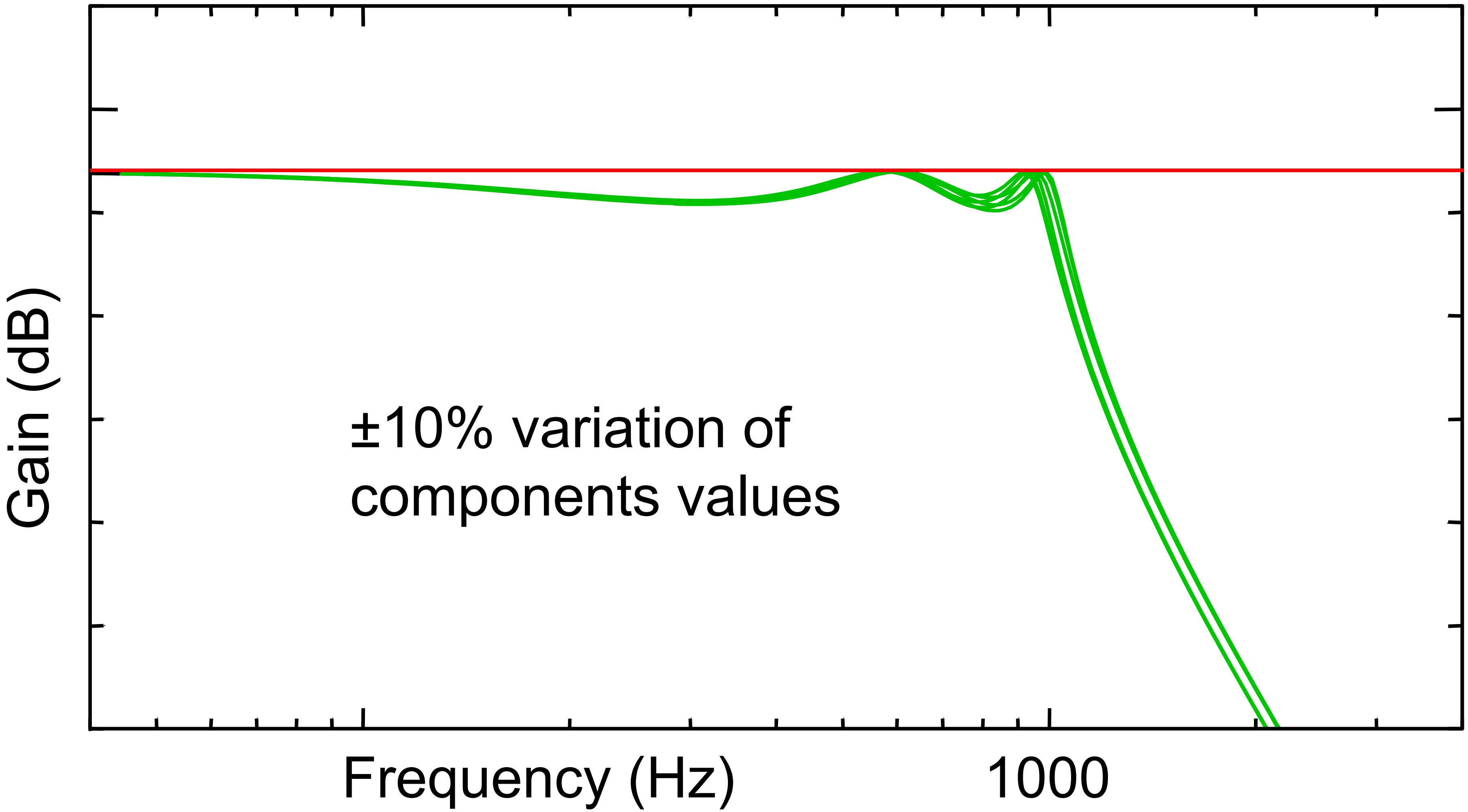
# Alternative Configuration

Scale frequency: divide inductor and capacitor values by $k$ where $k = 2\pi \times 5000 = 3.142 \times 10^5$



# Component Value Sensitivity

5th-order 3dB ripple low-pass Chebychev equally-terminated ladder filter with cut-off frequency 1000 Hz:



# Active Filters: Cascade Synthesis

The frequency response function is first split into second-order factors

$$H(j\omega) = \frac{a_0 + a_1(j\omega) + a_2(j\omega)^2 + .. + a_n(j\omega)^n}{b_0 + b_1(j\omega) + b_2(j\omega)^2 + .. + b_n(j\omega)^n}$$

$$= \frac{a_{10} + a_{11}(j\omega) + a_{12}(j\omega)^2}{b_{10} + b_{11}(j\omega) + b_{12}(j\omega)^2} \times \frac{a_{20} + a_{21}(j\omega) + a_{22}(j\omega)^2}{b_{20} + b_{21}(j\omega) + b_{22}(j\omega)^2} \times \ldots$$

$$= H_1(j\omega) \times H_2(j\omega) \times \ldots$$

Each factor is implemented using a second-order active filter

These filters are then connected in cascade

# Active Filters: Cascade Synthesis

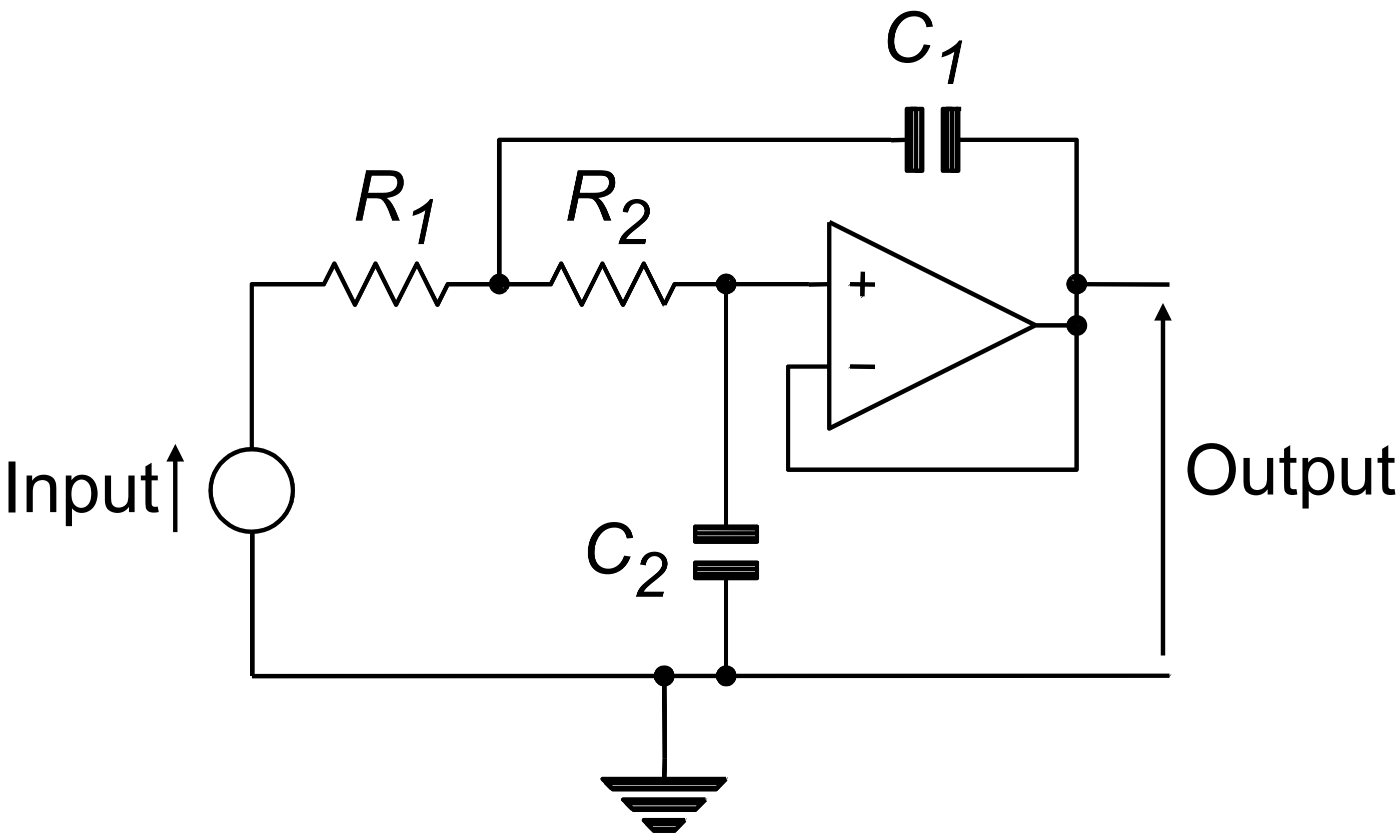
If the response to be implemented is derived from an all-pole approximation then the second-order factors will be of simple low-pass ($H_{lp}$), band-pass ($H_{bp}$) or high-pass ($H_{hp}$) form:

