(i) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{-\omega^2} - \frac{2e^{-j\omega} - 2}{j\omega^2}.$$

(j) $x(t)$ is periodic with period 2. Therefore,

$$X(j\omega) = \pi \sum_{k=-\infty}^{\infty} \tilde{X}(jk\pi)\delta(\omega - k\pi),$$

where $\tilde{X}(j\omega)$ is the Fourier transform of one period of $x(t)$. That is,

$$\tilde{X}(j\omega) = \frac{1}{1 - e^{-2}} \left[\frac{1 - e^{-2(1+j\omega)}}{1 + j\omega} - \frac{e^{-2}[1 - e^{-2(1+j\omega)}]}{1 - j\omega} \right].$$

4.22. (a) $x(t) = \begin{cases} e^{j2\pi t}, & |t| < 3 \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = \frac{1}{2}e^{-j\pi/3}\delta(t-4) + \frac{1}{2}e^{j\pi/3}\delta(t+4)$.

(c) The Fourier transform synthesis eq. (4.8) may be written as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)| e^{j\sphericalangle X(j\omega)} e^{j\omega t} d\omega.$$

From the given figure we have

$$x(t) = \frac{1}{\pi} \left[\frac{\sin(t-3)}{t-3} + \frac{\cos(t-3) - 1}{(t-3)^2} \right].$$

(d) $x(t) = \frac{2j}{\pi} \sin t + \frac{3}{\pi} \cos(2\pi t)$

(e) Using the Fourier transform synthesis equation (4.8),

$$x(t) = \frac{\cos 3t}{j\pi t} + \frac{\sin t - \sin 2t}{j\pi t^2}.$$

4.23. For the given signal $x_0(t)$, we use the Fourier transform analysis eq. (4.8) to evaluate the corresponding Fourier transform

$$X_0(j\omega) = \frac{1 - e^{-(1+j\omega)}}{1 + j\omega}.$$

(i) We know that

$$x_1(t) = x_0(t) + x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{2 - 2e^{-1}\cos\omega - 2\omega e^{-1}\sin\omega}{1 + \omega^2}.$$

(ii) We know that

$$x_2(t) = x_0(t) - x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_2(j\omega) = X_0(j\omega) - X_0(-j\omega) = j\left[\frac{-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega}{1+\omega^2}\right].$$

(iii) We know that

$$x_3(t) = x_0(t) + x_0(t+1).$$

Using the linearity and time shifting properties of the Fourier transform we have

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega}X_0(-j\omega) = \frac{1 + e^{j\omega} - e^{-1}(1+e^{-j\omega})}{1+j\omega}.$$

(iv) We know that

$$x_4(t) = tx_0(t).$$

Using the differentiation in frequency property

$$X_4(j\omega) = j\frac{d}{d\omega}X_0(j\omega).$$

Therefore,

$$X_4(j\omega) = \frac{1 - 2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1+j\omega)^2}.$$

4.24. (a) (i) For $\mathcal{R}e\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and odd. Therefore, signals in figures (a) and (c) have this property.

(ii) For $\mathcal{I}m\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and even. Therefore, signals in figures (e) and (f) have this property.

(iii) For there to exist a real α such that $e^{j\alpha\omega}X(j\omega)$ is real, we require that $x(t+\alpha)$ be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.

(iv) For this condition to be true, $x(0) = 0$. Therefore, signals in figures (a), (b), (c), (d), and (f) have this property.

(v) For this condition to be true the derivative of $x(t)$ has to be zero at $t = 0$. Therefore, signals in figures (b), (c), (e), and (f) have this property.

(vi) For this to be true, the signal $x(t)$ has to be periodic. Only the signal in figure (a) has this property.

(b) For a signal to satisfy only properties (i), (iv), and (v), it must be real and odd, and

$$x(t) = 0, \qquad x'(0) = 0.$$

The signal shown below is an example of that.

Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.

Figure S4.30

Therefore,

$$x(t) = \frac{2\sin t}{\pi t}.$$

(b) $X_1(j\omega)$ is as shown in Figure S4.30.

4.31. **(a)** We have

$$x(t) = \cos t \stackrel{FT}{\longleftrightarrow} X(j\omega) = \pi[\delta(\omega+1) + \delta(\omega-1)].$$

(i) We have

$$h_1(t) = u(t) \stackrel{FT}{\longleftrightarrow} H_1(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(ii) We have

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \stackrel{FT}{\longleftrightarrow} H_2(j\omega) = -2 + \frac{5}{2+j\omega}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(iii) We have

$$h_3(t) = 2te^{-t}u(t) \stackrel{FT}{\longleftrightarrow} H_2(j\omega) = \frac{2}{(1+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(b) An LTI system with impulse response

$$h_4(t) = \frac{1}{2}[h_1(t) + h_2(t)]$$

will have the same response to $x(t) = \cos(t)$. We can find other such impulse responses by suitably scaling and linearly combining $h_1(t)$, $h_2(t)$, and $h_3(t)$.

4.32. Note that $h(t) = h_1(t-1)$, where

$$h_1(t) = \frac{\sin 4t}{\pi t}.$$

The Fourier transform $H_1(j\omega)$ of $h_1(t)$ is as shown in Figure S4.32.

From the above figure it is clear that $h_1(t)$ is the impulse response of an ideal lowpass filter whose passband is in the range $|\omega| < 4$. Therefore, $h(t)$ is the impulse response of an ideal lowpass filter shifted by one to the right. Using the shift property,

$$H(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}.$$

(a) We have

$$X_1(j\omega) = \pi e^{j\frac{\pi}{12}}\delta(\omega-6) + \pi e^{j\frac{\pi}{12}}\delta(\omega+6).$$

It is clear that

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = 0 \Rightarrow y_1(t) = 0.$$

This result is equivalent to saying that $X_1(j\omega)$ is zero in the passband of $H(j\omega)$.

(b) We have

$$X_2(j\omega) = \frac{\pi}{j}\left[\sum_{k=0}^{\infty}(\frac{1}{2})^k\{\delta(\omega-3k) - \delta(\omega+3k)\}\right].$$

Therefore,

$$Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{\pi}{j}\left[(1/2)\{\delta(\omega-3) - \delta(\omega+3)\}e^{-j\omega}\right].$$

This implies that

$$y_2(t) = \frac{1}{2}\sin(3t-1).$$

We may have obtained the same result by noting that only the sinusoid with frequency 3 in $X_2(j\omega)$ lies in the passband of $H(j\omega)$.

