Chapter 3 Answers

3.1. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_3 e^{j3(2\pi/T)t} + a_{-3}e^{-j3(2\pi/T)t} \\ &= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4je^{j3(2\pi/8)t} - 4je^{-j3(2\pi/8)t} \\ &= 4\cos(\frac{\pi}{4}t) - 8\sin(\frac{6\pi}{8}t) \\ &= 4\cos(\frac{\pi}{4}t) + 8\cos(\frac{3\pi}{4}t + \frac{\pi}{2}) \end{aligned}$$

3.2. Using the Fourier series synthesis eq. (3.95).

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2}e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4}e^{-j4(2\pi/N)n} \\ &= 1 + e^{j(\pi/4)}e^{j2(2\pi/5)n} + e^{-j(\pi/4)}e^{-2j(2\pi/5)n} \\ &\quad +2e^{j(\pi/3)}e^{j4(2\pi/N)n} + 2e^{-j(\pi/3)}a_{-4}e^{-j4(2\pi/N)n} \\ &= 1 + 2\cos(\frac{4\pi}{5}n + \frac{\pi}{4}) + 4\cos(\frac{8\pi}{5}n + \frac{\pi}{3}) \\ &= 1 + 2\sin(\frac{4\pi}{5}n + \frac{3\pi}{4}) + 4\sin(\frac{8\pi}{5}n + \frac{5\pi}{6}) \end{aligned}$$

3.3. The given signal is

$$\begin{aligned} x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\ &= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t} \end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coeffcients of $x(t)$ are:

$$a_0 = 2, \qquad a_2 = a_{-2} = \frac{1}{2}, \qquad a_5 = a_{-5}^* = -2j$$

3.4. Since $\omega_0 = \pi$, $T = 2\pi/\omega_0 = 2$. Therefore,

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt$$

Now,

$$a_0 = \frac{1}{2}\int_0^1 1.5dt - \frac{1}{2}\int_1^2 1.5dt = 0$$

and for $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 1.5e^{-jk\pi t}dt - \frac{1}{2}\int_1^2 1.5e^{-jk\pi t}dt \\ &= \frac{3}{2k\pi j}[1 - e{-}jk\pi] \\ &= \frac{3}{k\pi}e^{-jk(\pi/2)}\sin(\frac{k\pi}{2}) \end{aligned}$$

3.5 Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundemental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of $x_1(1-t)$ and $x_1(t-1)$, it is also periodic with fundemental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Since $x_1(t) \overset{FS}{\longleftrightarrow} a_k$, using the results in Table 3.1 we have

$$x_1(t+1) \overset{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)}$$

$$x_1(t-1) \overset{FS}{\longleftrightarrow} a_k e^{-jk(2\pi/T_1)} \Rightarrow x_1(-t+1) \overset{FS}{\longleftrightarrow} a_{-k} e^{-jk(2\pi/T_1)}$$

Therefore,

$$x_1(t+1) + x_1(1-t) \overset{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)} + a_{-k} e^{-jk(2\pi/T_1)} = e^{-j\omega_1 k}(a_k + a_{-k})$$

3.6. (a) Comparing $x_1(t)$ with the Fourier series synthesis eq. (3.38), we obtain the Fourier series coefficients of $x_1(t)$ to be

$$a_k = \begin{cases} \left(\frac{1}{2}\right)^k, & 0 \le k \le 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_1(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a^*_{-k}$. Since this is not true for $x_1(t)$, the signal is **not real valued**.

Similarly, the Fourier series coefficients of $x_2(t)$ are

$$a_k = \begin{cases} \cos(k\pi), & 100 \le k \le 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_2(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a^*_{-k}$. Since this is true for $x_2(t)$, the signal is **real valued**.

Similarly, the Fourier series coefficients of $x_3(t)$ are

$$a_k = \begin{cases} j\sin(k\pi/2), & 100 \le k \le 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_3(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a^*_{-k}$. Since this is true for $x_3(t)$, the signal is **real valued**.

(b) For a signal to be even, its Fourier series coefficients must be even. This is true only for $x_2(t)$.

3.7. Given that

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

we have

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T}a_k.$$

Therefore,

$$a_k = \frac{b_k}{j(2\pi/T)k}, \qquad k \neq 0$$

3.22. (a) (i) $T = 1$, $a_0 = 0$, $a_k = \frac{j(-1)^k}{k\pi}$, $k \neq 0$.

(ii) Here,

$$x(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$$

$T = 6$, $a_0 = 1/2$, and

$$a_k = \begin{cases} 0, & k \text{ even} \\ \frac{6}{\pi^2 k^2} \sin(\frac{\pi k}{2}) \sin(\frac{\pi k}{6}), & k \text{ odd} \end{cases}.$$

(iii) $T = 3$, $a_0 = 1$, and

$$a_k = \frac{3j}{2\pi^2 k^2}[e^{jk2\pi/3} \sin(k2\pi/3) + 2e^{jk\pi/3} \sin(k\pi/3)], \qquad k \neq 0.$$

(iv) $T = 2$, $a_0 = -1/2$, $a_k = \frac{1}{2} - (-1)^k$, $k \neq 0$.

(v) $T = 6$, $\omega_0 = \pi/3$, and

$$a_k = \frac{\cos(2k\pi/3) - \cos(k\pi/3)}{jk\pi/3}.$$

Note that $a_0 = 0$ and $a_{k\ even} = 0$.