$$H_{lp}(j\omega) = \frac{a_0}{b_0 + b_1(j\omega) + b_2(j\omega)^2}$$

$$H_{bp}(j\omega) = \frac{a_1(j\omega)}{b_0 + b_1(j\omega) + b_2(j\omega)^2}$$

$$H_{hp}(j\omega) = \frac{a_2(j\omega)^2}{b_0 + b_1(j\omega) + b_2(j\omega)^2}$$

# Sallen-Key Low-Pass Filter




$$H_{lp}(j\omega) = \frac{1}{1 + 2T_2(j\omega) + T_1 T_2 (j\omega)^2}$$

$$\text{where}: \quad R_1 = R_2 \quad T_1 = R_1 C_1 \quad T_2 = R_2 C_2$$

# Resonance Frequency and Q-factor

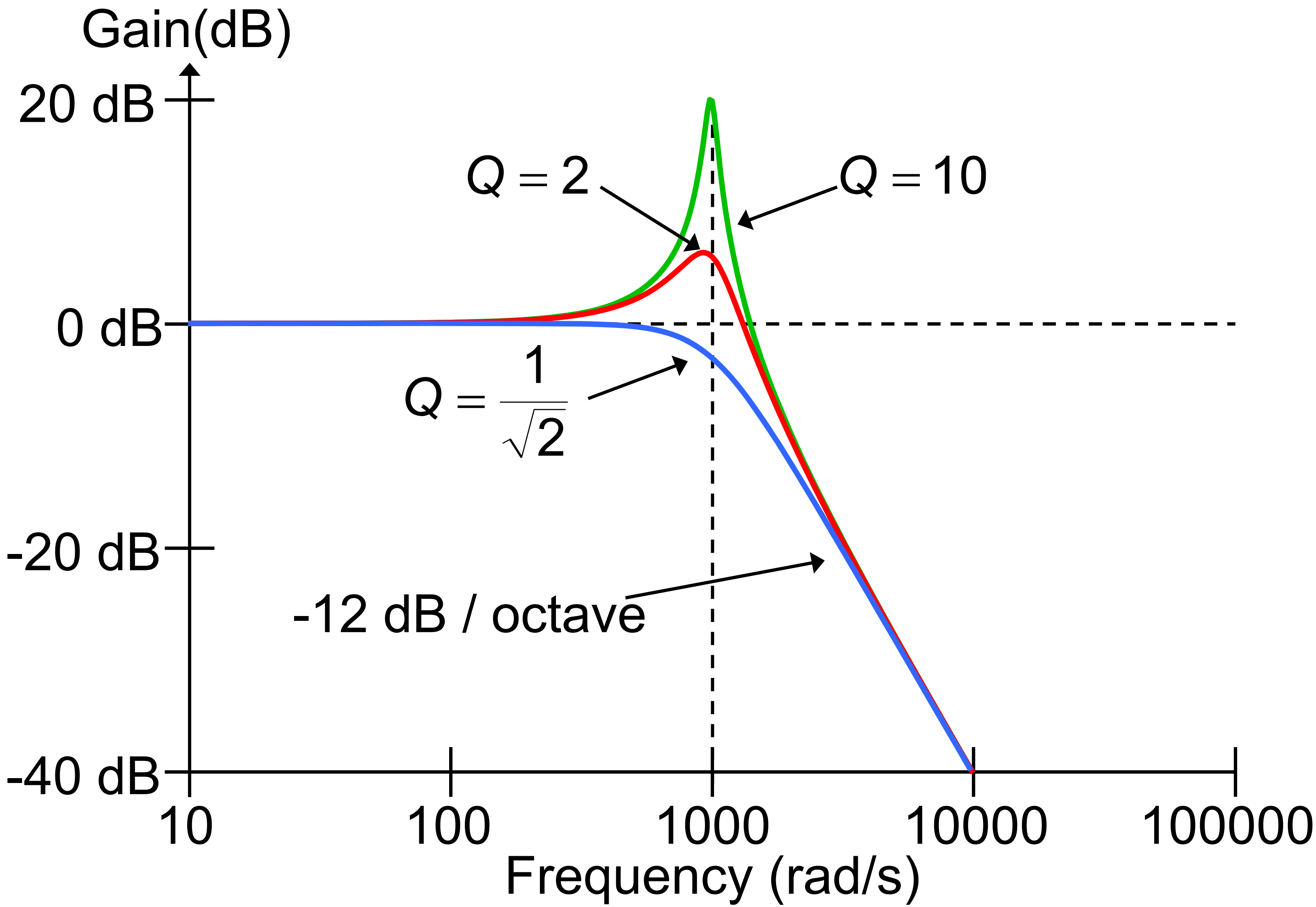
Sallen-Key:
$$H_{lp}(\mathrm{j}\omega) = \frac{1}{1 + 2T_2(\mathrm{j}\omega) + T_1T_2(\mathrm{j}\omega)^2}$$

Standard response:
$$H_{lp}(\mathrm{j}\omega) = \frac{1}{1 + \dfrac{\mathrm{j}\omega}{\omega_0 Q} + \dfrac{(\mathrm{j}\omega)^2}{\omega_0^2}}$$