Chapter 5 Answers

5.1. (a) Let $x[n] = (1/2)^{n-1}u[n-1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(1/2)^{n}e^{-j\omega(n+1)} \\ &= e^{-j\omega}\frac{1}{(1-(1/2)e^{-j\omega})} \end{aligned}$$

(b) Let $x[n] = (1/2)^{|n-1|}$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{0} (1/2)^{-(n-1)}e^{-j\omega n} + \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \end{aligned}$$

The second summation in the right-hand side of the above equation is exactly the same as the result of part (a). Now,

$$\sum_{n=-\infty}^{0} (1/2)^{-(n-1)}e^{-j\omega n} = \sum_{n=0}^{\infty}(1/2)^{(n+1)}e^{j\omega n} = \left(\frac{1}{2}\right)\frac{1}{1-(1/2)e^{j\omega}}.$$

Therefore,

$$X(e^{j\omega}) = \left(\frac{1}{2}\right)\frac{1}{1-(1/2)e^{j\omega}} + e^{-j\omega}\frac{1}{(1-(1/2)e^{-j\omega})} = \frac{0.75e^{-j\omega}}{1.25-\cos\omega}.$$

5.2. (a) Let $x[n] = \delta[n-1] + \delta[n+1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{-j\omega} + e^{j\omega} = 2\cos\omega \end{aligned}$$

(b) Let $x[n] = \delta[n+2] - \delta[n-2]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &= e^{2j\omega} - e^{-2j\omega} = 2j\sin(2\omega) \end{aligned}$$

5.3. We note from Section 5.2 that a periodic signal $x[n]$ with Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

has a Fourier transform

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 6$. The signal may be written as

$$x_1[n] = (1/2j)e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_1[n]$ in the range $-2 \le k \le 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}}, \qquad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_1 \delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6}) \\ &= (\pi/j)\{e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)\} \end{aligned}$$

(b) Consider the signal $x_2[n] = 2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 12$. The signal may be written as

$$x_1[n] = 2 + (1/2)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + (1/2)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})} = 2 + (1/2)e^{j\frac{\pi}{8}}e^{j\frac{2\pi}{12}n} + (1/2)e^{-j\frac{\pi}{8}}e^{-j\frac{2\pi}{12}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \le k \le 6$ as

$$a_0 = 2, \qquad a_1 = (1/2)e^{j\frac{\pi}{8}}, \qquad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_0\delta(\omega) + 2\pi a_1 \delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{12}) \\ &= 4\pi\delta(\omega) + \pi\{e^{j\pi/8}\delta(\omega - \pi/6) + e^{-j\pi/8}\delta(\omega + \pi/6)\} \end{aligned}$$

5.4. (a) Using the Fourier transform synthesis equation (5.8),

$$\begin{aligned}
x_1[n] &= (1/2\pi)\int_{-\pi}^{\pi} X_1(e^{j\omega})e^{j\omega n}d\omega \\
&= (1/2\pi)\int_{-\pi}^{\pi} [2\pi\delta(\omega) + \pi\delta(\omega - \pi/2) + \pi\delta(\omega + \pi/2)]e^{j\omega n}d\omega \\
&= e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n} \\
&= 1 + \cos(\pi n/2)
\end{aligned}$$

(b) Using the Fourier transform synthesis equation (5.8),

$$\begin{aligned}
x_2[n] &= (1/2\pi)\int_{-\pi}^{\pi} X_2(e^{j\omega})e^{j\omega n}d\omega \\
&= -(1/2\pi)\int_{-\pi}^{0} 2je^{j\omega n}d\omega + (1/2\pi)\int_{0}^{\pi} 2je^{j\omega n}d\omega \\
&= (j/\pi)\left[-\frac{1 - e^{-jn\pi}}{jn} + \frac{e^{jn\pi} - 1}{jn}\right] \\
&= -(4/(n\pi))\sin^2(n\pi/2)
\end{aligned}$$

5.5. From the given information,

$$\begin{aligned}
x[n] &= (1/2\pi)\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= (1/2\pi)\int_{-\pi}^{\pi} |X(e^{j\omega})|e^{j\sphericalangle\{X(e^{j\omega})\}}e^{j\omega n}d\omega \\
&= (1/2\pi)\int_{-\pi/4}^{\pi/4} e^{-\frac{3}{2}\omega}e^{j\omega n}d\omega \\
&= \frac{\sin(\frac{\pi}{4}(n - 3/2))}{\pi(n - 3/2)}
\end{aligned}$$

The signal $x[n]$ is zero when $\frac{\pi}{4}(n - 3/2)$ is a nonzero integer multiple of π or when $|n| \to \infty$. The value of $\frac{\pi}{4}(n - 3/2)$ can never be such that it is a nonzero integer multiple of π. Therefore, $x[n] = 0$ only for $n = \pm\infty$.

5.6. Throughout this problem, we assume that

$$x[n] \overset{FT}{\longleftrightarrow} X_1(e^{j\omega}).$$

(a) Using the time reversal property (Sec. 5.3.6), we have

$$x[-n] \overset{FT}{\longleftrightarrow} X(e^{-j\omega})$$

5.20. (a) Since the LTI system is causal and stable, a single input-output pair is sufficient to determine the frequency response of the system. In this case, the input is $x[n] = (4/5)^n u[n]$ and the output is $y[n] = n(4/5)^n u[n]$. The frequency response is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of $x[n]$ and $y[n]$ respectively. Using Table 5.2, we have

$$x[n] = \left(\frac{4}{5}\right)^n u[n] \stackrel{FT}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}.$$

Using the differentiation in frequency property (Table 5.1, Property 5.3.8), we have

$$y[n] = n\left(\frac{4}{5}\right)^n u[n] \stackrel{FT}{\longleftrightarrow} Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} = \frac{(4/5)e^{-j\omega}}{(1 - \frac{4}{5}e^{-j\omega})^2}.$$