(vi) $T = 4$, $\omega_0 = \pi/2$, $a_0 = 3/4$ and

$$a_k = \frac{e^{-jk\pi/2} \sin(k\pi/2) + e^{-jk\pi/4} \sin(k\pi/4)}{k\pi}, \qquad \forall k.$$

(b) $T = 2$, $a_k = \frac{-1^k}{2(1+jk\pi)}[e - e^{-1}]$ for all k.

(c) $T = 3$, $\omega_0 = 2\pi/3$, $a_0 = 1$ and

$$a_k = \frac{2e^{-j\pi k/3}}{\pi k} \sin(2\pi k/3) + \frac{e^{-j\pi k}}{\pi k} \sin(\pi k).$$

3.23. (a) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/4)}{k\pi}.$$

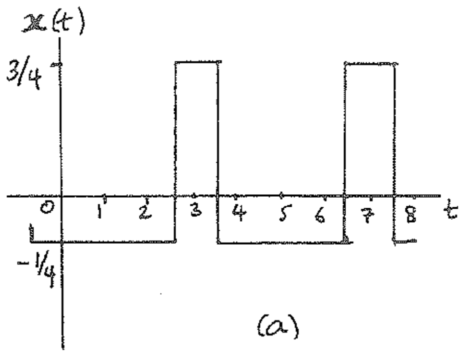
From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & 1/2 < |t| < 2 \end{cases}.$$

Now, note that $b_0 = 1/4$. Let us define another signal $z(t) = -1/4$ whose only nonzero FS coefficient is $c_0 = -1/4$. The signal $p(t) = y(t) + z(t)$ will have FS coefficients

$$d_k = a_k + c_k = \begin{cases} 0, & k = 0 \\ \frac{\sin(k\pi/4)}{k\pi}, & \text{otherwise.} \end{cases}$$

Now note that $a_k = d_k e^{j(\pi/2)k}$. Therefore, the signal $x(t) = p(t+1)$ which is as shown in Figure S2.23(a).



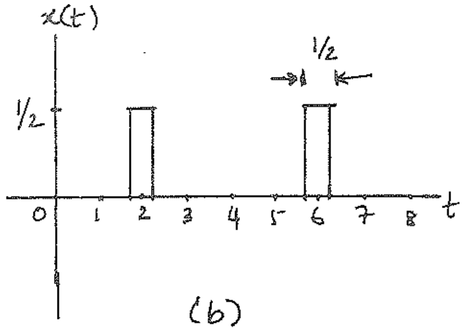


Figure S3.23

(b) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/8)}{2k\pi}.$$

From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1/2, & |t| < 1/4 \\ 0, & 1/4 < |t| < 2 \end{cases}.$$

Now note that $a_k = b_k e^{j\pi k}$. Therefore, the signal $x(t) = y(t+2)$ which is as shown in Figure S2.23(b).

(c) The only nonzero FS coefficients are $a_1 = a_{-1}^* = j$ and $a_2 = a_{-2}^* = 2j$. Using the FS synthesis equation, we get

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_2 e^{j2(2\pi/T)t} + a_{-2}e^{-j2(2\pi/T)t} \\ &= je^{j(2\pi/4)t} - je^{-j(2\pi/4)t} + 2je^{j2(2\pi/4)t} - 2je^{-j2(2\pi/4)t} \\ &= -2\sin(\frac{\pi}{2}t) - 4\sin(\pi t) \end{aligned}$$

(d) The FS coefficients a_k may be written as the sum of two sets of FS coefficients b_k and c_k, where

$$b_k = 1, \quad \text{for all } k$$

and

$$c_k = \begin{cases} 1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

The FS coefficients b_k correspond to the signal

$$y(t) = \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$

and the FS coefficients c_k correspond to the signal

$$z(t) = \sum_{k=-\infty}^{\infty} e^{j(\pi/2)t}\delta(t - 2k).$$

Therefore,

$$x(t) = y(t) + p(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k) + \sum_{k=-\infty}^{\infty} e^{j(\pi/2)t}\delta(t-2k).$$

3.24. **(a)** We have

$$a_0 = \frac{1}{2}\int_0^1 t dt + \frac{1}{2}\int_1^2 (2-t)dt = 1/2.$$

(b) The signal $g(t) = dx(t)/dt$ is as shown in Figure S3.24.

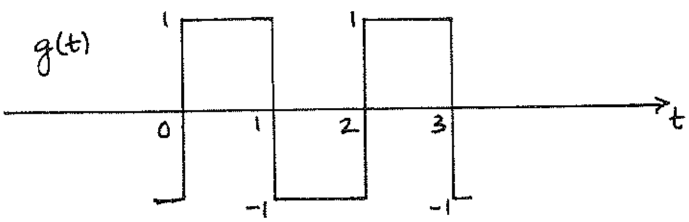


Figure S3.24

The FS coefficients b_k of $g(t)$ may be found as follows:

$$b_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$$

and

$$\begin{aligned} b_k &= \frac{1}{2}\int_0^1 e^{-j\pi kt}dt - \frac{1}{2}\int_1^2 e^{-j\pi kt}dt \\ &= \frac{1}{j\pi k}[1 - e^{-j\pi k}]. \end{aligned}$$

(c) Note that

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\pi a_k.$$

Therefore,

$$a_k = \frac{1}{jk\pi}b_k = -\frac{1}{\pi^2 k^2}\{1 - e^{-j\pi k}\}.$$

3.25. **(a)** The nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1}^* = 1/2j$.