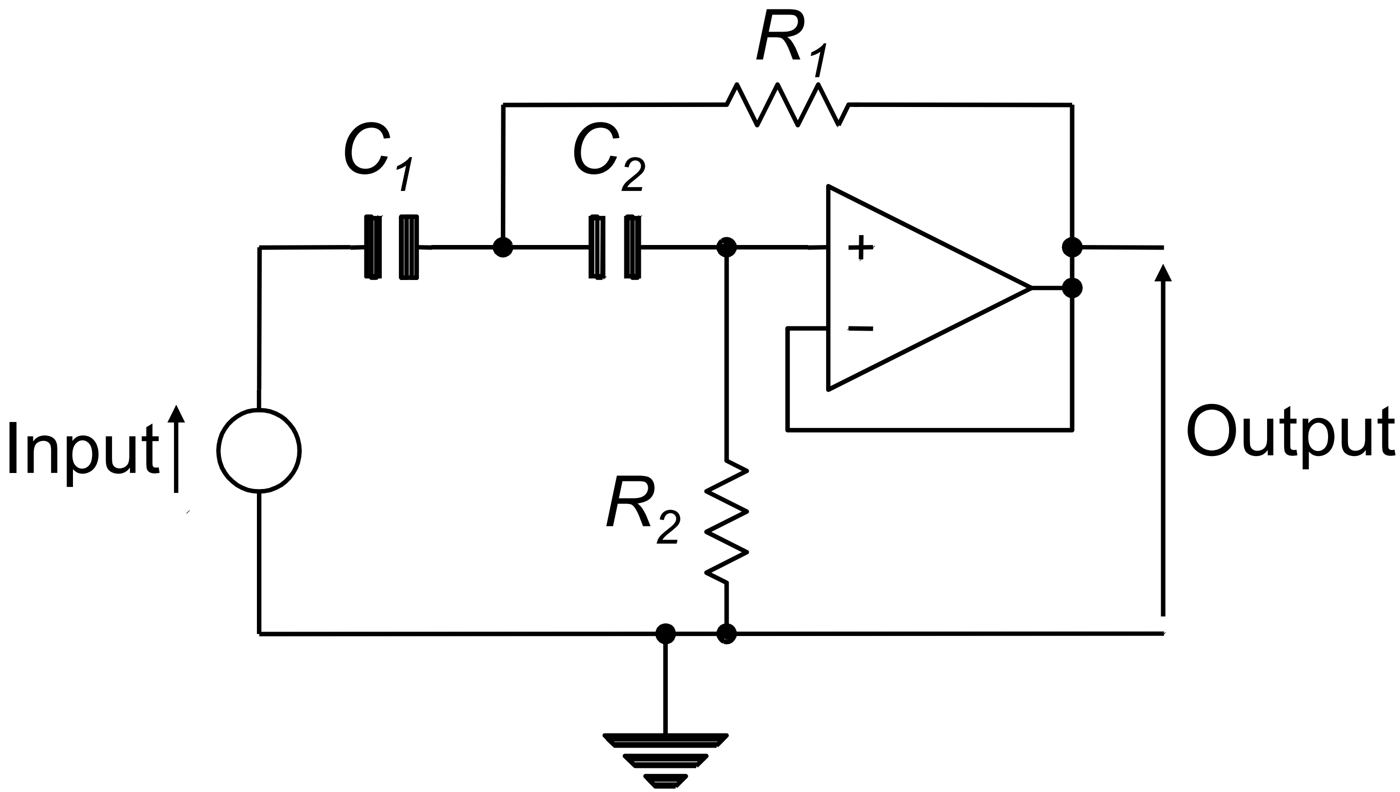
Thus:
$$\frac{1}{\omega_0^2} = T_1T_2 \qquad \frac{1}{\omega_0 Q} = 2T_2$$

or:
$$\omega_0 = \frac{1}{\sqrt{T_1T_2}} \qquad Q = \frac{1}{2\omega_0 T_2} = \frac{1}{2}\sqrt{\frac{T_1}{T_2}}$$

# Sallen-Key Low-Pass Filter



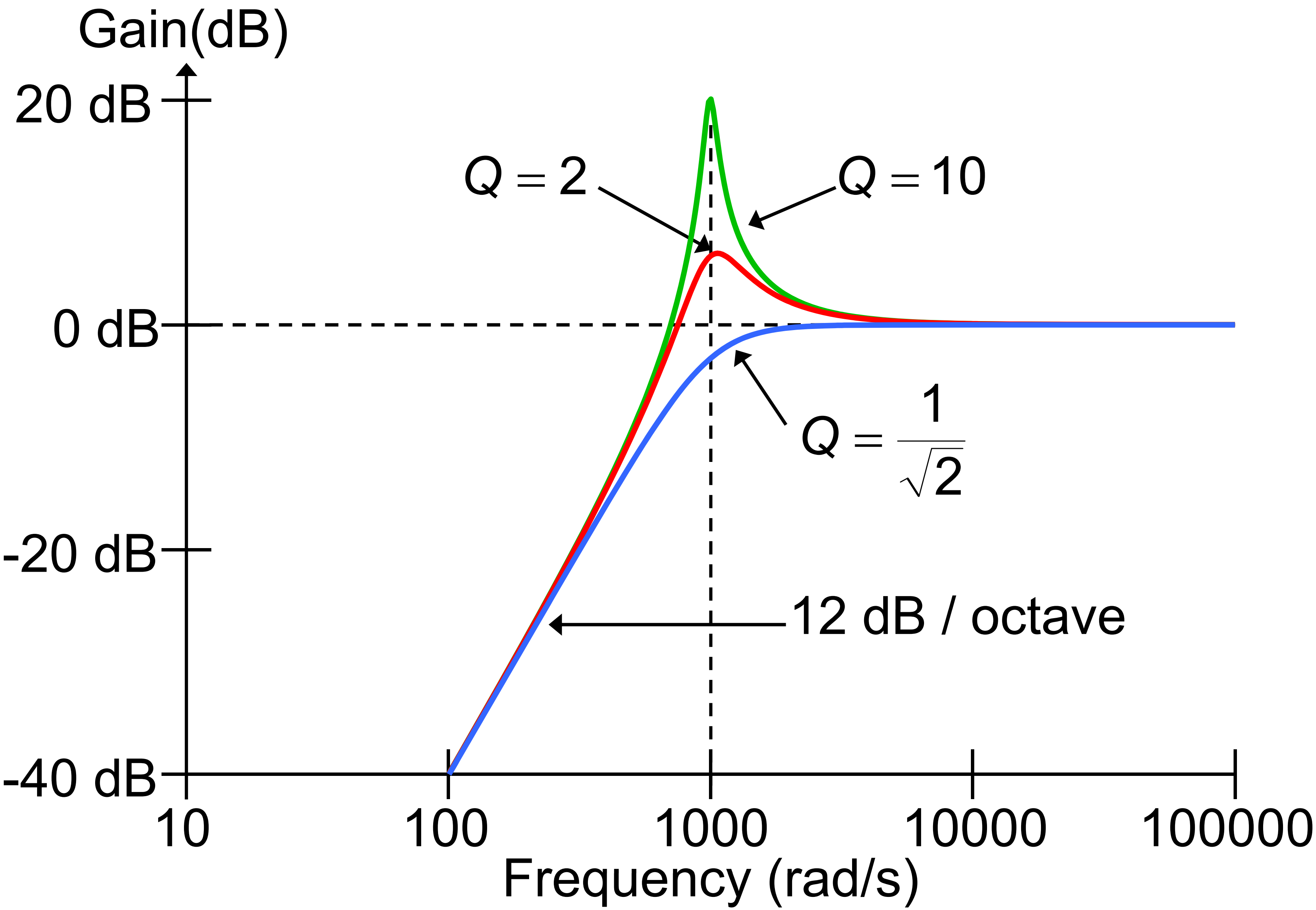
# Sallen-Key High-Pass Filter




$$H_{hp}(j\omega) = \frac{T_1 T_2 (j\omega)^2}{1 + 2T_1(j\omega) + T_1 T_2 (j\omega)^2}$$