Therefore,

$$H(e^{j\omega}) = \frac{(4/5)e^{-j\omega}}{1 - \frac{4}{5}e^{-j\omega}}.$$

(b) Since $H(ej\omega) = Y(e^{j\omega})/X(e^{j\omega})$, we may write

$$Y(e^{j\omega})\left[1 - \frac{4}{5}e^{-j\omega}\right] = X(e^{j\omega})\left[(4/5)e^{-j\omega}\right].$$

Taking the inverse Fourier tranform of both sides

$$y[n] - \frac{4}{5}y[n-1] = \frac{4}{5}x[n].$$

5.21 (a) The given signal is

$$x[n] = u[n-2] - u[n-6] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5].$$

Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}.$$

(b) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{-1} (\frac{1}{2})^{-n} e^{-j\omega n} \\ &= \sum_{n=1}^{\infty} (\frac{1}{2}e^{j\omega})^n \\ &= \frac{e^{j\omega}}{2} \frac{1}{(1 - \frac{1}{2}e^{j\omega})} \end{aligned}$$

(c) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{-2} (\frac{1}{3})^{-n} e^{-j\omega n} \\ &= \sum_{n=2}^{\infty} (\frac{1}{3} e^{j\omega})^n \\ &= \frac{e^{2j\omega}}{9} \frac{1}{(1-\frac{1}{3}e^{j\omega})} \end{aligned}$$

(d) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{0} 2^n \sin(\pi n/4) e^{-j\omega n} \\ &= -\sum_{n=0}^{\infty} 2^{-n} \sin(\pi n/4) e^{j\omega n} \\ &= -\frac{1}{2j} \sum_{n=0}^{\infty} [(1/2)^n e^{j\pi n/4} e^{j\omega n} - (1/2)^n e^{-j\pi n/4} e^{j\omega n}] \\ &= -\frac{1}{2j} \left[\frac{1}{1-(1/2)e^{j\pi/4}e^{j\omega}} - \frac{1}{1-(1/2)e^{-j\pi/4}e^{j\omega}} \right] \end{aligned}$$

(e) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} (1/2)^{|n|} \cos[\pi(n-1)/8] e^{-j\omega n} \\ &= \frac{1}{2} \left[\frac{e^{-j\pi/8}}{1-(1/2)e^{j\pi/8}e^{-j\omega}} + \frac{e^{j\pi/8}}{1-(1/2)e^{-j\pi/8}e^{-j\omega}} \right] \\ &+ \frac{1}{4} \left[\frac{e^{j\pi/4}e^{j\omega}}{1-(1/2)e^{j\pi/8}e^{j\omega}} + \frac{e^{-j\pi/4}e^{j\omega}}{1-(1/2)e^{-j\pi/8}e^{j\omega}} \right] \end{aligned}$$

(f) The given signal is

$$x[n] = -3\delta[n+3] - 2\delta[n+2] - \delta[n+1] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3].$$

Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = -3e^{3j\omega} - 2e^{2j\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}.$$

(g) The given signal is

$$x[n] = \sin(\pi n/2) + \cos(n) = \frac{1}{2j}[e^{j\pi n/2} - e^{-j\pi n/2}] + \frac{1}{2}[e^{jn} + e^{-jn}].$$

Therefore,

$$X(e^{j\omega}) = \frac{\pi}{j}[\delta(\omega - \pi/2) - \delta(\omega + \pi/2)] + \pi[\delta(\omega - 1) + \delta(\omega + 1)], \qquad \text{in } 0 \le |\omega| < \pi.$$

(h) The given signal is

$$\begin{aligned} x[n] &= \sin(5\pi n/3) + \cos(7\pi n/3) \\ &= -\sin(\pi n/3) + \cos(\pi n/3) \\ &= -\frac{1}{2j}[e^{j\pi n/3} - e^{-j\pi n/3}] + \frac{1}{2}[e^{j\pi n/3} + e^{-j\pi n/3}]. \end{aligned}$$

Therefore,

$$X(e^{j\omega}) = -\frac{\pi}{j}[\delta(\omega - \pi/3) - \delta(\omega + \pi/3)] + \pi[\delta(\omega - \pi/3) + \delta(\omega + \pi/3)], \qquad \text{in } 0 \le |\omega| < \pi.$$

(i) $x[n]$ is periodic with period 6. The Fourier series coefficients of $x[n]$ are given by

$$\begin{aligned} a_k &= \frac{1}{6}\sum_{n=0}^{5} x[n]e^{-j(2\pi/6)kn} \\ &= \frac{1}{6}\sum_{n=0}^{4} e^{-j(2\pi/6)kn} \\ &= \frac{1}{6}\left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}}\right] \end{aligned}$$

Therefore, from the results of Section 5.2

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\left(\frac{1}{6}\right)\left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}}\right]\delta(\omega - \frac{2\pi}{6} - 2\pi l).$$

(j) Using the Fourier transform analysis eq. (5.9) we obtain

$$\left(\frac{1}{3}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} \frac{4}{5 - 3\cos\omega}.$$

Using the differentiation in frequency property of the Fourier transform,

$$n\left(\frac{1}{3}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} -j\frac{12\sin\omega}{(5 - 3\cos\omega)^2}.$$

Therefore,

$$x[n] = n\left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{3}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} \frac{4}{5 - 3\cos\omega} - j\frac{12\sin\omega}{(5 - 3\cos\omega)^2}.$$

(k) We have

$$x_1[n] = \frac{\sin(\pi n/5)}{\pi n} \stackrel{FT}{\longleftrightarrow} X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \le |\omega| < \pi \end{cases}.$$

Also,

$$x_2[n] = \cos(7\pi n/2) = \cos(\pi n/2) \stackrel{FT}{\longleftrightarrow} X_2(e^{j\omega}) = \pi\{\delta(\omega - \pi/2) + \delta(\omega + \pi/2)\},$$

in the range $0 \le |\omega| < \pi$. Therefore, if $x[n] = x_1[n]x_2[n]$, then

$$X(e^{j\omega}) = \text{Periodic convolution of } X_1(e^{j\omega}) \text{ and } X_2(e^{j\omega}).$$