(c) Using the multiplication property, we know that

$$z(t) = x(t)y(t) \overset{FS}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Therefore,

$$c_k = a_k * b_k = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2].$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2}^* = (1/4j)$.

(d) We have

$$z(t) = \sin(4t)\cos(4t) = \frac{1}{2}\sin(8t).$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2} = (1/4j)$.

3.26. (a) If $x(t)$ is real, then $x(t) = x^*(t)$. This implies that for $x(t)$ real $a_k = a_{-k}^*$. Since this is not true in this case problem, $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. Since this is true for this case, $x(t)$ is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T_0}a_k.$$

Therefore,

$$b_k = \begin{cases} 0, & k = 0 \\ -k(1/2)^{|k|}(2\pi/T_0), & \text{otherwise} \end{cases}.$$

Since b_k is not even, $g(t)$ is not even.

3.27. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2}e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4}e^{-j4(2\pi/N)n} \\ &= 2 + 2e^{j\pi/6}e^{j(4\pi/5)n} + 2e^{-j\pi/6}e^{-j(4\pi/5)n} + e^{j\pi/3}e^{j(8\pi/5)n} + e^{-j\pi/3}e^{-j(8\pi/5)n} \\ &= 2 + 4\cos[(4\pi n/5) + \pi/6] + 2\cos[(8\pi n/5) + \pi/3] \\ &= 2 + 4\sin[(4\pi n/5) + 2\pi/3] + 2\sin[(8\pi n/5) + 5\pi/6] \end{aligned}$$

3.28. (a) $N = 7$,

$$a_k = \frac{1}{7}\frac{e^{-j4\pi k/7}\sin(5\pi k/7)}{\sin(\pi k/7)}.$$

(b) $N = 6$, a_k over one period $(0 \le k \le 5)$ may be specified as: $a_0 = 4/6$,

$$a_k = \frac{1}{6}e^{-j\pi k/2}\frac{\sin(\frac{2\pi k}{3})}{\sin(\frac{\pi k}{6})}, \qquad 1 \le k \le 5.$$

(c) $N=6$,

$$a_k = 1 + 4\cos(\pi k/3) - 2\cos(2\pi k/3).$$

(d) $N = 12$, a_k over one period ($0 \le k \le 11$) may be specified as: $a_1 = \frac{1}{4j} = a_{11}^*$, $a_5 = -\frac{1}{4j} = a_7^*$, $a_k = 0$ otherwise.

(e) $N=4$.

$$a_k = 1 + 2(-1)^k(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}).$$

(f) $N = 12$,

$$\begin{aligned} a_k &= 1 + (1 - \frac{1}{\sqrt{2}})2\cos(\frac{\pi k}{6}) + 2(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}) \\ &+ 2(1 + \frac{1}{\sqrt{2}})\cos(\frac{5\pi k}{6}) + 2(-1)^k + 2\cos(\frac{2\pi k}{3}). \end{aligned}$$

3.29. (a) $N = 8$. Over one period ($0 \le n \le 7$),

$$x[n] = 4\delta[n-1] + 4\delta[n-7] + 4j\delta[n-3] - 4j\delta[n-5].$$

(b) $N = 8$. Over one period ($0 \le n \le 7$),

$$x[n] = \frac{1}{2j}\left[\frac{-e^{j\frac{3\pi n}{4}}\sin\{\frac{7}{2}(\frac{\pi n}{4} + \frac{\pi}{3})\}}{\sin\{\frac{1}{2}(\frac{\pi n}{4} + \frac{\pi}{3})\}} + \frac{e^{j\frac{3\pi n}{4}}\sin\{\frac{7}{2}(\frac{\pi n}{4} - \frac{\pi}{3})\}}{\sin\{\frac{1}{2}(\frac{\pi n}{4} - \frac{\pi}{3})\}}\right].$$

(c) $N = 8$. Over one period ($0 \le n \le 7$),

$$x[n] = 1 + (-1)^n + 2\cos(\frac{\pi n}{4}) + 2\cos(\frac{3\pi n}{4}).$$

(d) $N = 8$. Over one period ($0 \le n \le 7$),

$$x[n] = 2 + 2\cos\left(\frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{2}\right) + \frac{1}{2}\cos\left(\frac{3\pi n}{4}\right).$$

3.30. (a) The nonzero FS coefficients of $x(t)$ are $a_0 = 1$, $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1}^* = e^{-j\pi/4}/2$.

(c) Using the multiplication property, we know that

$$z[n] = x[n]y[n] \stackrel{FS}{\longleftrightarrow} c_k = \sum_{l=-2}^{2} a_l b_{k-l}.$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are $c_0 = \cos(\pi/4)/2$, $c_1 = c_{-1}^* = e^{-j\pi/4}/2$, $c_2 = c_{-2}^* = e^{-j\pi/4}/4$.

(d) We have

$$\begin{aligned} z[n] &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)\cos\left(\frac{2\pi}{6}n\right) \\ &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \frac{1}{2}\left[\sin(\frac{4\pi}{6}n + \frac{\pi}{4}) + \sin(\frac{\pi}{4})\right] \end{aligned}$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are $c_0 = \cos(\pi/4)/2$, $c_1 = c_{-1}^* = e^{-j\pi/4}/2$, $c_2 = c_{-2}^* = e^{-j\pi/4}/4$.

