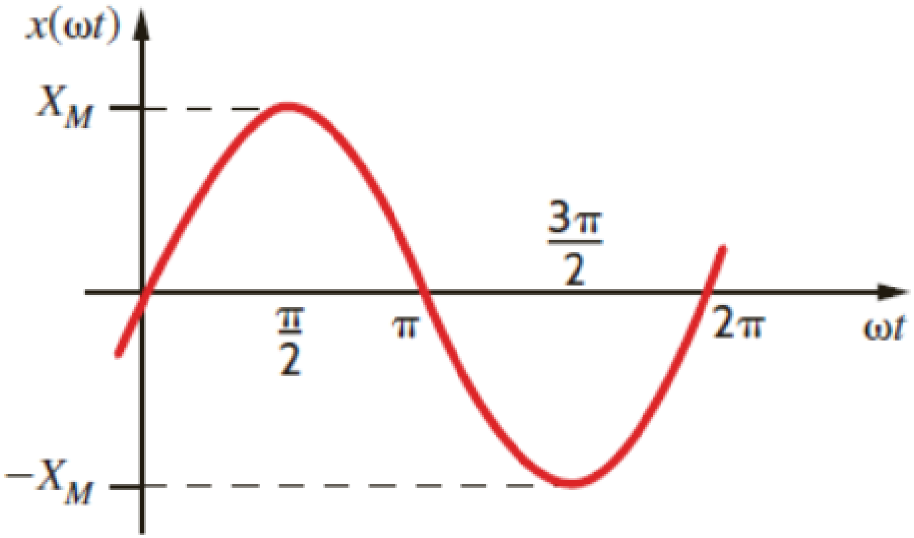
AC Steady State Analysis → Chapter #8

- **Basic Characteristics of Sinusoidal Functions**
- **Phasor / Inverse Phasor Transformations & Diagrams**
- **Impedance and Admittance → R, L, C**
- **Equivalent Impedance / Equivalent Admittance**
- **Equivalent Circuit in Frequency Domain**
- **Apply Circuit Analysis Techniques to Frequency Domain Circuits**

Sinusoids

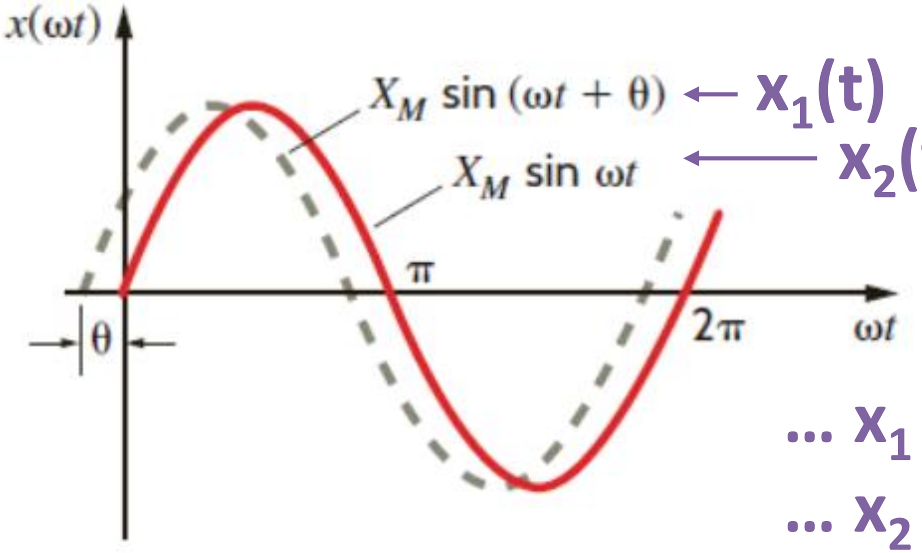
$$x(t) = X_M \cdot \sin(\omega t + \theta)$$

- X_m → amplitude / maximum value
- ω → radian / angular frequency
- θ → phase angle



- T → period
- $f = \frac{1}{T}$ → # cycles per second

$$\omega = \frac{2\pi}{T} = 2\pi f$$



← $x_1(t)$

← $x_2(t)$

… x_1 leads x_2 by θ

… x_2 lags x_1 by θ

Trigonometric Identities

$$-\cos(\omega t) = \cos(\omega t \mp 180^{o})$$

$$-\sin(\omega t) = \sin(\omega t \mp 180^{o})$$

$$\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

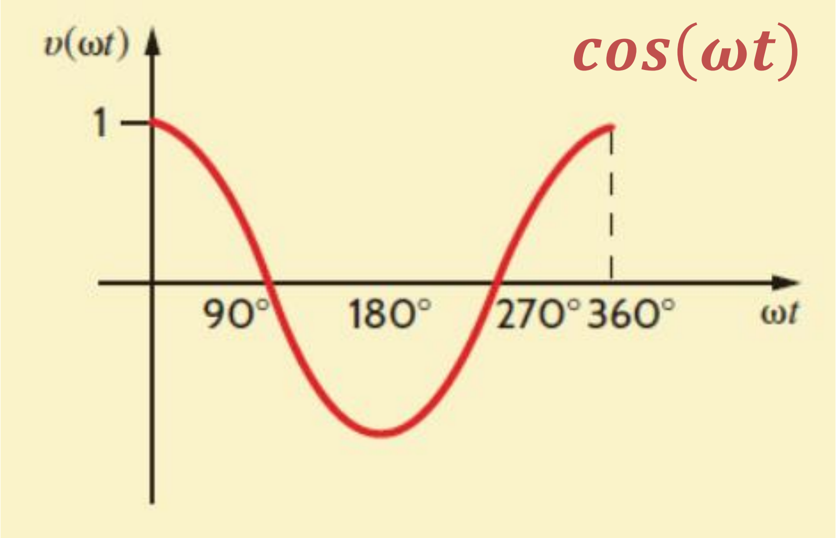
$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Example 8.1