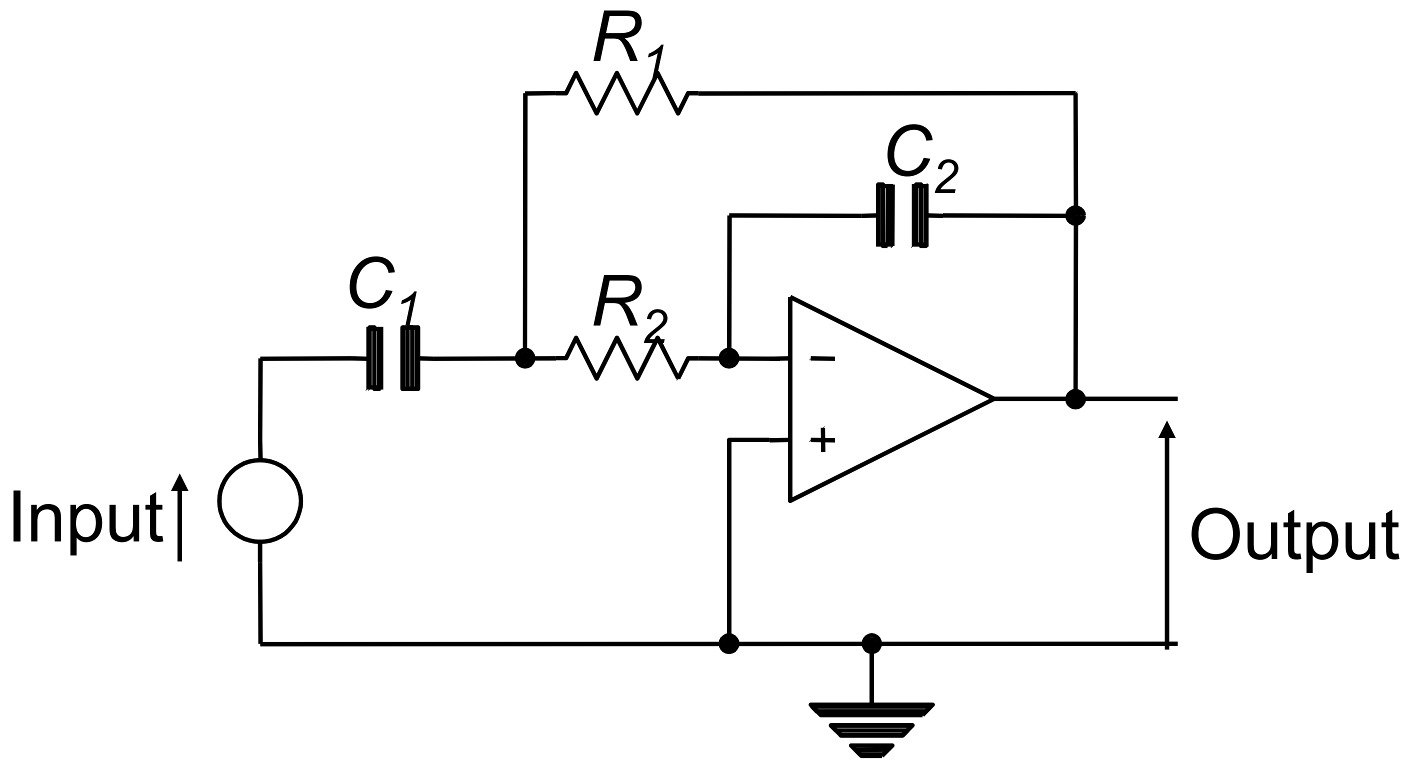
where : $\quad C_1 = C_2 \quad T_1 = R_1 C_1 \quad T_2 = R_2 C_2$

# Sallen-Key High-Pass Filter



# Rauch Band-Pass Filter

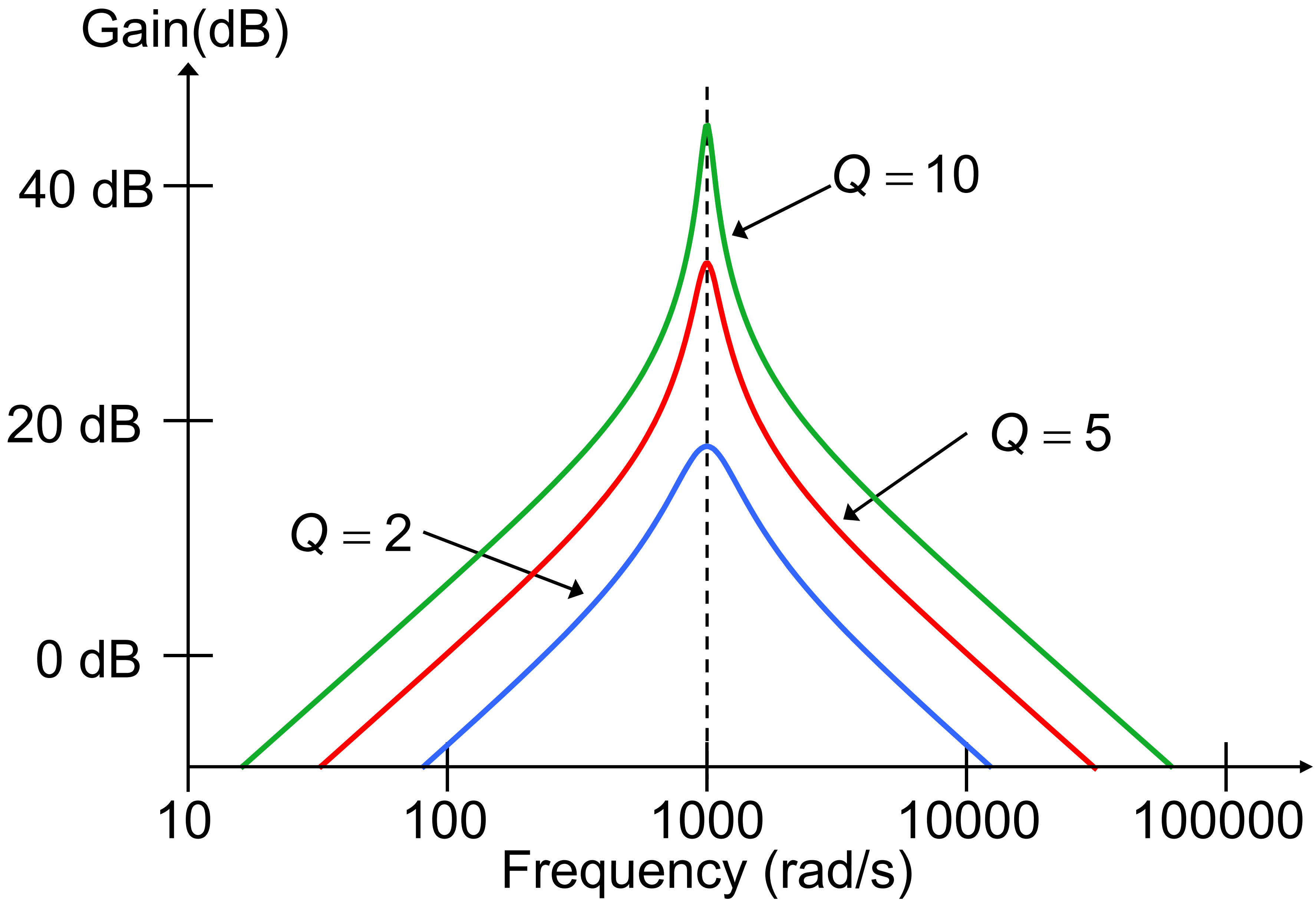





$$H_{bp}(j\omega) = \frac{-T_1(j\omega)}{1 + 2T_2(j\omega) + T_1T_2(j\omega)^2}$$

$$\text{where:} \quad R_1 = R_2 \quad T_1 = R_1C_1 \quad T_2 = R_2C_2$$

# Rauch Band-Pass Filter



# All-Pole Cascade Synthesis

Design example: 5th-order Chebychev low-pass filter has numerator and denominator coefficients:

| | |
|---|---|
| $a_0 = 0.06265$ | $b_0 = 0.06265$ |
| $a_1 = 0.0$ | $b_1 = 6.493 \times 10^{-5}$ |
| $a_2 = 0.0$ | $b_2 = 1.390 \times 10^{-8}$ |
| $a_3 = 0.0$ | $b_3 = 5.705 \times 10^{-12}$ |
| $a_4 = 0.0$ | $b_4 = 3.686 \times 10^{-16}$ |
| $a_5 = 0.0$ | $b_5 = 1.021 \times 10^{-19}$ |

The frequency response is factored into second order sections by finding the poles (there are no zeros)

# All-Pole Cascade Synthesis

The poles are the roots of the equation obtained by setting the denominator polynomial to zero:

$$-9.011\times10^{+2} + j3.751\times10^{+3}$$
$$-9.011\times10^{+2} - j3.751\times10^{+3}$$
$$-3.463\times10^{+2} + j6.070\times10^{+3}$$
$$-3.463\times10^{+2} - j6.070\times10^{+3}$$
$$-1.115\times10^{+3} + j0.0$$

The frequency response is of 5th-order so that the factors are two conjugate pairs of complex roots and a single real root

# All-Pole Cascade Synthesis

Conjugate pole pairs are combined:

$-9.011\times10^{+2} + j3.751\times10^{+3}$
$-9.011\times10^{+2} - j3.751\times10^{+3}$

$$D(j\omega) = (j\omega + 9.011\times10^{2} - j3.751\times10^{3})\times$$
$$(j\omega + 9.011\times10^{2} + j3.751\times10^{3})$$
$$= (j\omega)^2$$
$$+ (9.011\times10^{2} + 9.011\times10^{2})(j\omega)$$
$$+ (9.011\times10^{2})^2 + (3.751\times10^{3})^2$$
$$= (j\omega)^2 + 1.802\times10^{+3}(j\omega) + 1.488\times10^{+7}$$

# All-Pole Cascade Synthesis

The denominator of the 1st-order section:

$$D(\mathrm{j}\omega) = (\mathrm{j}\omega - p) = \mathrm{j}\omega + 1.115 \times 10^{+3}$$

Dividing through by $1.115 \times 10^{+3}$:

$$D(\mathrm{j}\omega) = 1 + 8.966 \times 10^{-4} (\mathrm{j}\omega)$$

Frequency response function of 1st-order filter:

$$H(\mathrm{j}\omega) = \frac{1}{1 + RC(\mathrm{j}\omega)}$$

Thus:

$$RC = 8.966 \times 10^{-4}$$

Let $R$=10 k$\Omega$; then $C$=89.66 nF.