3.31. (a) $g[n]$ is as shown in Figure S3.31. Clearly, $g[n]$ has a fundamental period of 10.

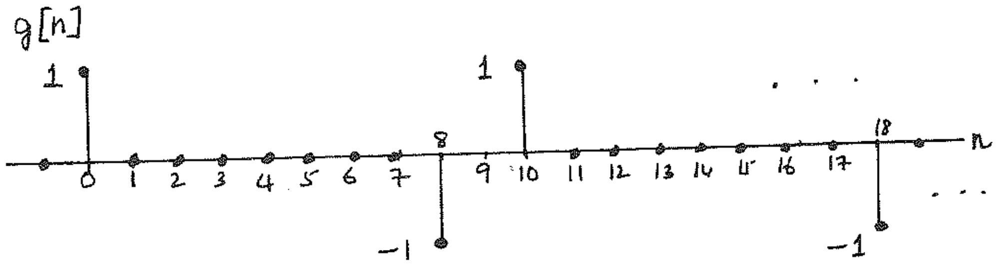


Figure S3.31

(b) The Fourier series coefficiennts of $g[n]$ are $b_k = (1/10)[1 - e^{-j(2\pi/10)8k}]$.

(c) Since $g[n] = x[n] - x[n-1]$, the FS coeffcients a_k and b_k must be related as

$$b_k = a_k - e^{-j(2\pi/10)k}a_k.$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-j(2\pi/10)k}} = \frac{(1/10)[1 - e^{-j(2\pi/10)8k}]}{1 - e^{-j(2\pi/10)k}}.$$

3.32. (a) The four equations are

$$a_0 + a_1 + a_2 + a_3 = 1, \qquad a_0 + ja_1 - a_2 - ja_3 = 0$$

$$a_0 - a_1 + a_2 - a_3 = 2, \qquad a_0 - ja_1 - a_2 + ja_3 = -1.$$

Solving, we get $a_0 = 1/2$, $a_1 = -\frac{1+j}{4}$, $a_2 = -1$, $a_3 = -\frac{1-j}{4}$.

(b) By direct calculation,

$$a_k = \frac{1}{4}[1 + 2e^{-jk\pi} - e^{-jk3\pi/2}].$$

This is the same as the answer we obtained in part (a) for $0 \le k \le 3$.

Chapter 4 Answers

4.1 (a) Let $x(t) = e^{-2(t-1)}u(t-1)$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

(b) Let $x(t) = e^{-2|t-1|}$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2|t-1|}e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt + \int_{-\infty}^{1} e^{2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) + e^{-j\omega}/(2-j\omega) \\ &= 4e^{-j\omega}/(4+\omega^2) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

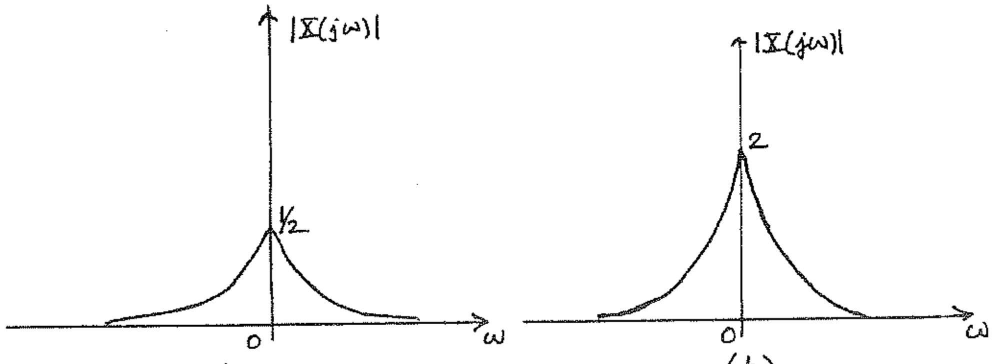


Figure S4.1

4.2 (a) Let $x_1(t) = \delta(t+1) + \delta(t-1)$. Then the Fourier transform $X_1(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X_1(j\omega) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)]e^{-j\omega t}dt \\ &= e^{j\omega} + e^{-j\omega} = 2\cos\omega \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

(b) The signal $x_2(t) = u(-2-t) + u(t-2)$ is as shown in the figure below. Clearly,

$$\frac{d}{dt}\{u(-2-t) + u(t-2)\} = \delta(t-2) - \delta(t+2)$$

Therefore,

$$\begin{aligned} X_2(j\omega) &= \int_{-\infty}^{\infty} [\delta(t-2) - \delta(t+2)] e^{-j\omega t} dt \\ &= e^{-2j\omega} - e^{2j\omega} = -2j\sin(2\omega) \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