Plot the waveforms for the following functions:

- $v_1(t) = \cos(\omega t + 45^o)$
- $v_2(t) = cos(\omega t + 225^o)$
- $v_3(t) = cos(\omega t - 315^o)$



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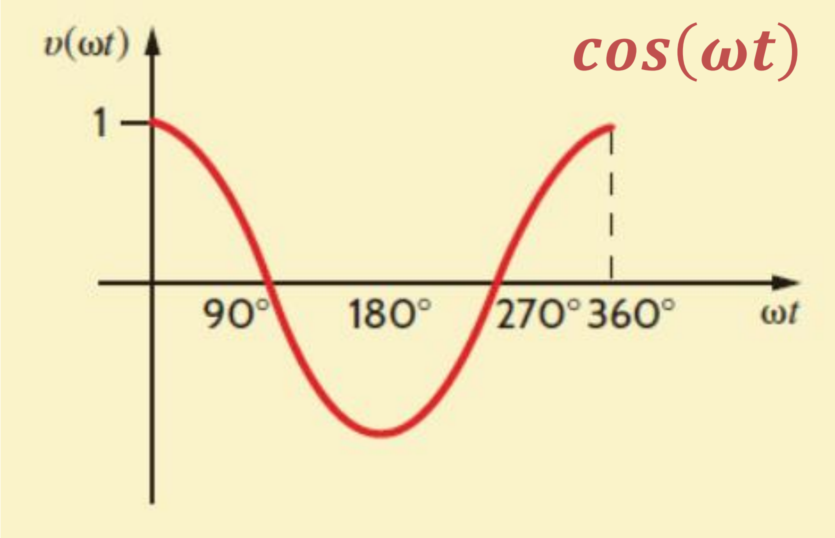
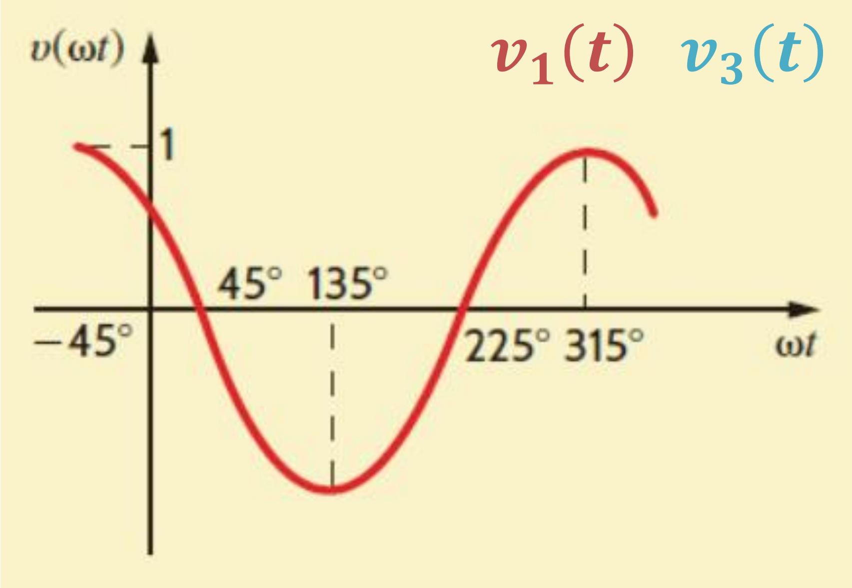
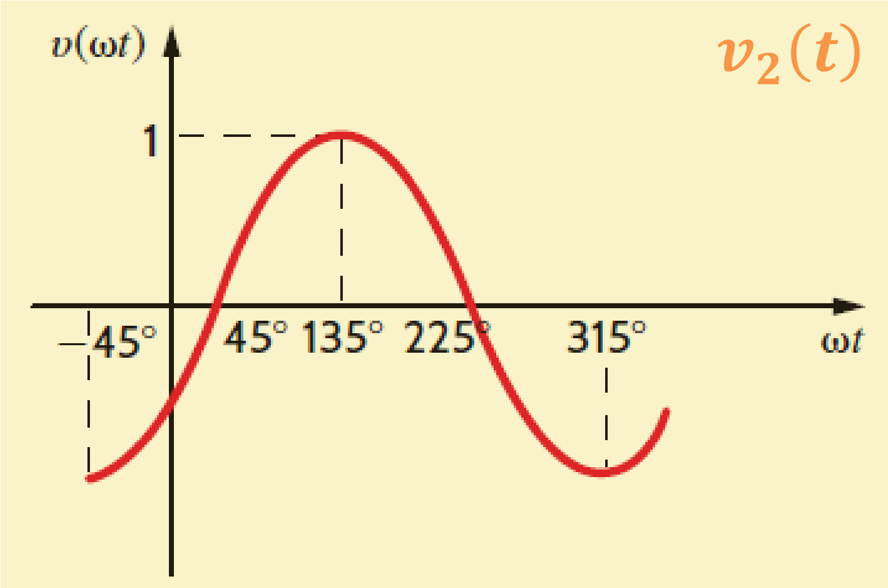