Using the mechanics of periodic convolution demosntrated in Example 5.15, we obtain in the range $0 \le |\omega| < \pi$,

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{3\pi}{10} < |\omega| < \frac{7\pi}{10} \\ 0, & \text{otherwise} \end{cases} .$$

5.22 (a) Using the Fourier transform synthesis eq. (5.8), we obtain

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega \\ &= \frac{1}{\pi n}[\sin(3\pi n/4) - \sin(\pi n/4)] \end{aligned}$$

(b) Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] - 4\delta[n-3] + \delta[n-10].$$

(c) Using the Fourier transform synthesis eq. (5.8), we obtain

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/2} e^{j\omega n} d\omega \\ &= \frac{(-1)^{n+1}}{\pi(n - \frac{1}{2})} \end{aligned}$$

(d) The given Fourier transform is

$$\begin{aligned} X(e^{j\omega}) &= \cos^2\omega + \sin^2(3\omega) \\ &= \frac{1 + \cos(2\omega)}{2} + \frac{1 - \cos(3\omega)}{2} \\ &= 1 + \frac{1}{4}e^{2j\omega} + \frac{1}{4}e^{-2j\omega} + -\frac{1}{4}e^{3j\omega} - \frac{1}{4}e^{-3j\omega} \end{aligned}$$

Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-3] - \frac{1}{4}\delta[n+3].$$

ain

(e) This is the Fourier transform of a periodic signal with fundamental frequency $\pi/2$. Therefore, its fundamental period is 4. Also, the Fourier series coefficients of this signal are $a_k = (-1)^k$. Therefore, the signal is given by

$$x[n] = \sum_{k=0}^{3}(-1)^k e^{jk(\pi/2)n} = 1 - e^{j\pi n/2} + e^{j\pi n} - e^{j3\pi n/2}.$$

(f) The given Fourier transform may be written as

$$\begin{aligned} X(e^{j\omega}) &= e^{-j\omega}\sum_{n=0}^{\infty}(1/5)^n e^{-j\omega n} - (1/5)\sum_{n=0}^{\infty}(1/5)^n e^{-j\omega n} \\ &= 5\sum_{n=1}^{\infty}(1/5)^n e^{-j\omega n} - (1/5)\sum_{n=0}^{\infty}(1/5)^n e^{-j\omega n} \end{aligned}$$

Comparing each of the two terms in the right-hand side of the above equation with the Fourier transform analysis eq. (5.9) we obtain

$$x[n] = \left(\frac{1}{5}\right)^{n-1} u[n-1] - \left(\frac{1}{5}\right)^{n+1} u[n].$$

(g) The given Fourier transform may be written as

$$X(e^{j\omega}) = \frac{2/9}{1 - \frac{1}{2}e^{-j\omega}} + \frac{7/9}{1 + \frac{1}{4}e^{-j\omega}}.$$

Therefore,

$$x[n] = \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{9}\left(-\frac{1}{4}\right)^n u[n].$$

(h) The given Fourier transform may be written as

$$X(e^{j\omega}) = 1 + \frac{1}{3}e^{-j\omega} + \frac{1}{3^2}e^{-j2\omega} + \frac{1}{3^3}e^{-j3\omega} + \frac{1}{3^4}e^{-j4\omega} + \frac{1}{3^5}e^{-j5\omega}.$$

Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{9}\delta[n-2] + \frac{1}{27}\delta[n-3] + \frac{1}{81}\delta[n-4] + \frac{1}{243}\delta[n-5].$$

5.23. (a) We have from eq. (5.9)

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 6.$$

(b) Note that $y[n] = x[n+2]$ is an even signal. Therefore, $Y(e^{j\omega})$ is real and even. This implies that $\sphericalangle Y(e^{j\omega}) = 0$. Furthermore, from the time shifting property of the Fourier transform we have $Y(e^{j\omega}) = e^{j2\omega}X(e^{j\omega})$. Therefore, $\sphericalangle X(e^{j\omega}) = e^{-j2\omega}$.

Chapter 6 Answers

6.1. The signal $x(t)$ may be broken up into a sum of the two complex exponentials $x_1(t) = (1/2)e^{j\omega_0 t+\phi_0}$ and $x_2(t) = (1/2)e^{-j\omega_0 t-\phi_0}$. Since complex exponentials are Eigen functions of LTI systems, we know that when $x_1(t)$ passes through the LTI system, the output is

$$\begin{aligned} y_1(t) &= x_1(t)H(j\omega_0) = x_1(t)|H(j\omega_0)|e^{j\sphericalangle H(j\omega_0)} \\ &= (1/2)|H(j\omega_0)|e^{j(\omega_0 t+\phi_0+\sphericalangle H(j\omega_0))} \end{aligned}$$

Similarly, when the input is $x_2(t)$, the output is

$$y_2(t) = (1/2)|H(-j\omega_0)|e^{-j(\omega_0 t+\phi_0-\sphericalangle H(-j\omega_0))}.$$

But since $h[n]$ is given to be real, $|H(j\omega_0)| = |H(-j\omega_0)|$ and $\sphericalangle H(j\omega_0) = -\sphericalangle H(-j\omega_0)$. Therefore,

$$y_2(t) = (1/2)|H(j\omega_0)|e^{-j(\omega_0 t+\phi_0+\sphericalangle H(j\omega_0))}.$$

Using linearity we may argue that when the input to the LTI system is $x(t) = x_1(t) + x_2(t)$, the output will be $y(t) = y_1(t) = y_2(t)$. Therefore,

$$y(t) = |H(j\omega_0)|\cos(\omega_0 t + \phi_0 + \sphericalangle H(j\omega_0)) = |H(j\omega_0)|\cos\left(\omega_0(t - \frac{-\sphericalangle H(j\omega_0)}{\omega_0}) + \phi_0\right)$$

(a) From $y(t)$, we have $A = |H(j\omega_0)|$.

(b) From $y(t)$, we have $t_0 = \frac{-\sphericalangle H(j\omega_0)}{\omega_0}$.