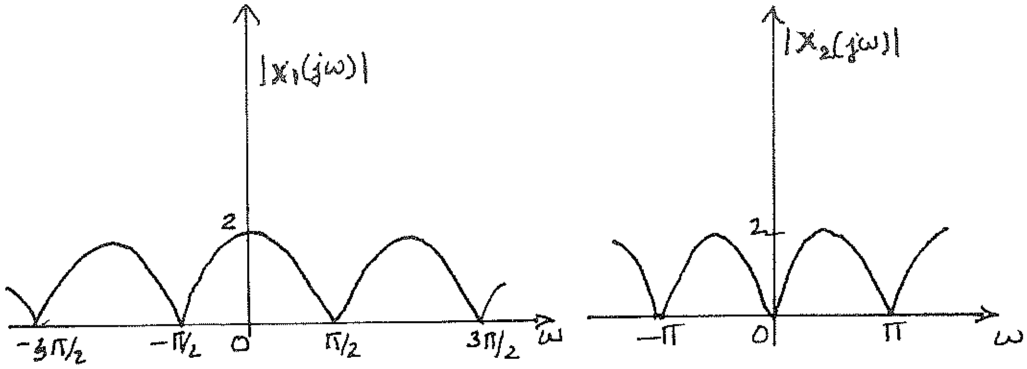


Figure S4.2

4.3. (a) The signal $x_1(t) = \sin(2\pi t + \pi/4)$ is periodic with a fundamental period of $T = 1$. This translates to a fundamental frequency of $\omega_0 = 2\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_1(t) &= \frac{1}{2j}\left(e^{j(2\pi t+\pi/4)} - e^{-j(2\pi t+\pi/4)}\right) \\ &= \frac{1}{2j}e^{j\pi/4}e^{j2\pi t} - \frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_1(t)$ are

$$a_1 = \frac{1}{2j}e^{j\pi/4}e^{j2\pi t}, \qquad a_{-1} = -\frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_1(t)$, the corresponding Fourier transform $X_1(j\omega)$ is given by

$$\begin{aligned} X_1(j\omega) &= 2\pi a_1 \delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0) \\ &= (\pi/j)e^{j\pi/4}\delta(\omega - 2\pi) - (\pi/j)e^{-j\pi/4}\delta(\omega + 2\pi) \end{aligned}$$

(b) The signal $x_2(t) = 1 + \cos(6\pi t + \pi/8)$ is periodic with a fundamental period of $T = 1/3$. This translates to a fundamental frequency of $\omega_0 = 6\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_2(t) &= 1 + \frac{1}{2}\left(e^{j(6\pi t+\pi/8)} - e^{-j(6\pi t+\pi/8)}\right) \\ &= 1 + \frac{1}{2}e^{j\pi/8}e^{j6\pi t} + \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_2(t)$ are

$$a_0 = 1, \qquad a_1 = \frac{1}{2}e^{j\pi/8}e^{j6\pi t}, \qquad a_{-1} = \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_2(t)$, the corresponding Fourier transform $X_2(j\omega)$ is given by

$$\begin{aligned} X_2(j\omega) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0) \\ &= 2\pi\delta(\omega) + \pi e^{j\pi/8}\delta(\omega - 6\pi) + \pi e^{-j\pi/8}\delta(\omega + 6\pi) \end{aligned}$$

4.4. (a) The inverse Fourier transform is

$$\begin{aligned} x_1(t) &= (1/2\pi)\int_{-\infty}^{\infty}[2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)]e^{j\omega t}d\omega \\ &= (1/2\pi)[2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}] \\ &= 1 + (1/2)e^{j4\pi t} + (1/2)e^{-j4\pi t} = 1 + \cos(4\pi t) \end{aligned}$$

(b) The inverse Fourier transform is

$$\begin{aligned} x_2(t) &= (1/2\pi)\int_{-\infty}^{\infty}X_2(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{0}^{2}2e^{j\omega t}d\omega + (1/2\pi)\int_{-2}^{0}(-2)e^{j\omega t}d\omega \\ &= (e^{j2t} - 1)/(\pi jt) - (1 - e^{-j2t})/(\pi jt) \\ &= -(4j\sin^2 t)/(\pi t) \end{aligned}$$

4.5. From the given information,

$$\begin{aligned} x(t) &= (1/2\pi)\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-\infty}^{\infty}|X(j\omega)|e^{j\sphericalangle\{X(j\omega)\}}e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-3}^{3}2e^{-\frac{3}{2}\omega+\pi}e^{j\omega t}d\omega \\ &= \frac{-2}{\pi(t-3/2)}\sin[3(t-3/2)] \end{aligned}$$

The signal $x(t)$ is zero when $3(t - 3/2)$ is a nonzero integer multiple of π. This gives

$$t = \frac{k\pi}{2} + \frac{3}{2}, \qquad \text{for } k \in \mathcal{I}, \text{ and } k \neq 0.$$

4.6. Throughout this problem, we assume that

$$x(t) \stackrel{FT}{\longleftrightarrow} X_1(j\omega).$$

(a) Using the time reversal property (Sec. 4.3.5), we have

$$x(-t) \stackrel{FT}{\longleftrightarrow} X(-j\omega)$$

Using the time shifting property (Sec. 4.3.2) on this, we have

$$x(-t+1) \stackrel{FT}{\longleftrightarrow} e^{-j\omega t}X(-j\omega) \qquad \text{and} \qquad x(-t-1) \stackrel{FT}{\longleftrightarrow} e^{j\omega t}X(-j\omega)$$

Therefore,

$$\begin{aligned} x_1(t) = x(-t+1) + x(-t-1) &\stackrel{FT}{\longleftrightarrow} e^{-j\omega t}X(-j\omega) + e^{j\omega t}X(-j\omega) \\ &\stackrel{FT}{\longleftrightarrow} 2X(-j\omega)\cos\omega \end{aligned}$$