Example 8.2

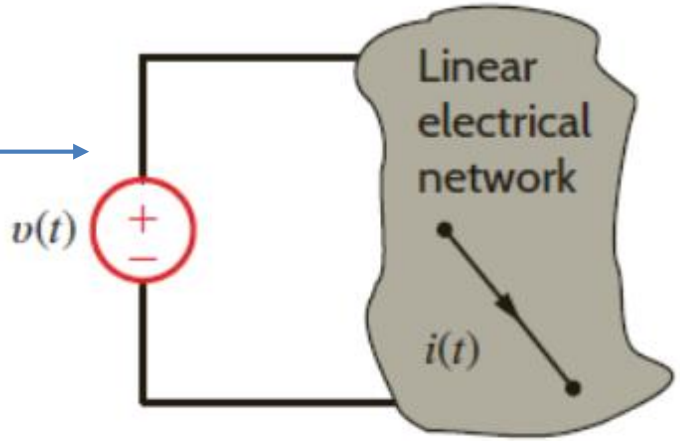
Determine the frequency and the phase angle between the two voltages:

- $v_1(t) = 12\sin(1000 \cdot t + 60^o)\ V\ and$
- $v_2(t) = -6\cos(1000 \cdot t + 30^o)\quad V.$

Sinusoidal Forcing Functions

Constant Forcing Function → Linear Network

∴ Constant Steady State Response



Sinusoidal Forcing Function → Linear Network

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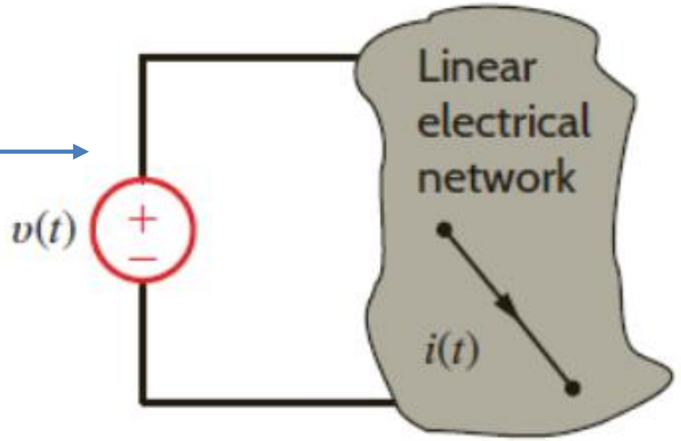
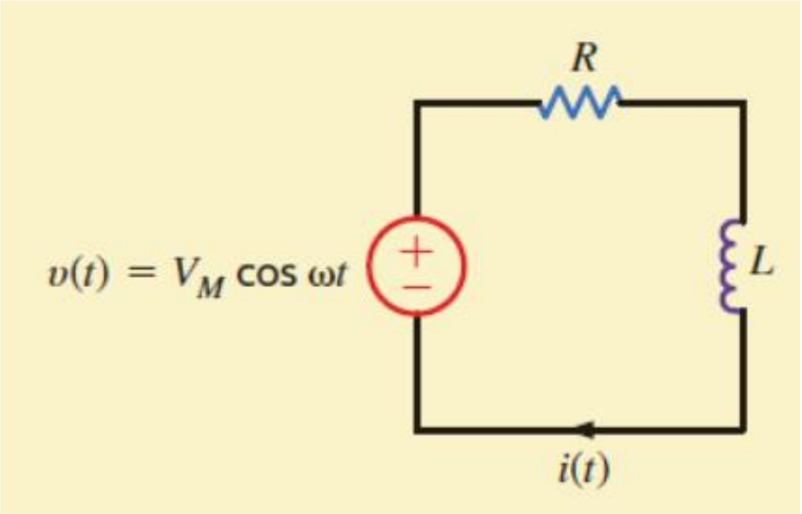


Sinusoidal Forcing Function → Linear Network

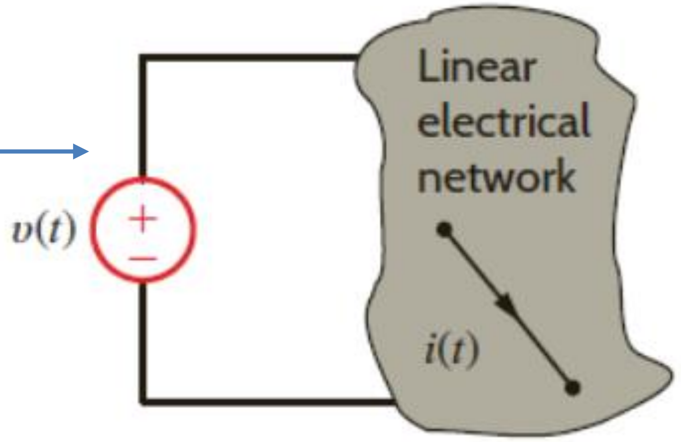
∴ Sinusoidal Steady State Response

Example 8.3

Derive the expression for the current i(t).