6.2. The signal $x[n]$ may be broken up into a sum of the two complex exponentials $x_1[n] = (1/2j)e^{j\omega_0 n+\phi_0}$ and $x_2[n] = (-1/2j)e^{-j\omega_0 n-\phi_0}$. Since complex exponentials are Eigen functions of LTI systems, we know that when $x_1[n]$ passes through the LTI system, the output is

$$\begin{aligned} y_1[n] &= x_1[n]H(e^{j\omega_0}) = x_1[n]|H(e^{j\omega_0})|e^{j\sphericalangle H(e^{j\omega_0})} \\ &= (1/2j)|H(e^{j\omega_0})|e^{j(\omega_0 n+\phi_0+\sphericalangle H(e^{j\omega_0}))} \end{aligned}$$

Similarly, when the input is $x_2[n]$, the output is

$$y_2[n] = (-1/2j)|H(e^{-j\omega_0})|e^{-j(\omega_0 n+\phi_0-\sphericalangle H(e^{-j\omega_0}))}.$$

But since $h(t)$ is given to be real, $|H(e^{j\omega_0})| = |H(e^{-j\omega_0})|$ and $\sphericalangle H(e^{j\omega_0}) = -\sphericalangle H(e^{-j\omega_0})$. Therefore,

$$y_2[n] = (-1/2j)|H(e^{j\omega_0})|e^{-j(\omega_0 t+\phi_0+\sphericalangle H(e^{j\omega_0}))}.$$

Using linearity we may argue that when the input to the LTI system is $x[n] = x_1[n] + x_2[n]$, the output will be $y[n] = y_1[n] + y_2[n]$. Therefore,

$$y[n] = |H(e^{j\omega_0})|\sin(\omega_0 n + \phi_0 + \sphericalangle H(e^{j\omega_0})) = |H(e^{j\omega_0})|\sin\left(\omega_0(n - \frac{-\sphericalangle H(e^{j\omega_0})}{\omega_0}) + \phi_0\right)$$

Now note that if we require that $y[n] = |H(e^{j\omega_0})|x[n-n_0]$, then $n_0 = -\sphericalangle H(e^{j\omega_0})/\omega_0$ has to be an integer. Therefore, $\sphericalangle H(e^{j\omega_0}) = -n_0\omega_0$. Now also, note that if we add an integer multiple of 2π to this $\sphericalangle H(e^{j\omega_0})$, it does not make any difference. Therefore, we require in general that $\sphericalangle H(e^{j\omega_0}) = -n_0(\omega_0 + 2k\pi)$.

6.3. (a) We have

$$|H(j\omega)| = \frac{|1-j\omega|}{|1+j\omega|} = \frac{\sqrt{1+\omega^2}}{\sqrt{1+\omega^2}} = 1.$$

Therefore, $A = 1$.

(b) We have

$$\sphericalangle H(j\omega) = \tan^{-1}(-\omega) - \tan^{-1}(\omega) = 2\tan^{-1}(\omega).$$

Therefore, the group delay is

$$\tau(\omega) = -\frac{d}{d\omega}\sphericalangle H(j\omega) = \frac{2}{1+\omega^2}.$$

Clearly, $\tau(\omega) > 0$ for $\omega > 0$. Therefore, statement 2 is true.

6.4 (a) The signal $\cos(\pi n/2)$ can be broken up into a sum of two complex exponentials $x_1[n] = (1/2)e^{j\pi n/2}$ and $x_2[n] = (1/2)e^{-j\pi n/2}$. From the given information, we know that when $x_1[n]$ passes through the given LTI system, it experiences a delay of 2 samples. Since the system has a real impulse response, it has an even group delay function. Therefore, the complex exponential $x_2[n]$ with frequency $-\omega_0$ also experiences a group delay of 2 samples. The output $y[n]$ of the LTI system when the input is $x[n] = x_1[n] + x_2[n]$ is therefore

$$y[n] = 2x_1[n-2] + 2x_2[n-2] = 2\cos\left(\frac{\pi}{2}(n-2)\right) = 2\cos\left(\frac{\pi}{2}n - \pi\right)$$

(b) The signal $x[n] = \sin(\frac{7\pi}{2}n + \frac{\pi}{4})$ is the same as $-\sin(\frac{\pi}{2}n - \frac{\pi}{4})$. This signal may once again be broken up into complex exponentials of frequency $\pi/2$ and $-\pi/2$. We may then use an argument similar to the one used in part (a) to argue that the output $y[n]$ is

$$\begin{aligned} y[n] &= 2x[n-2] = 2\sin\left(\frac{7\pi}{2}(n-2) + \frac{\pi}{4}\right) \\ &= 2\sin\left(\frac{7\pi}{2}n - 7\pi + \frac{\pi}{4}\right) \\ &= 2\sin\left(\frac{7\pi}{2}n - \pi + \frac{\pi}{4}\right) \\ &= 2\sin\left(\frac{7\pi}{2}n - \frac{3\pi}{4}\right) \end{aligned}$$

6.5 The frequency response $H(j\omega)$ is as shown in Figure S6.5.

(a) Consider the signal $h_1(t) = \sin(\omega_c t)/(\pi t)$. Its Fourier transform $H_1(j\omega)$ is as shown in Figure S6.5.

Clearly,

$$H(j\omega) = H_1(j(\omega - 2\omega_c)) + H_1(j(\omega + 2\omega_c)).$$

6.6

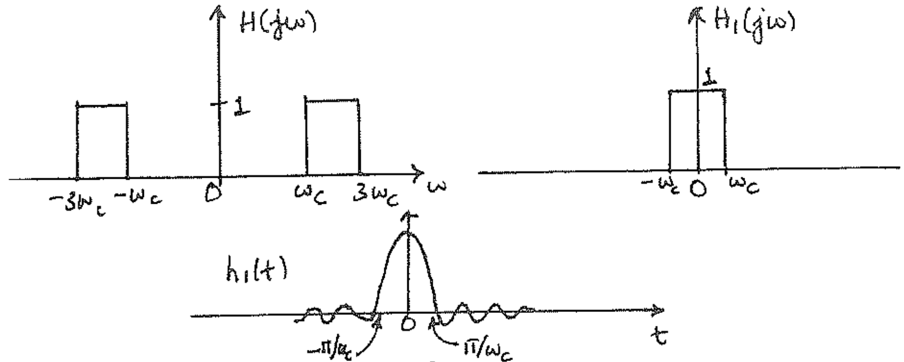

Figure S6.5

Taking the inverse Fourier transform, we have

$$\begin{aligned} h(t) &= h_1(t)e^{j2\omega_c t} + h_1(t)e^{-j2\omega_c t} \\ &= 2h_1(t)\cos(2\omega_c t) \end{aligned}$$

Therefore, $g(t) = \cos(2\omega_c t)$.