(b) Using the time scaling property (Sec. 4.3.5), we have

$$x(3t) \stackrel{FT}{\longleftrightarrow} \frac{1}{3}X\left(j\frac{\omega}{3}\right)$$

Using the time shifting property on this, we have

$$x_2(t) = x(3(t-2)) \stackrel{FT}{\longleftrightarrow} e^{-2j\omega}\frac{1}{3}X\left(j\frac{\omega}{3}\right)$$

(c) Using the differentiation in time property (Sec. 4.3.4), we have

$$\frac{dx(t)}{dt} \stackrel{FT}{\longleftrightarrow} j\omega X(j\omega)$$

Applying this property again, we have

$$\frac{d^2x(t)}{dt^2} \stackrel{FT}{\longleftrightarrow} -\omega^2 X(j\omega).$$

Using the time shifting property, we have

$$x_3(t) = \frac{d^2x(t-1)}{dt^2} \stackrel{FT}{\longleftrightarrow} -\omega^2 X(j\omega)e^{-j\omega t}.$$

4.7. **(a)** Since $X_1(j\omega)$ is not conjugate symmetric, the corresponding signal $x_1(t)$ is **not real**. Since $X_1(j\omega)$ is neither even nor odd, the corresponding signal $x_1(t)$ is **neither even nor odd**.

(b) The Fourier transform of a real and odd signal is purely imaginary and odd. Therefore, we may conclude that the Fourier transform of a purely imaginary and odd signal is real and odd. Since $X_2(j\omega)$ is real and odd, we may therefore conclude that the corresponding signal $x_2(t)$ is **purely imaginary and odd**.

4.8.

(c) Consider a signal $y_3(t)$ whose magnitude of the Fourier transform is $|Y_3(j\omega)| = A(\omega)$, and whose phase of the Fourier transform is $\sphericalangle\{Y_3(j\omega)\} = 2\omega$. Since $|Y_3(j\omega)| = |Y_3(-j\omega)|$ and $\sphericalangle\{Y_3(j\omega)\} = -\sphericalangle\{Y_3(-j\omega)\}$, we may conclude that the signal $y_3(t)$ is real (See Table 4.1, Property 4.3.3).

Now, consider the signal $x_3(t)$ with Fourier transform $X_3(j\omega) = Y_3(j\omega)e^{j\pi/2} = jY_3(j\omega)$. Using the result from the previous paragraph and the linearity property of the Fourier transform, we may conclude that $x_3(t)$ has to **imaginary**. Since the Fourier transform $X_3(j\omega)$ is neither purely imaginary nor purely real, the signal $x_3(t)$ is **neither even nor odd**.

(d) Since $X_4(j\omega)$ is both real and even, the corresponding signal $x_4(t)$ is **real and even**.

4.8. (a) The signal $x(t)$ is as shown in the Figure S4.8.

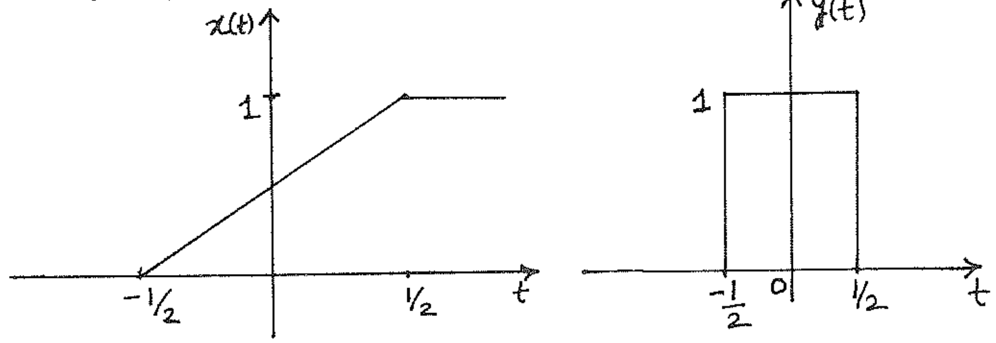


Figure S4.8

We may express this signal as

$$x(t) = \int_{-\infty}^{t} y(t)dt,$$

where $y(t)$ is the rectangular pulse shown in Figure S4.8. Using the integration property of the Fourier transform, we have

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{j\omega}Y(j\omega) + \pi Y(j0)\delta(\omega)$$

We know from Table 4.2 that

$$Y(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

Therefore,

$$X(j\omega) = \frac{2\sin(\omega/2)}{j\omega^2} + \pi\delta(\omega)$$

(b) If $g(t) = x(t) - 1/2$, then the Fourier transform $G(j\omega)$ of $g(t)$ is given by

$$G(j\omega) = X(j\omega) - (1/2)2\pi\delta(\omega) = \frac{2\sin(\omega/2)}{j\omega^2}.$$

4.11. We know that

$$x(3t) \stackrel{FT}{\longleftrightarrow} \frac{1}{3}X(j\frac{\omega}{3}), \qquad h(3t) \stackrel{FT}{\longleftrightarrow} \frac{1}{3}H(j\frac{\omega}{3})$$

Therefore,

$$G(j\omega) = \mathcal{F}T\{x(3t) * h(3t)\} = \frac{1}{9}X(j\frac{\omega}{3})H(j\frac{\omega}{3})$$

Now note that

$$Y(j\omega) = \mathcal{F}T\{x(t) * h(t)\} = X(j\omega)H(j\omega)$$

From this, we may write

$$Y(j\frac{\omega}{3}) = X\left(j\frac{\omega}{3}\right)H\left(j\frac{\omega}{3}\right)$$

Using this in eq. (**), we have

$$G(j\omega) = \frac{1}{9}Y(j\frac{\omega}{3})$$

and

$$g(t) = \frac{1}{3}y(3t).$$

Therefore, $A = \frac{1}{3}$ and $B = 3$.