Sinusoidal Forcing Functions



Constant Forcing Function → Linear Network
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Sinusoidal Forcing Function → Linear Network
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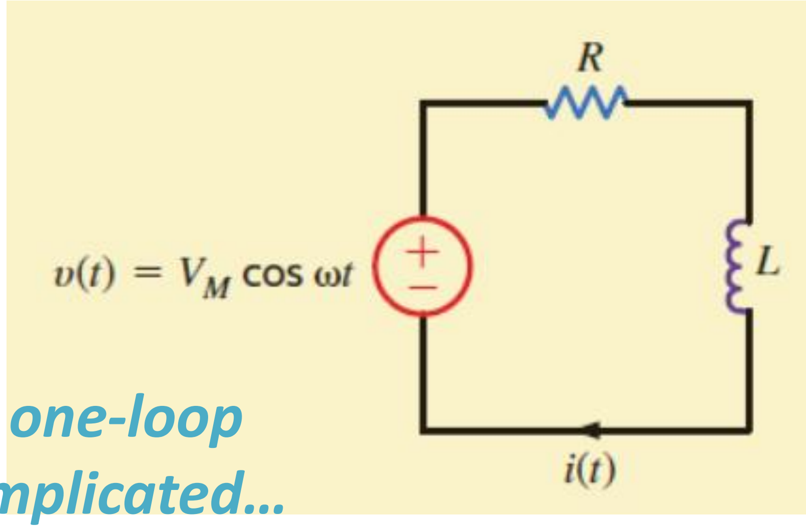

$$v(t) = R \cdot i(t) + L\frac{di(t)}{dt}$$

$$where \;\; i(t) = I_M \cos(\omega t + \varphi)$$

Solution to the one-loop circuit is very complicated…

Sinusoidal Time Functions ↔ Complex Numbers

This relationship leads to algebraic set of equations for currents and voltages in a network in which the coefficients of the variables are complex numbers.

Euler's Equation

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$Re(e^{j\omega t})$ *real part*

$Im(e^{j\omega t})$ *imaginary part*

$$j = \sqrt{-1}$$

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$$j = \sqrt{-1}$$

Forcing Function

$$v(t) = V_M e^{j\omega t}$$

$$v(t) = V_M\cos(\omega t) + jV_M\sin(\omega t)$$

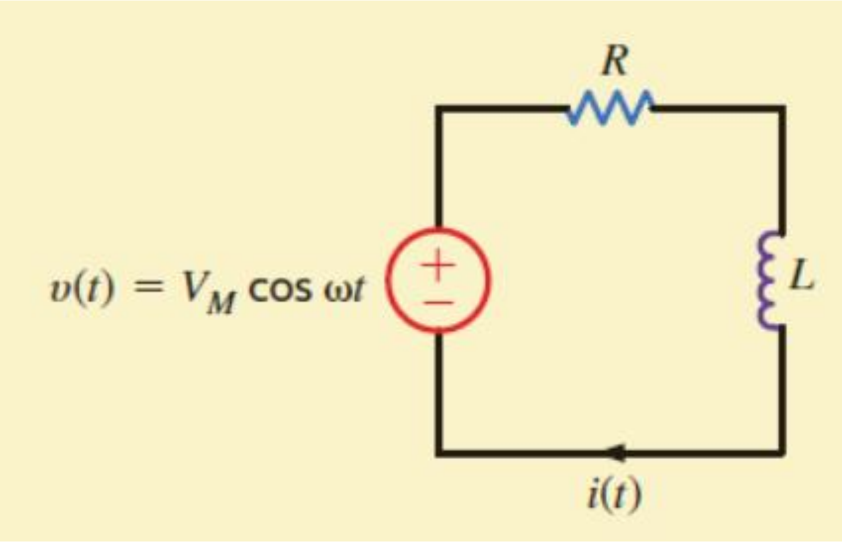
As a consequence of linearity, the principle of superposition applies, hence …

$$i(t) = I_M\cos(\omega t + \varphi) + jI_M\sin(\omega t + \varphi)$$

$$i(t) = I_M e^{j(\omega t + \varphi)}$$

Example 8.4

Derive the expression for the current i(t) using the forcing function $v(t) = V_M e^{j\omega t}$ instead.



KVL

$$v(t) = R \cdot i(t) + L\frac{di(t)}{dt}$$

response

$$i(t) = I_M e^{j(\omega t+\varphi)}$$

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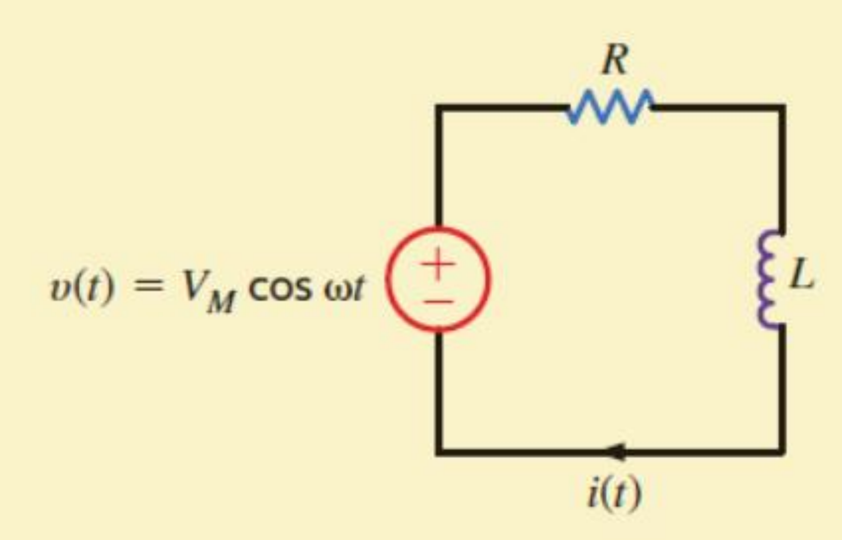