# All-Pole Cascade Synthesis

Denominator of the 1st 2nd-order section:

$$D(j\omega) = (j\omega)^2 + 1.802\times10^{+3}(j\omega) + 1.488\times10^{+7}$$

Dividing through by $1.488\times10^{+7}$:

$$D(j\omega) = 1 + 1.211\times10^{-4}(j\omega) + 6.718\times10^{-8}(j\omega)^2$$

Frequency response function of 2nd-order filter:

$$H_{lp}(j\omega) = \frac{1}{1 + 2T_2(j\omega) + T_1T_2(j\omega)^2}$$

Thus: $2T_2 = 1.211\times10^{-4} \quad \rightarrow \quad T_2 = 6.054\times10^{-5}$

$$T_1T_2 = 6.718\times10^{-8} \quad \rightarrow \quad T_1 = 1.110\times10^{-3}$$

Let $R_1$=$R_2$=10 kΩ; $C_1$=111.0 nF, $C_2$=6.054nF

# All-Pole Cascade Synthesis

Denominator of the 1st 2nd-order section:

$$D(j\omega) = (j\omega)^2 + 6.926 \times 10^{+2}(j\omega) + 3.696 \times 10^{+7}$$

Dividing through by $3.696{\times}10^{+7}$:

$$D(j\omega) = 1 + 1.874 \times 10^{-5}(j\omega) + 2.706 \times 10^{-8}(j\omega)^2$$

Frequency response function of 2nd-order filter:

$$H_{lp}(j\omega) = \frac{1}{1 + 2T_2(j\omega) + T_1T_2(j\omega)^2}$$

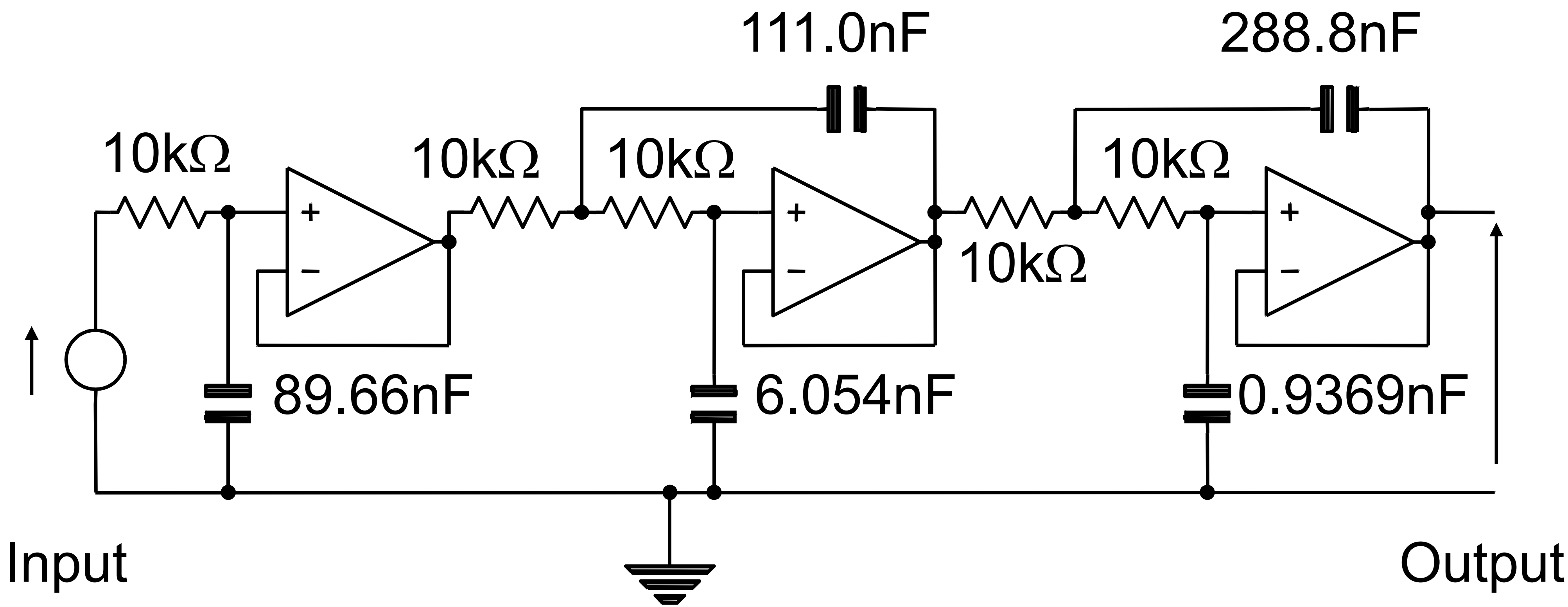
Thus: $2T_2 = 1.874 \times 10^{-5} \quad \rightarrow \quad T_2 = 9.369 \times 10^{-6}$

$$T_1T_2 = 2.706 \times 10^{-8} \quad \rightarrow \quad T_1 = 2.888 \times 10^{-3}$$

Let $R_1$=$R_2$=10 kΩ; $C_1$=288.8 nF, $C_2$=0.9369nF

# All-Pole Cascade Synthesis

Complete cascade synthesis of Chebychev active filter:



# General Cascade Synthesis

If the response is not derived from an all-pole approximation then the second-order factors will be of general form:

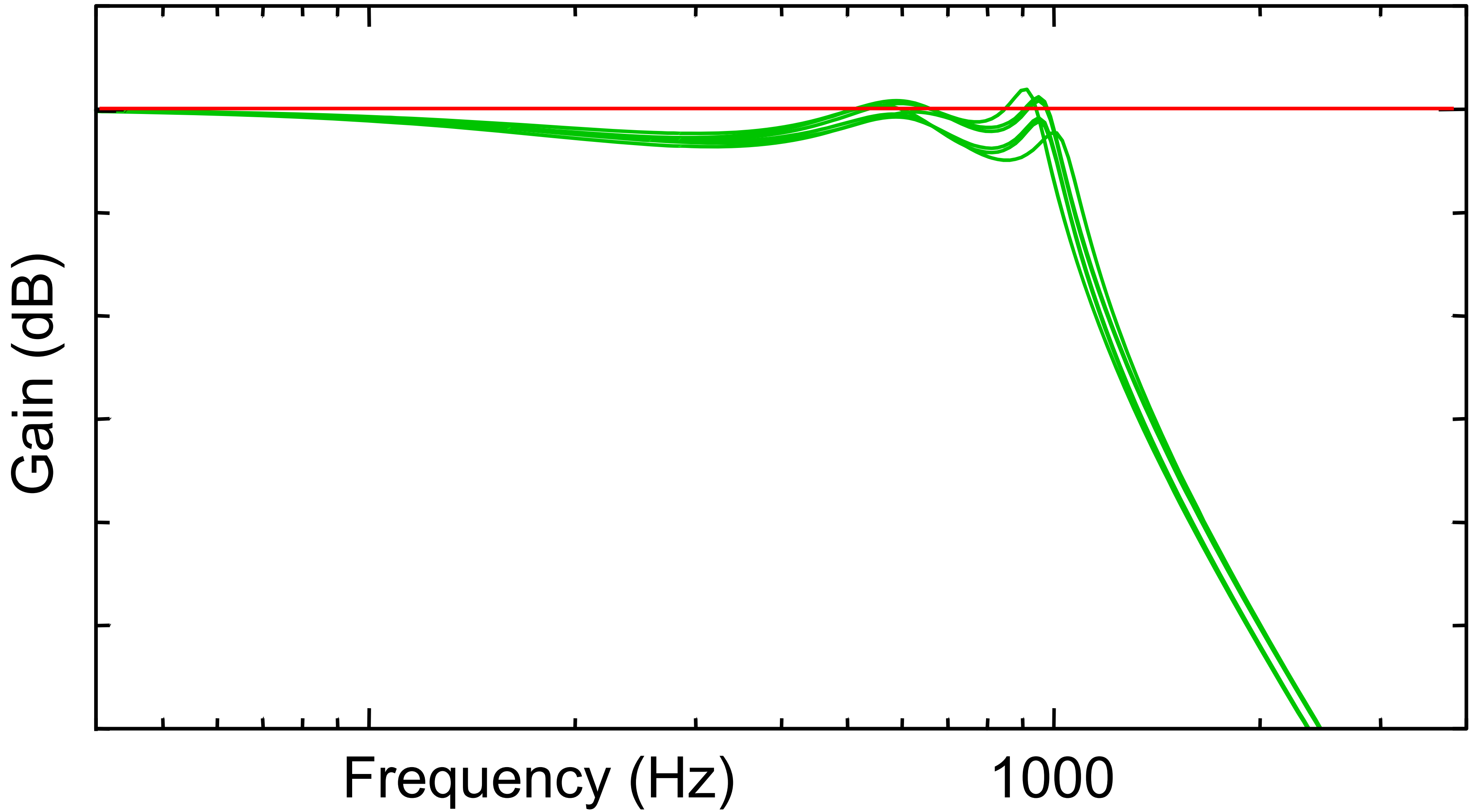
$$H(\mathrm{j}\omega) = \frac{a_0 + a_1(\mathrm{j}\omega) + a_2(\mathrm{j}\omega)^2}{b_0 + b_1(\mathrm{j}\omega) + b_2(\mathrm{j}\omega)^2}$$

Rauch or Sallen-Key second-order filters are unsuitable and a more complex filter configuration must be used

The ring-of-three filter (aka the bi-quad or state-variable filter) can be used

# Component Value Sensitivity

5th-order Chebychev low-pass implemented as a cascade of active Sallen-Key sections:



# Copying Methods

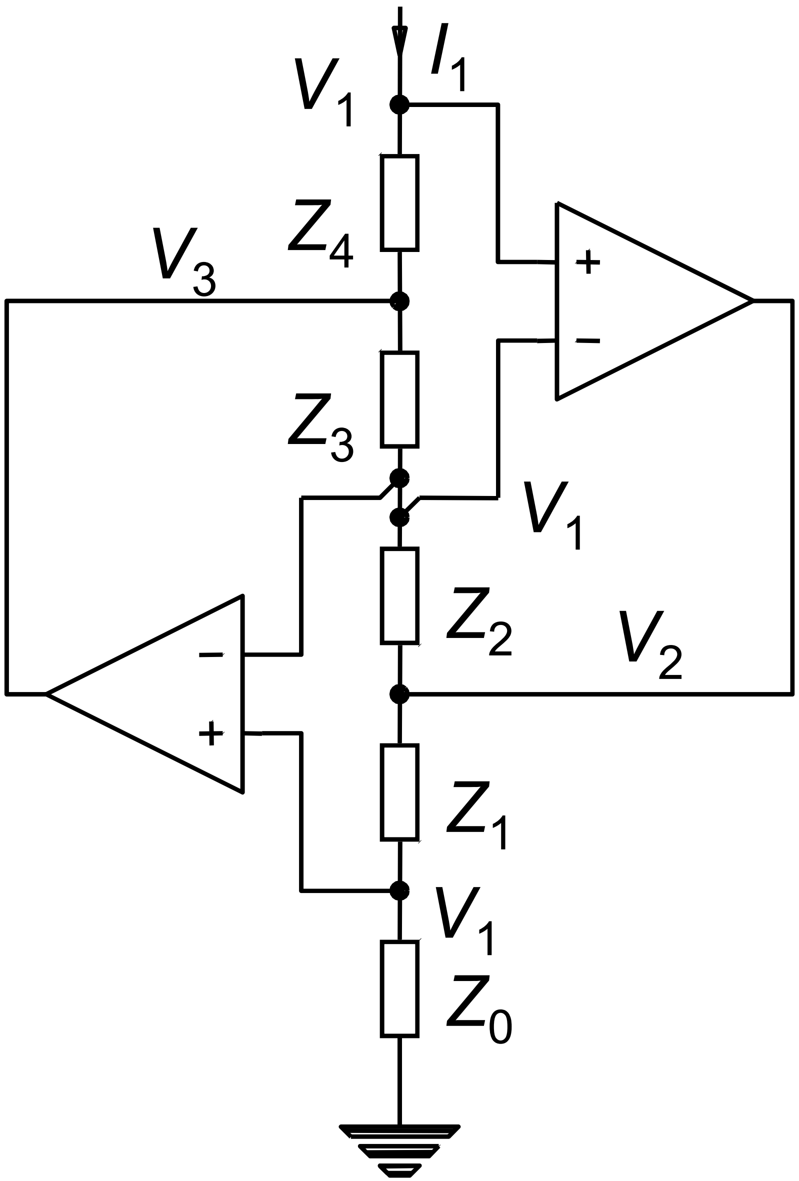
Copying methods are ways of designing active filters with the same low sensitivity properties of passive equally-terminated ladder filters

The starting point for all copying methods is a prototype passive filter

This is then copied in some way which preserves the desirable properties of the passive filters but which eliminates the inductors

The copying method that will be described here is based on inductor simulation

# Positive Immittance Converters




$$V_1 = V_3 + I_1 Z_4$$

$$V_1 = V_3 \frac{Z_2}{Z_2 + Z_3} + V_2 \frac{Z_3}{Z_2 + Z_3}$$

$$V_1 = V_2 \frac{Z_0}{Z_0 + Z_1}$$

# Positive Immittance Converters

University of Reading

$$V_1 = V_3 + I_1 Z_4$$

$$V_1 = V_3 \frac{Z_2}{Z_2 + Z_3} + V_2 \frac{Z_3}{Z_2 + Z_3}$$

$$V_1 = V_2 \frac{Z_0}{Z_0 + Z_1}$$

Substitute to remove $V_2$ and $V_3$:

$$V_1(Z_2 + Z_3) = V_3 Z_2 + V_2 Z_3$$

$$= V_3 Z_2 + V_1 \frac{Z_3(Z_1 + Z_0)}{Z_0}$$

$$= Z_2(V_1 - I_1 Z_4) + V_1 \frac{Z_3(Z_1 + Z_0)}{Z_0}$$

# Positive Immittance Converters

Multiply both sides by $Z_0$:

$$V_1(Z_0Z_2 + Z_0Z_3) = V_1Z_0Z_2 - I_1Z_0Z_2Z_4 + V_1(Z_1Z_3 + Z_0Z_3)$$

Remove cancelling terms:

$$V_1Z_1Z_3 = I_1Z_0Z_2Z_4$$

The impedance $Z_i$ of the PIC is given by:

$$Z_i = \frac{V_1}{I_1} = \frac{Z_0Z_2Z_4}{Z_1Z_3}$$

# Inductor Simulation

Let $Z_0$, $Z_1$, $Z_2$, and $Z_4$ be resistors of value $R$, and $Z_3$ be a capacitor of value $C$:

$$Z_i = \frac{Z_0 Z_2 Z_4}{Z_1 Z_3} = \frac{R^3}{R / j\omega C} = j\omega C R^2$$

or:

$$Z_i = j\omega L \quad \text{where } L = CR^2$$

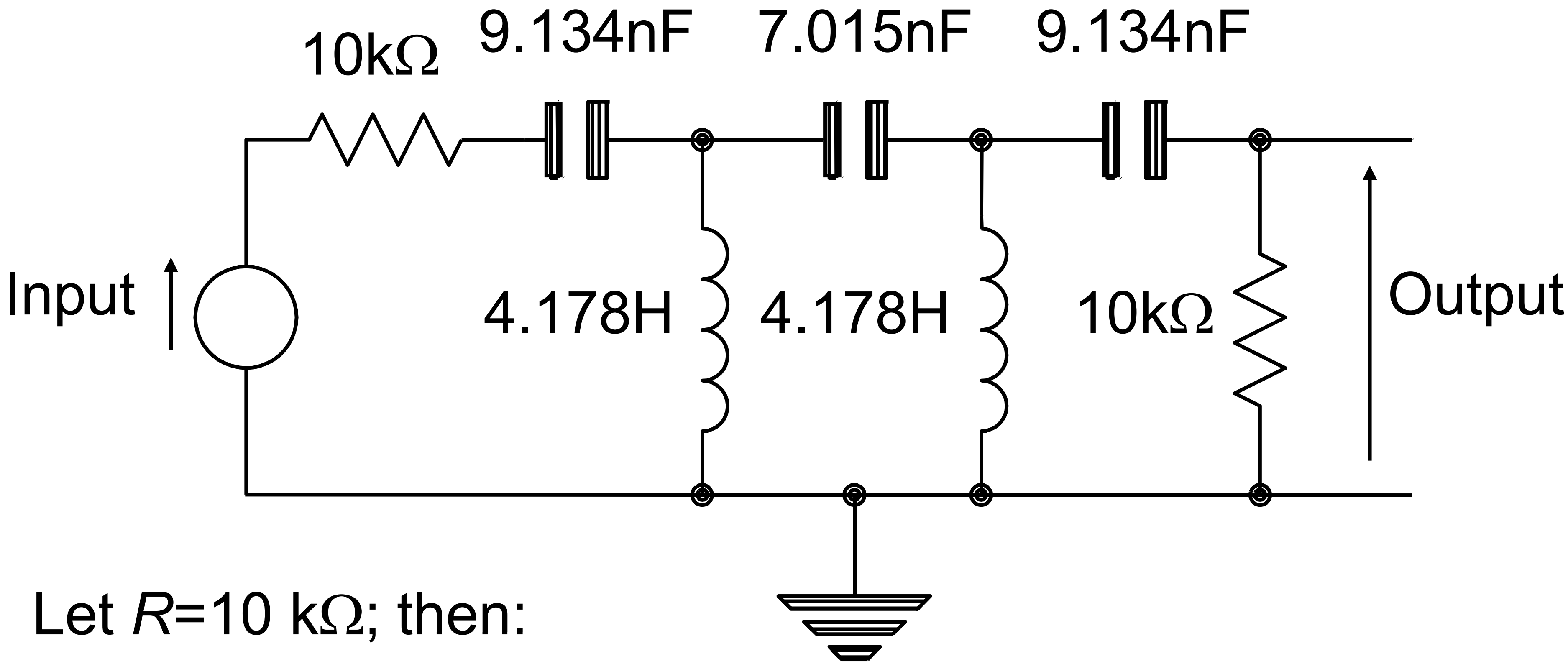
The PIC therefore simulates a grounded inductor of value $CR^2$

By correct choice of $R$ and $C$ the PIC can be made to simulate any required inductance.

# Copying Methods

University of Reading

5th-order Chebychev high-pass passive filter:

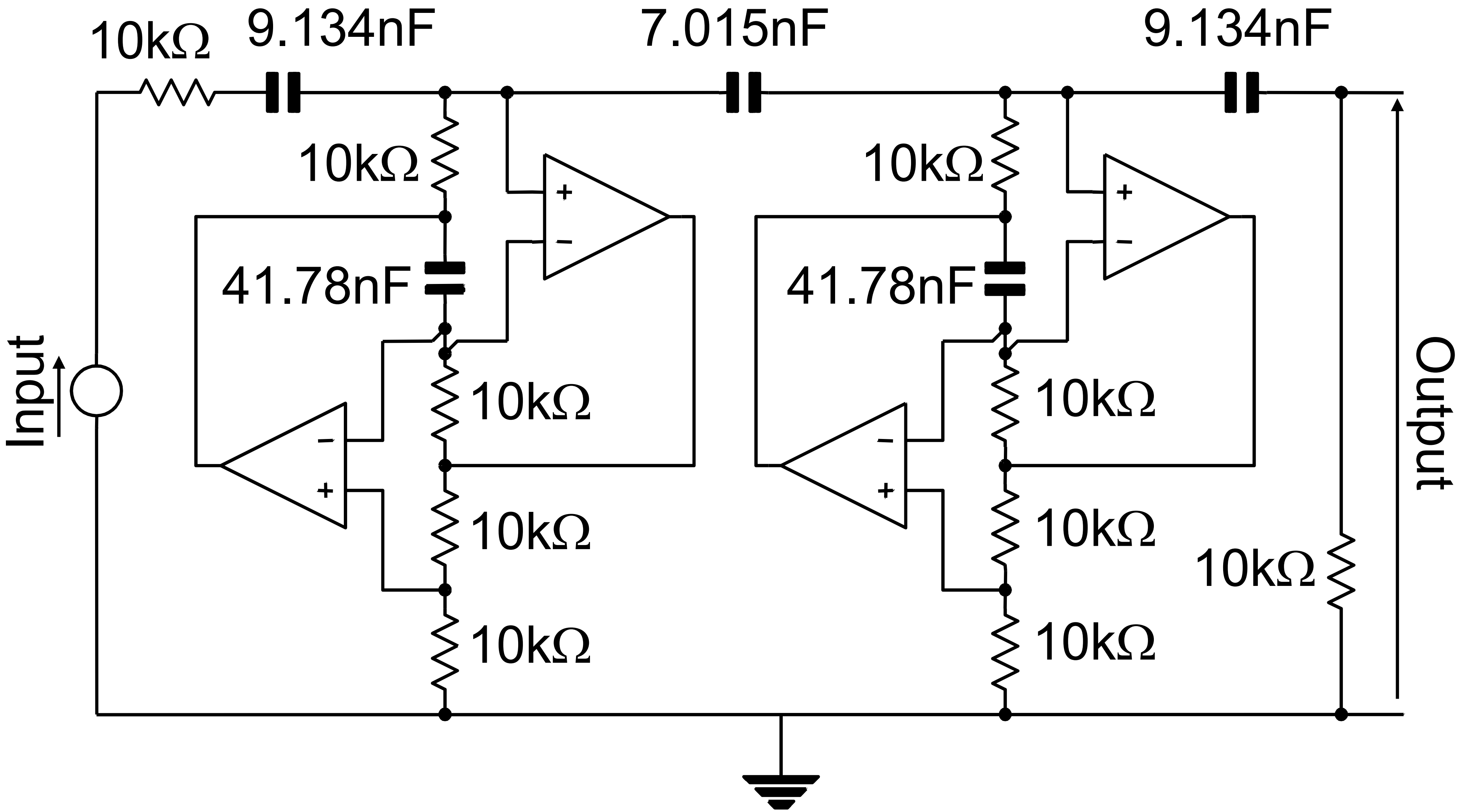



Let $R$=10 kΩ; then:

$$CR^2 = 4.178\text{H} \quad \rightarrow \quad C = 4.178 \times 10^{-8}\,\text{F}$$