4.12. **(a)** From Example 4.2 we know that

$$e^{-|t|} \stackrel{FT}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

Using the differentiation in frequency property, we have

$$te^{-|t|} \stackrel{FT}{\longleftrightarrow} j\frac{d}{d\omega}\left\{\frac{2}{1+\omega^2}\right\} = -\frac{4j\omega}{(1+\omega^2)^2}.$$

(b) The duality property states that if

$$g(t) \stackrel{FT}{\longleftrightarrow} G(j\omega)$$

then

$$G(t) \stackrel{FT}{\longleftrightarrow} 2\pi g(j\omega).$$

Now, since

$$te^{-|t|} \stackrel{FT}{\longleftrightarrow} -\frac{4j\omega}{(1+\omega^2)^2}$$

we may use duality to write

$$-\frac{4jt}{(1+t^2)^2} \stackrel{FT}{\longleftrightarrow} 2\pi\omega e^{-|\omega|}$$

Multiplying both sides by j, we obtain

$$\frac{4t}{(1+t^2)^2} \stackrel{FT}{\longleftrightarrow} j2\pi\omega e^{-|\omega|}.$$

We see that $G(j\omega)$ is periodic with a period of 8. Using the multiplication property, we know that

$$X(j\omega) = \frac{1}{2\pi}\left[\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\} * G(j\omega)\right]$$

If we denote $\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\}$ by $A(j\omega)$, then

$$\begin{aligned} X(j\omega) &= (1/2\pi)[A(j\omega) * 8\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 8k) \\ &= 4 \sum_{k=-\infty}^{\infty} A(j\omega - 8k) \end{aligned}$$

$X(j\omega)$ may thus be viewed as a replication of $4A(j\omega)$ every 8 rad/sec. This is obviously periodic.

Using Table 4.2, we obtain

$$A(j\omega) = \begin{cases} 1, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, we may specify $X(j\omega)$ over one period as

$$X(j\omega) = \begin{cases} 4, & |\omega| \le 1 \\ 0, & 1 < |\omega \le 4 \end{cases}$$

4.17. (a) From Table 4.1, we know that a real and odd signal signal $x(t)$ has a purely imaginary and odd Fourier transform $X(j\omega)$. Let us now consider the purely imaginary and odd signal $jx(t)$. Using linearity, we obtain the Fourier transform of this signal to be $jX(j\omega)$. The function $jX(j\omega)$ will clearly be real and odd. Therefore, the given statement is **false**.

(b) An odd Fourier transform corresponds to an odd signal, while an even Fourier transform corresponds to an even signal. The convolution of an even Fourier transform with an odd Fourier may be viewed in the time domain as a multiplication of an even and odd signal. Such a multiplication will always result in an odd time signal. The Fourier transform of this odd signal will always be odd. Therefore, the given statement is **true**.

4.18. Using Table 4.2, we see that the rectangular pulse $x_1(t)$ shown in Figure S4.18 has a Fourier transform $X_1(j\omega) = \sin(3\omega)/\omega$. Using the convolution property of the Fourier transform, we may write

$$x_2(t) = x_1(t) * x_1(t) \xleftrightarrow{FT} X_2(j\omega) = X_1(j\omega)X_1(j\omega) = \left(\frac{\sin(3\omega)}{\omega}\right)^2$$

The signal $x_2(t)$ is shown in Figure S4.18. Using the shifting property, we also note that

$$\frac{1}{2}x_2(t+1) \xleftrightarrow{FT} \frac{1}{2}e^{j\omega}\left(\frac{\sin(3\omega)}{\omega}\right)^2$$

and

$$\frac{1}{2}x_2(t-1) \overset{FT}{\longleftrightarrow} \frac{1}{2}e^{-j\omega}\left(\frac{\sin(3\omega)}{\omega}\right)^2.$$

Adding the two above equations, we obtain

$$h(t) = \frac{1}{2}x_2(t+1) + \frac{1}{2}x_2(t-1) \overset{FT}{\longleftrightarrow} \cos(\omega)\left(\frac{\sin(3\omega)}{\omega}\right)^2.$$

The signal $h(t)$ is as shown in Figure S4.18. We note that $h(t)$ has the given Fourier transform $H(j\omega)$.