KVL

$$v(t) = R \cdot i(t) + L\frac{di(t)}{dt}$$

response

$$i(t) = I_M e^{j(\omega t + \varphi)}$$

Differential Equation

$$V_M e^{j\omega t} = RI_M e^{j(\omega t+\varphi)} + L\frac{d}{dt}\left(I_M e^{j(\omega t+\varphi)}\right)$$

Solution

$$\therefore I_M = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\therefore \varphi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$i(t) = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

Phasors

Representation of a complex number with just the magnitude and the phase angle.

- ***Forcing Function***

$$v(t) = V_M e^{j(\omega t+\theta)} = V_M e^{j\omega t} e^{j\theta}$$

- ***Steady State Response***

$$i(t) = I_M e^{j(\omega t+\varphi)} = I_M e^{j\omega t} e^{j\varphi}$$

$e^{j(\omega t)} \rightarrow$ **common to every term**

Phasors

Representation of a complex number with just the magnitude and the phase angle.

- ***Forcing Function***

$$v(t) = V_M e^{j(\omega t+\theta)} = V_M e^{j\omega t} e^{j\theta}$$

- ***Steady State Response***

$$i(t) = I_M e^{j(\omega t+\varphi)} = I_M e^{j\omega t} e^{j\varphi}$$

$e^{j(\omega t)} \rightarrow$ common to every term

Assuming…

$$v(t) = V_M \cos(\omega t + \theta) = Re\left[V_M e^{j(\omega t+\theta)}\right]$$

$$i(t) = I_M \cos(\omega t + \varphi) = Re\left[V_M e^{j(\omega t+\varphi)}\right]$$

$$V = Re\left[V_M \langle\theta\ e^{j\omega t}\right] = V_M\langle\theta$$

$$I = Re\left[I_M \langle\varphi\ e^{j\omega t}\right] = I_M\langle\varphi$$

phasor

TABLE 8.1 Phasor representation

TIME DOMAIN	FREQUENCY DOMAIN
$A \cos(\omega t \pm \theta)$	$A\underline{/\pm\theta}$
$A \sin(\omega t \pm \theta)$	$A\underline{/\pm\theta - 90°}$

Learning Assessment

E8.3 – Convert the following voltage functions to phasors

- $v_1(t) = 12\cos(377t - 425^o)\ V$
- $v_2(t) = 18\, sin(2513t + 4.2^o)\ V$

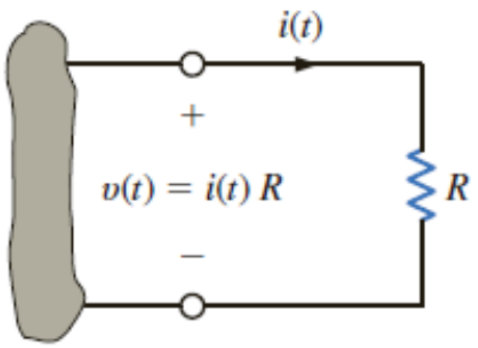
E8.4 – Convert the following phasor to the time domain if the frequency is 400Hz.

- $V_3 = 10\ \langle 20^o\ V$
- $V_4 = 12\ \langle -60^o\ V$

Phasor Relationships → Resistor

- $v(t) = V_M e^{j(\omega t+\theta_v)}$
- $i(t) = I_M e^{j(\omega t+\theta_i)}$

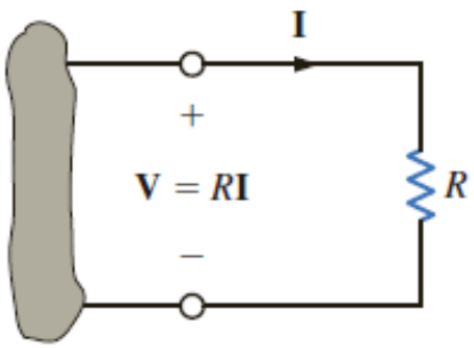
- *Time Domain*



$$v(t) = R \cdot i(t)$$

$$V_M e^{j(\omega t+\theta_v)} = R \cdot I_M e^{j(\omega t+\theta_i)}$$

- *Frequency Domain*



Phasor Relationships → Resistor

- $v(t) = V_M e^{j(\omega t + \theta_v)}$
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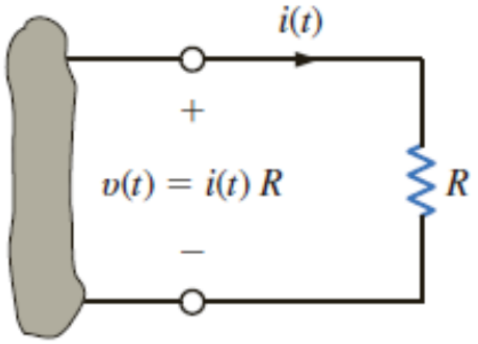
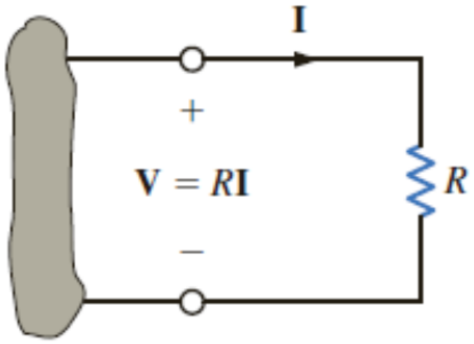
- ***Time Domain***