(b) The impulse response $h_1(t)$ is as shown in Figure S6.5. As ω_c increases, it is clear that the significant central lobe of $h_1(t)$ becomes more concentrated around the origin. Consequently $h(t) = 2h_1(t)\cos(2\omega_c t)$ also becomes more concentrated about the origin.

6.6 The frequency response $H(e^{j\omega})$ is as shown in Figure S6.6.

(a) Consider the signal $h_1[n] = \sin(\omega_c n)/(\pi n)$. Its Fourier transform $H_1(e^{j\omega})$ is as shown in the figure below.

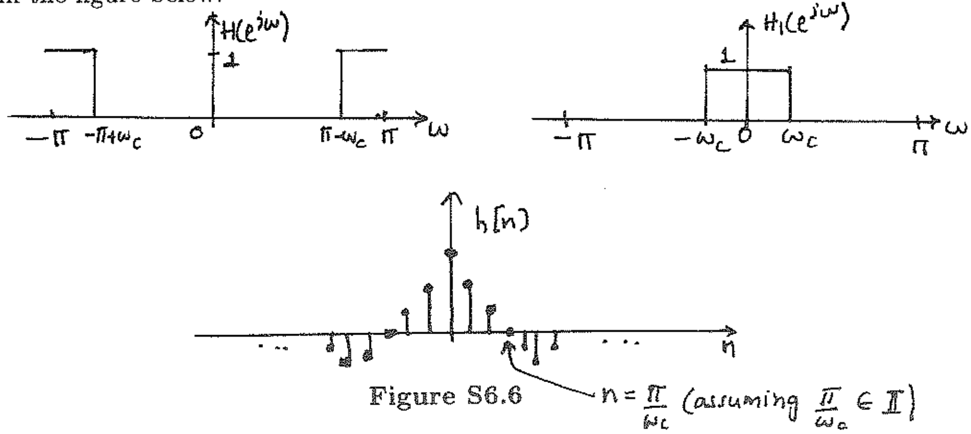

Figure S6.6

Clearly,

$$H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}).$$

Taking the inverse Fourier transform, we have

$$h[n] = h_1[n]e^{j\pi n} = h_1[n](-1)^n.$$

Therefore, $g[n] = (-1)^n$.

(b) The impulse response $h_1[n]$ is as shown in Figure S6.6. As ω_c increases, it is clear that the significant central lobe of $h_1[n]$ becomes more concentrated around the origin. Consequently $h[n] = h_1[n](-1)^n$ also becomes more concentrated about the origin.

6.7. The frequency response magnitude $|H(j\omega)|$ is as shown in Figure S6.7. The frequency response of the bandpass filter $G(j\omega)$ will be given by

$$\begin{aligned} G(j\omega) &= \mathcal{FT}\{2h(t)\cos(4000\pi t) \\ &= H(j(\omega - 4000\pi)) + H(j(\omega + 4000\pi)) \end{aligned}$$

This is as shown in Figure S6.7

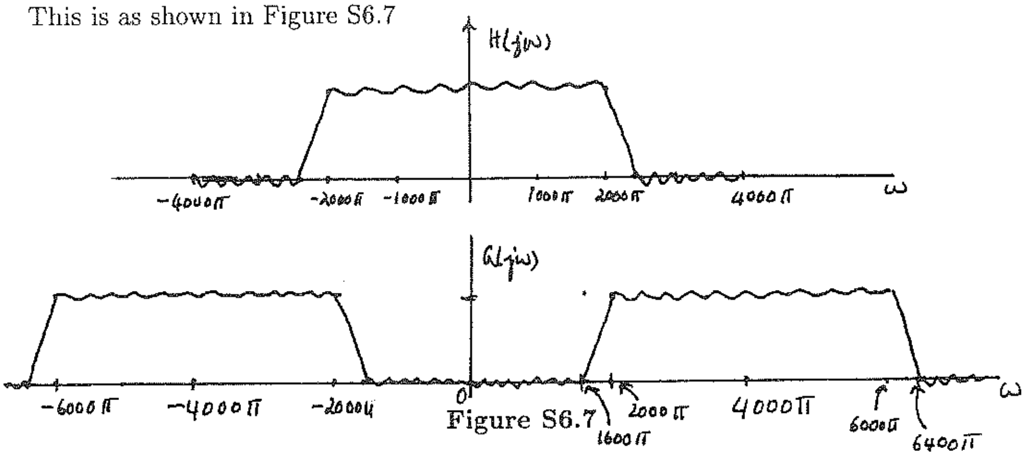

Figure S6.7

(a) From the figure, it is obvious that the passband edges are at 2000π rad/sec and 6000π rad/sec. This translates to 1000 Hz and 3000 Hz, respectively.

(b) From the figure, it is obvious that the stopband edges are at 1600π rad/sec and 6400π rad/sec. This translates to 800 Hz and 3200 Hz, respectively.

6.8. Taking the Fourier transform of both sides of the first difference equation and simplifying, we obtain the frequency response $H(e^{j\omega})$ of the first filter.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 - \sum_{k=1}^{N} a_k e^{-j\omega k}}.$$

Taking the Fourier transform of both sides of the second difference equation and simplifying, we obtain the frequency response $H_1(e^{j\omega})$ of the second filter.

$$H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 - \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}}.$$

6.13. Using an approach similar to the one used in the previous problem, we obtain

$$H(j\omega) = \frac{320}{(j\omega + 2)(j\omega + 80)}.$$

(a) Let us assume that we desire to construct this system by cascading two systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$, respectively. We require that

$$H(j\omega) = H_1(j\omega)H_2(j\omega).$$

We see that $H_1(j\omega)$ and $H_2(j\omega)$ may be defined in different ways to obtain $H(j\omega)$. For instance

$$H_1(j\omega) = \frac{40}{(j\omega + 2)} \quad \text{and} \quad H_2(j\omega) = \frac{8}{(j\omega + 80)}$$

and

$$H_1(j\omega) = \frac{32}{(j\omega + 2)} \quad \text{and} \quad H_2(j\omega) = \frac{10}{(j\omega + 80)}$$

are both valid combinations.