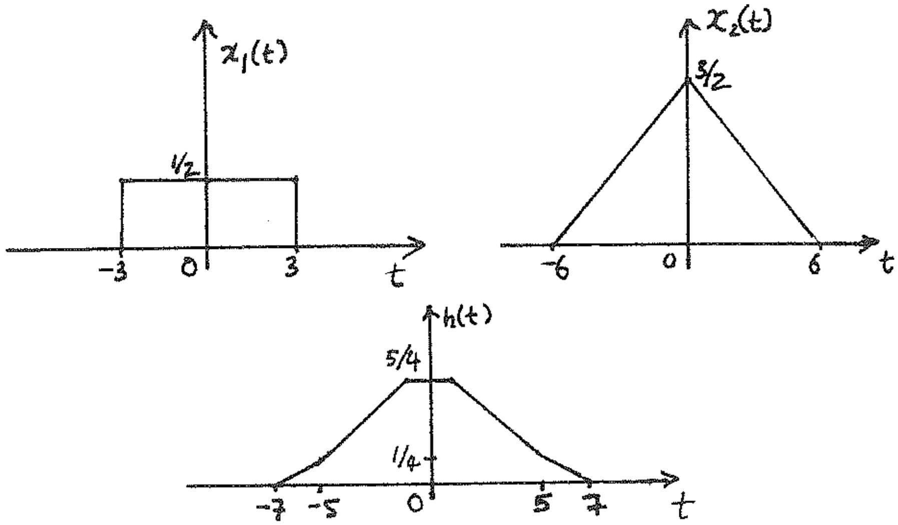


Figure S4.18

Mathematically $h(t)$ may be expressed as

$$h(t) = \begin{cases} \frac{5}{4}, & |t| < 1 \\ -\frac{|t|}{4} + \frac{3}{2}, & 1 \le |t| \le 5 \\ -\frac{|t|}{8} + \frac{7}{8}, & 5 < |t| \le 7 \\ 0, & \text{otherwise} \end{cases}$$

4.19. We know that

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}.$$

Since it is given that $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$, we can compute $Y(j\omega)$ to be

$$Y(j\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{1}{(3+j\omega)(4+j\omega)}.$$

Since, $H(j\omega) = 1/(3+j\omega)$, we have

$$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = 1/(4+j\omega)$$

Taking the inverse Fourier transform of $X(j\omega)$, we have

$$x(t) = e^{-4t}u(t).$$

4.20. From the answer to Problem 3.20, we know that the frequency response of the circuit is

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}.$$

Breaking this up into partial fractions, we may write

$$H(j\omega) = -\frac{1}{j\sqrt{3}}\left[\frac{-1}{\frac{1}{2} - \frac{\sqrt{3}}{2}j + j\omega} + \frac{-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}j + j\omega}\right]$$

Using the Fourier transform pairs provided in Table 4.2, we obtain the Fourier transform of $H(j\omega)$ to be

$$h(t) = -\frac{1}{j\sqrt{3}}\left[-e^{(-\frac{1}{2}+\frac{\sqrt{3}}{2}j)t} + +e^{(-\frac{1}{2}-\frac{\sqrt{3}}{2}j)t}\right]u(t).$$

Simplifying,

$$h(t) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)u(t).$$

4.21. (a) The given signal is

$$e^{-\alpha t}\cos(\omega_0 t)u(t) = \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} - \frac{1}{2(\alpha - j\omega_0 + j\omega)}.$$

(b) The given signal is

$$x(t) = e^{-3t}\sin(2t)u(t) + e^{3t}\sin(2t)u(-t).$$

We have

$$x_1(t) = e^{-3t}\sin(2t)u(t) \stackrel{FT}{\longleftrightarrow} X_1(j\omega) = \frac{1/2j}{3 - j2 + j\omega} - \frac{1/2j}{3 + j2 + j\omega}.$$

Also,

$$x_2(t) = e^{3t}\sin(2t)u(-t) = -x_1(-t) \stackrel{FT}{\longleftrightarrow} X_2(j\omega) = -X_1(-j\omega) = \frac{1/2j}{3 - j2 - j\omega} - \frac{1/2j}{3 + j2 - j\omega}.$$

Therefore,

$$X(j\omega) = X_1(j\omega + X_2(j\omega) = \frac{3j}{9 + (\omega+2)^2} - \frac{3j}{9 + (\omega-2)^2}.$$

(c) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{2\sin\omega}{\omega} + \frac{\sin\omega}{\pi-\omega} - \frac{\sin\omega}{\pi+\omega}.$$

(d) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{1}{1-\alpha e^{-j\omega T}}.$$

(e) We have

$$x(t) = (1/2j)te^{-2t}e^{j4t}u(t) - (1/2j)te^{-2t}e^{-j4t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1/2j}{(2-j4+j\omega)^2} - \frac{1/2j}{(2+j4-j\omega)^2}.$$

(f) We have

$$x_1(t) = \frac{\sin \pi t}{\pi t} \stackrel{FT}{\longleftrightarrow} X_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}.$$

Also

$$x_2(t) = \frac{\sin 2\pi(t-1)}{\pi(t-1)} \stackrel{FT}{\longleftrightarrow} X_2(j\omega) = \begin{cases} e^{-2\omega}, & |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

$$x(t) = x_1(t)x_2(t) \stackrel{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{2\pi}\{X_1(j\omega) * X_2(j\omega)\}.$$

Therefore,

$$X(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < \pi \\ (1/2\pi)(3\pi+\omega)e^{-j\omega}, & -3\pi < \omega < -\pi \\ (1/2\pi)(3\pi-\omega)e^{-j\omega}, & \pi < \omega < 3\pi \\ 0, & \text{otherwise} \end{cases}.$$

(g) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{2j}{\omega}\left[\cos 2\omega - \frac{\sin\omega}{\omega}\right].$$

(h) If

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k),$$

then

$$x(t) = 2x_1(t) + x_1(t-1).$$

Therefore,

$$X(j\omega) = X_1(j\omega)[2+e^{-\omega}] = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)[2+(-1)^k].$$