$$v(t) = R \cdot i(t)$$

$$V_M e^{j(\omega t+\theta_v)} = R \cdot I_M e^{j(\omega t+\theta_i)}$$

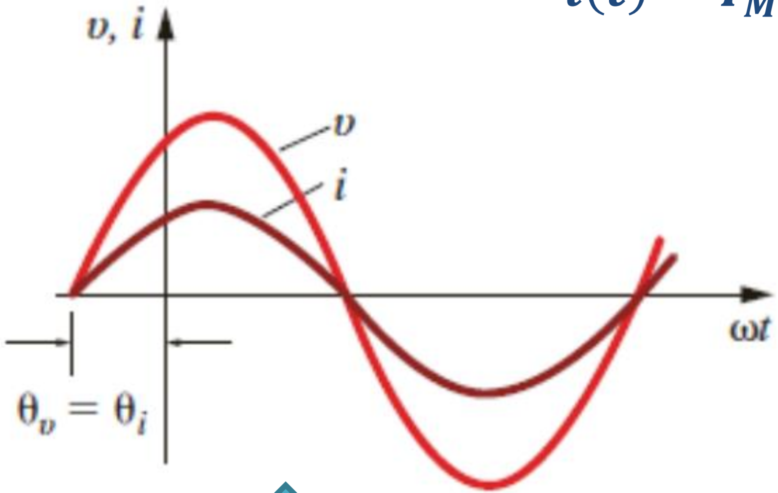
- ***Frequency Domain***

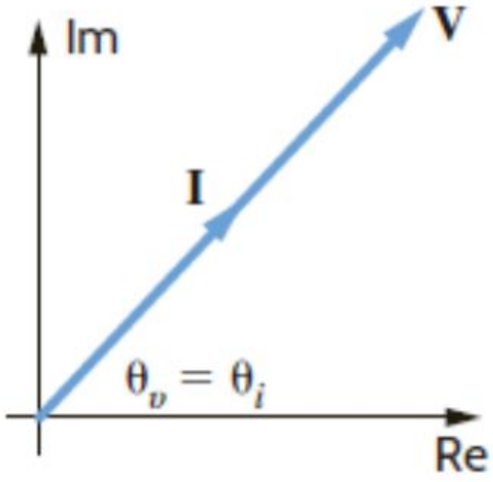


$$\therefore V_M e^{j\theta_v} = R \cdot I_M e^{j\theta_i}$$

$$V_M \angle \theta_v = R \cdot I_M \angle \theta_i$$

$\therefore \theta_v = \theta_i \rightarrow v(t)$ *and* $i(t)$ *are in phase*





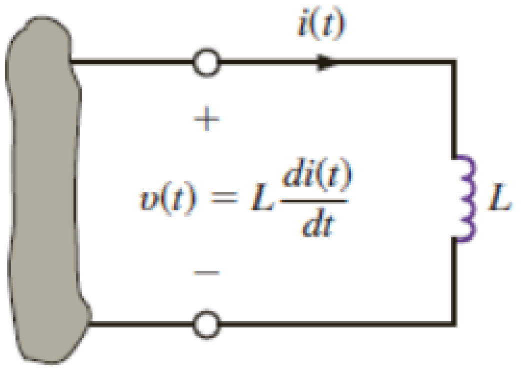
Example 8.6

If the voltage $v(t) = 24\cos(377t + 75^o)\ V$ is applied to a 6-Ω resistor, find the resultant current.

Phasor Relationships → Inductor

- $v(t) = V_M e^{j(\omega t + \theta_v)}$
- $i(t) = I_M e^{j(\omega t + \theta_i)}$

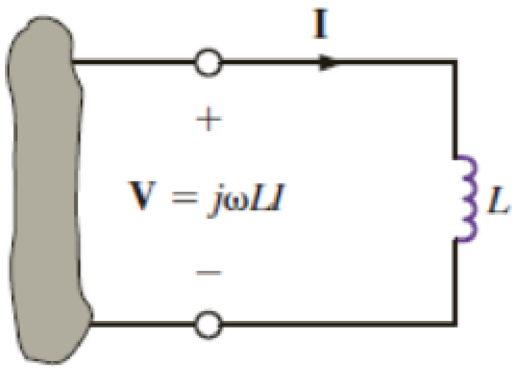
- *Time Domain*



$$v(t) = L \cdot \frac{di(t)}{dt}$$

$$V_M e^{j(\omega t + \theta_v)} = L \cdot \frac{d}{dt} I_M e^{j(\omega t + \theta_i)}$$