(b) Let us assume that we desire to construct this system by connecting two systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$ in parallel. We require that

$$H(j\omega) = H_1(j\omega) + H_2(j\omega).$$

Using partial fraction expansion on $H(j\omega)$, we obtain

$$H(j\omega) = \frac{160/39}{(j\omega + 2)} - \frac{160/39}{(j\omega + 80)}$$

From the above expression it is clear that we can define $H_1(j\omega)$ and $H_2(j\omega)$ in only one way.

6.14. Using an approach similar to the one used in Problem 6.12, we have

$$H(j\omega) = \frac{50000(j\omega + 0.2)^2}{(j\omega + 50)(j\omega + 10)}.$$

The inverse to this system has a frequency response

$$H_I(j\omega) = \frac{1}{H(j\omega)} = \frac{0.2 \times 10^{-4}(j\omega + 50)(j\omega + 10)}{(j\omega + 0.2)^2}.$$

6.15. We will use the results from Section 6.5 in this problem.

(a) We may write the frequency response of the system described by the given differential equation as

$$H_1(j\omega) = \frac{1}{(j\omega)^2 + 4j\omega + 4}.$$

6.19. Let us first find the differential equation governing the input and output of this circuit. Current through resistor and inductor = Current through capacitor = $C\frac{dy(t)}{dt}$.
Voltage across resistor = $RC\frac{dy(t)}{dt}$.
Voltage across inductor = $LC\frac{d^2y(t)}{dt^2}$.
Total input voltage = Voltage across inductor + Voltage across resistor + Voltage across capacitor
Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t).$$

The frequency response of this circuit is therefore

$$H(j\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + 1}.$$

We may rewrite this to be

$$H(j\omega) = \frac{1}{(\frac{j\omega}{1/\sqrt{LC}})^2 + 2(R/2)\sqrt{C/L}\frac{j\omega}{1/\sqrt{LC}} + 1}.$$

Therefore, the damping constant $\zeta = (R/2)\sqrt{C/L}$. In order for the step response to have no oscillations, we must have $\zeta \geq 1$. Therefore, we require

$$R \geq 2\sqrt{\frac{L}{C}}.$$

6.20. Let us call the given impulse response $h[n]$. It is easily observed that the signal $h_1[n] = h[n+2]$ is real and even. Therefore, (using properties of the Fourier transform) we know that the Fourier transform $H_1(e^{j\omega})$ of $h_1[n]$ is real and even. Therefore $H_1(e^{j\omega})$ has zero phase. We also know that the Fourier transform $H(e^{j\omega}) = H_1(e^{j\omega})e^{-2j\omega}$. Since $H_1(e^{j\omega})$ is zero phase, we have

$$\sphericalangle H(e^{j\omega}) = -2\omega.$$

Therefore, the group delay is

$$\tau(\omega) = \frac{d}{d\omega}\sphericalangle H(e^{j\omega}) = 2.$$

6.21. Note that in all parts of this problem $Y(j\omega) = H(j\omega)X(j\omega) = -2j\omega X(j\omega)$. Therefore, $y(t) = -2dx(t)/dt$.

(a) Here, $x(t) = e^{jt}$. Therefore, $y(t) = -2dx(t)/dt = -2je^{jt}$. This part could also have been solved by noting that complex exponentials are Eigen functions of LTI systems. Then, when $x(t) = e^{jt}$, $y(t)$ should be $y(t) = H(j1)e^{jt} = -2je^{jt}$.

(b) Here, $x(t) = \sin(\omega_0 t)u(t)$. Then, $dx(t)/dt = \omega_0\cos(\omega_0 t)u(t) + \sin(\omega_0 t)\delta(t) = \omega_0\cos(\omega_0 t)u(t)$. Therefore, $y(t) = -2dx(t)/dt = -2\omega_0\cos(\omega_0 t)u(t)$.

(c) Here, $Y(j\omega) = X(j\omega)H(j\omega) = -2/(6+j\omega)$. Taking the inverse Fourier transform we obtain $y(t) = -2e^{-6t}u(t)$.

(d) Here, $X(j\omega) = 1/(2 + j\omega)$. From this we obtain $x(t) = e^{-2t}u(t)$. Therefore, $y(t) = -2dx(t)/dt = 4e^{-2t}u(t) - 2\delta(t)$.

6.22. Note that

$$H(j\omega) = \begin{cases} \frac{j\omega}{3\pi}, & -3\pi \le \omega \le 3\pi \\ 0, & \text{otherwise} \end{cases}.$$

(a) Since $x(t) = \cos(2\pi t + \theta)$, $X(j\omega) = e^{j\theta}\pi\delta(\omega - 2\pi) + e^{-j\theta}\pi\delta(\omega + 2\pi)$. This is zero outside the region $-3\pi < \omega < 3\pi$. Thus, $Y(j\omega) = H(j\omega)X(j\omega) = (j\omega/3\pi)X(j\omega)$. This implies that $y(t) = (1/3\pi)dx(t)/dt = (-2/3)\sin(2\pi t + \theta)$.

(b) Since $x(t) = \cos(4\pi t + \theta)$, $X(j\omega) = e^{j\theta}\pi\delta(\omega - 4\pi) + e^{-j\theta}\pi\delta(\omega + 4\pi)$. Therefore, the nonzero portions of $X(j\omega)$ lie outside the range $-3\pi < \omega < 3\pi$. This implies that $Y(j\omega) = X(j\omega)H(j\omega) = 0$. Therefore, $y(t) = 0$.