# Copying Methods

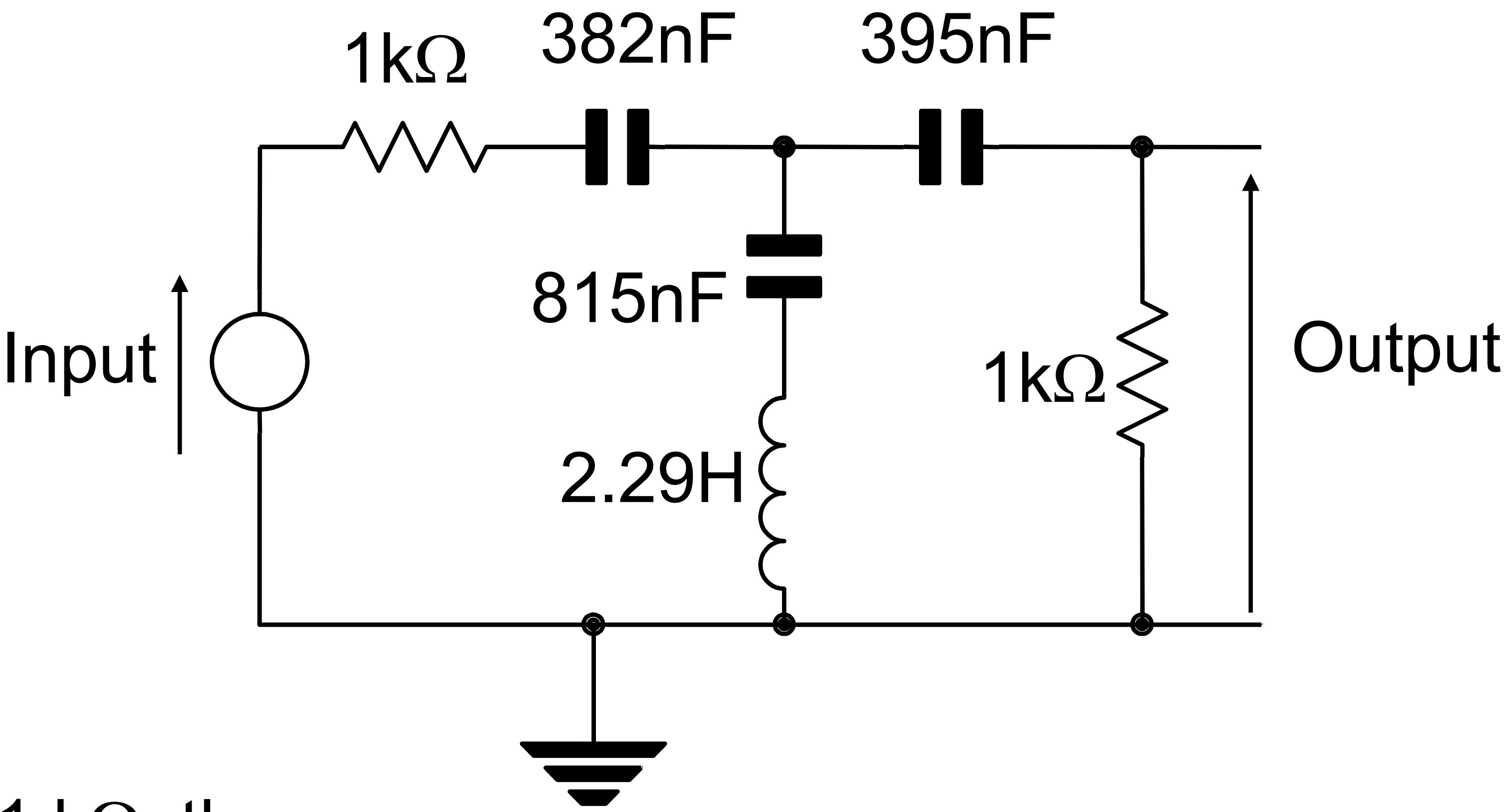
5th-order Chebychev high-pass active filter:



# Copying Methods

University of Reading

3th-order elliptic high-pass passive filter:

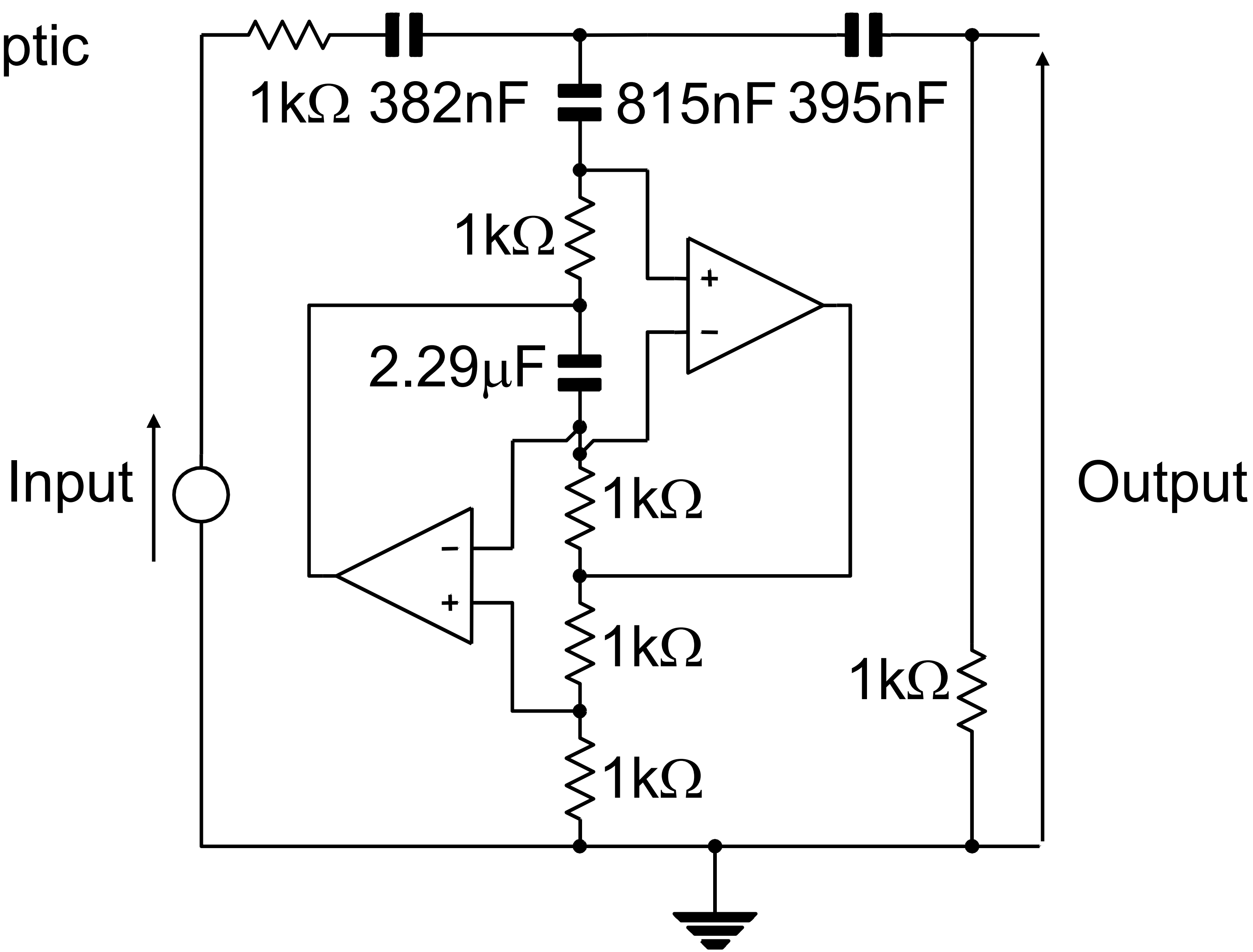



Let $R$=1 kΩ; then:

$$CR^2 = 2.29\text{H} \quad \rightarrow \quad C = 2.29 \times 10^{-6}\,\text{F}$$

# Copying Methods

3th-order elliptic high-pass active filter:



# Analogue Filter Design