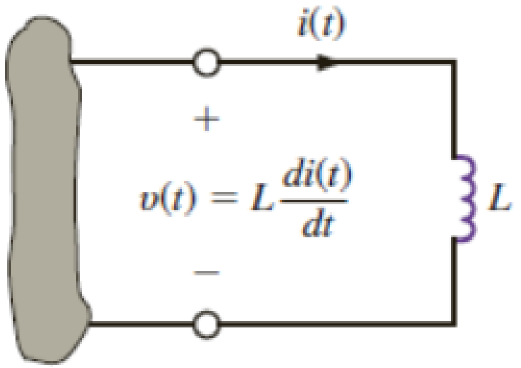
- *Frequency Domain*



Phasor Relationships → Inductor

- $v(t) = V_M e^{j(\omega t + \theta_v)}$
- $i(t) = I_M e^{j(\omega t + \theta_i)}$

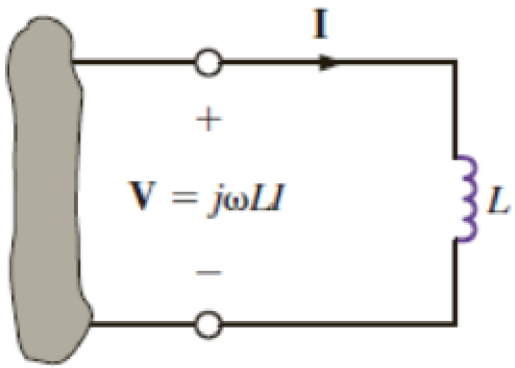
- *Time Domain*



$$v(t) = L \cdot \frac{di(t)}{dt}$$

$$V_M e^{j(\omega t + \theta_v)} = L \cdot \frac{d}{dt} I_M e^{j(\omega t + \theta_i)}$$

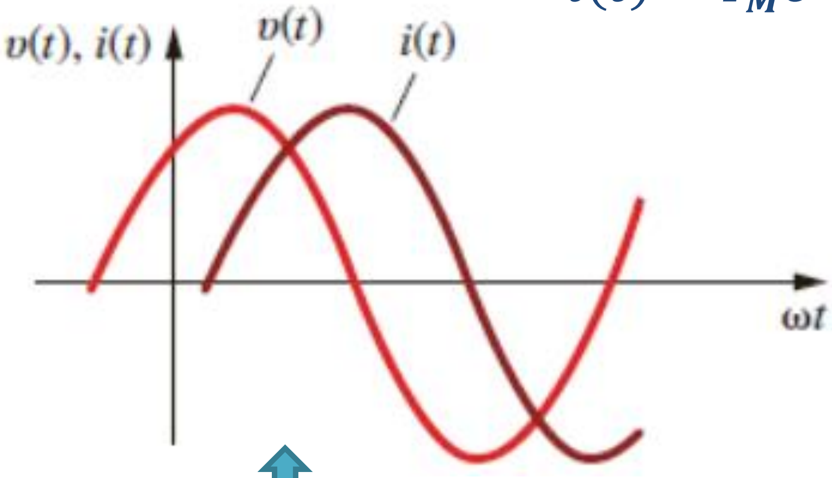
- *Frequency Domain*

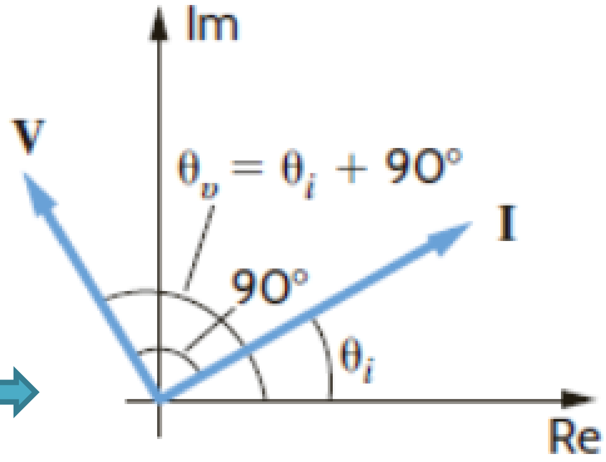


$$\therefore V_M e^{j\theta_v} = j\omega L \cdot I_M e^{j\theta_i}$$

$$V_M \angle \theta_v = j\omega L \cdot I_M \angle \theta_i$$

$$\therefore \theta_v = \theta_i + 90^o \rightarrow v(t)\ lead\ i(t)\ by\ 90^o$$

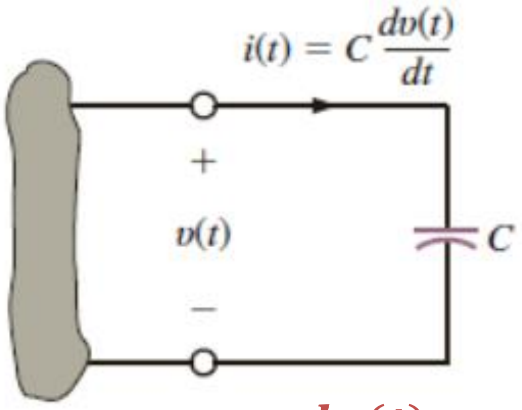




Phasor Relationships → Capacitance

- $v(t) = V_M e^{j(\omega t + \theta_v)}$
- $i(t) = I_M e^{j(\omega t + \theta_i)}$

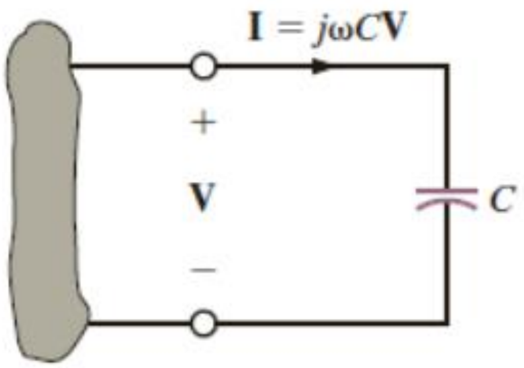
- *Time Domain*



$$i(t) = C \cdot \frac{dv(t)}{dt}$$

$$I_M e^{j(\omega t + \theta_i)} = C \cdot \frac{d}{dt} V_M e^{j(\omega t + \theta_v)}$$

- *Frequency Domain*



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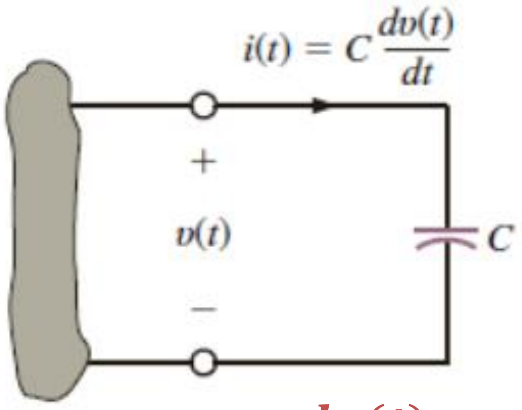
- ***Time Domain***



$$i(t) = C \cdot \frac{dv(t)}{dt}$$

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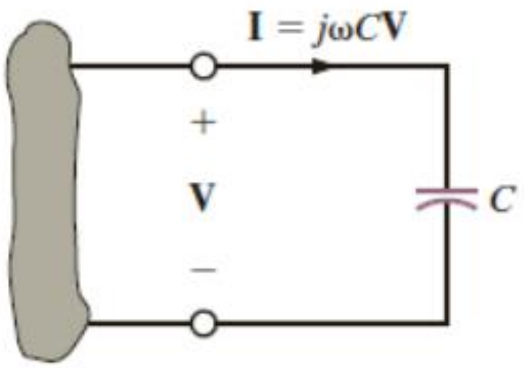
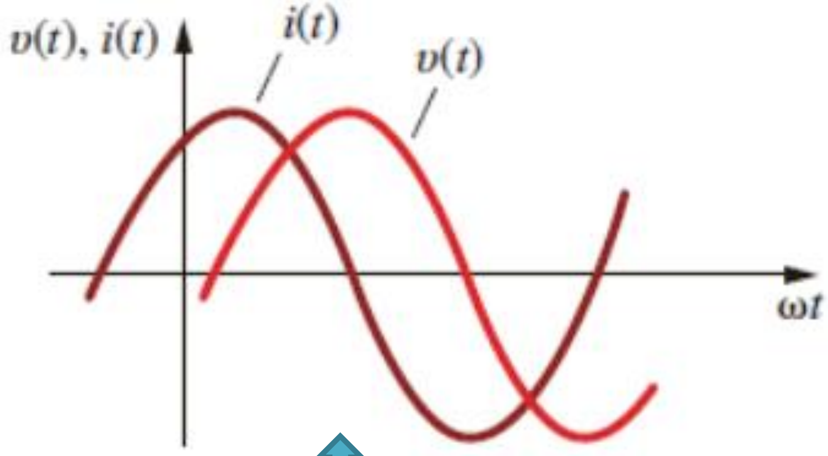
- ***Frequency Domain***

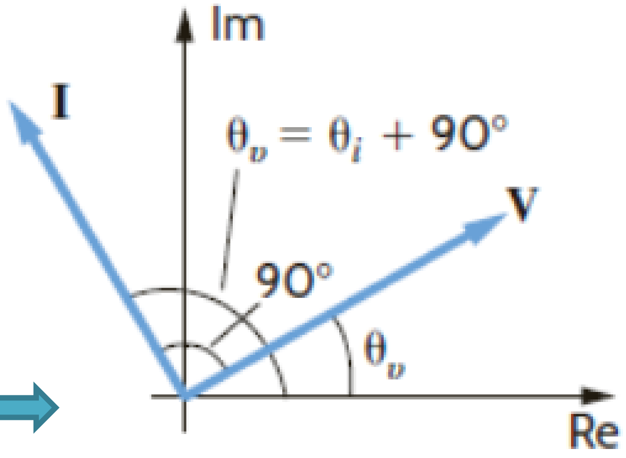




$$\therefore I_M e^{j\theta_i} = j\omega C \cdot V_M e^{j\theta_v}$$

$$I_M \angle \theta_i = j\omega C \cdot V_M \angle \theta_v$$

$$\therefore \theta_v = \theta_i - 90^o \rightarrow v(t) \text{ lags } i(t) \text{ by } 90^o$$



Example 8.8

If the voltage $v(t) = 100\cos(314t + 15^o)\ V$ is applied to a 100-μF capacitor, find the resultant current.