(c) The Fourier series coefficients of the signal $x(t)$ are given by

$$a_k = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-jk\omega_0 t},$$

where $T_0 = 1$ and $\omega_0 = 2\pi/T_0 = 2\pi$. Also,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0).$$

The only impulses of $X(j\omega)$ which lie in the region $-3\pi < \omega < 3\pi$ are at $\omega = 0,\ 2\pi,$ and 2π. Defining the signal $x_{lp}(t) = a_0 + a_1 e^{j2\pi t} + a_{-1}e^{-j2\pi t}$, we note that $y(t) = (1/3\pi)dx_{lp}(t)/dt$. We can also easily show that $a_0 = 1/\pi$, $a_1 = a_{-1}^* = -1/(4j)$. Putting these into the expression for $x_{lp}(t)$ we obtain $x_{lp}(t) = (1/\pi) + (1/2)\sin(2\pi t)$. Finally, $y(t) = (1/3\pi)dx_{lp}(t)/dt = (1/3)\cos(2\pi t)$.

6.23. (a) From the given information, we have

$$H_a(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

Using Table 4.2, we get

$$h_a(t) = \frac{\sin(\omega_c t)}{\pi t}.$$

(b) Here,

$$H_b(j\omega) = H_a(j\omega)e^{j\omega T}.$$

Using Table 4.1, we get

$$h_b(t) = h_a(t + T).$$

Therefore,

$$h_b(t) = \frac{\sin[\omega_c(t + T)]}{\pi(t + T)}.$$

(c) Let us consider a frequency response $H_0(j\omega)$ given by

$$H_0(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c/2 \\ 0, & \text{otherwise} \end{cases}.$$

Clearly,

$$H_c(j\omega) = \frac{1}{2\pi}[H_0(j\omega) * W(j\omega)],$$

where

$$W(j\omega) = j2\pi\delta(\omega - \omega_c/2) - j2\pi\delta(\omega + \omega_c/2).$$

Therefore, from Table 4.1

$$h_c(t) = h_0(t)w(t) = \left[\frac{\sin(\omega_c t/2)}{\pi t}\right][-2\sin(\omega_c t/2)].$$

6.24. If $\tau(\omega) = k_1$, where k_1 is a constant, then

$$\sphericalangle H(j\omega) = -k_1\omega + k_2 \qquad \text{(S6.24-1)}$$

where k_2 is another constant.

(a) Note that if $h(t)$ is real, the phase of the Fourier transform $\sphericalangle H(j\omega)$ has to be an odd function. Therefore, the value of k_2 in eq. (S6.24-1) will be zero.

Also, let us define $H_0(j\omega) = |H(j\omega)|$. Then,

$$h_0(t) = \frac{\sin(200\pi t)}{\pi t}.$$

(i) Here $k_1 = 5$. Hence, $\sphericalangle H(j\omega) = -5\omega$. Then,

$$H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)} = H_0(j\omega)e^{-j5\omega}.$$

Therefore,

$$h(t) = h_0(t-5) = \frac{\sin[200\pi(t-5)]}{\pi(t-5)}.$$

(ii) Here $k_1 = 5/2$. Hence, $\sphericalangle H(j\omega) = -(5/2)\omega$. Then,

$$H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)} = H_0(j\omega)e^{-j(5/2)\omega}.$$

Therefore,

$$h(t) = h_0(t-5/2) = \frac{\sin[200\pi(t-5/2)]}{\pi(t-5/2)}.$$

(iii) Here $k_1 = -5/2$. Hence, $\sphericalangle H(j\omega) = (5/2)\omega$. Then,

$$H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)} = H_0(j\omega)e^{j(5/2)\omega}.$$

Therefore,

$$h(t) = h_0(t+5/2) = \frac{\sin[200\pi(t+5/2)]}{\pi(t+5/2)}.$$

(b) If $h(t)$ is not specified to be real, then $\sphericalangle H(j\omega)$ does not have to be an odd function. Therefore, the value of k_2 in eq. (S6.24-1) does not have to be zero. Given only $|H(j\omega)|$ and $\tau(\omega)$, k_2 cannot be determined uniquely. Therefore, $h(t)$ cannot be determined uniquely.

6.25. (a) We may write $H_a(j\omega)$ as

$$H_a(j\omega) = \frac{(1-j\omega)}{(1+j\omega)(1-j\omega)} = \frac{1-j\omega}{2}.$$

Therefore,

$$\sphericalangle H_a(j\omega) = \tan^{-1}[-\omega].$$

and

$$\tau_a(\omega) = -\frac{d\sphericalangle H_a(j\omega)}{d\omega} = \frac{1}{1+\omega^2}.$$

Since $\tau_a(0) = 1 \neq 2 = \tau_a(1)$, $\tau_a(\omega)$ is not a constant for all ω. Therefore, the frequency response has nonlinear phase.

(b) In this case, $H_b(j\omega)$ is the frequency response of a system which is a cascade combination of two systems, each of which has a frequency response $H_a(j\omega)$. Therefore,

$$\sphericalangle H_b(j\omega) = \sphericalangle H_a(j\omega) + \sphericalangle H_a(j\omega)$$

and

$$\tau_b(\omega) = -2\frac{d\sphericalangle H_a(j\omega)}{d\omega} = \frac{2}{1+\omega^2}.$$

Since $\tau_b(0) = 2 \neq 4 = \tau_b(1)$, $\tau_b(\omega)$ is not a constant for all ω. Therefore, the frequency response has nonlinear phase.

(c) In this case, $H_c(j\omega)$ is again the frequency response of a system which is a cascade combination of two systems. The first system has a frequency response $H_a(j\omega)$, while the second system has a frequency response $H_0(j\omega) = 1/(2+j\omega)$. Therefore,

$$\sphericalangle H_b(j\omega) = \sphericalangle H_a(j\omega) + \sphericalangle H_0(j\omega)$$

and

$$\tau_c(\omega) = -\frac{d\sphericalangle H_a(j\omega)}{d\omega} - \frac{d\sphericalangle H_0(j\omega)}{d\omega} = \frac{1}{1+\omega^2} + \frac{2}{4+\omega^2}.$$

Since $\tau_c(0) = (3/2) \neq (3/5) = \tau_c(1)$, $\tau_b(\omega)$ is not a constant for all ω. Therefore, the frequency response has nonlinear phase.

6.26. (a) Note that $H(j\omega) = 1 - H_0(j\omega)$, where $H_0(j\omega)$ is

$$H_0(j\omega) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

$$h(t) = \delta(t) - h_0(t).$$

6

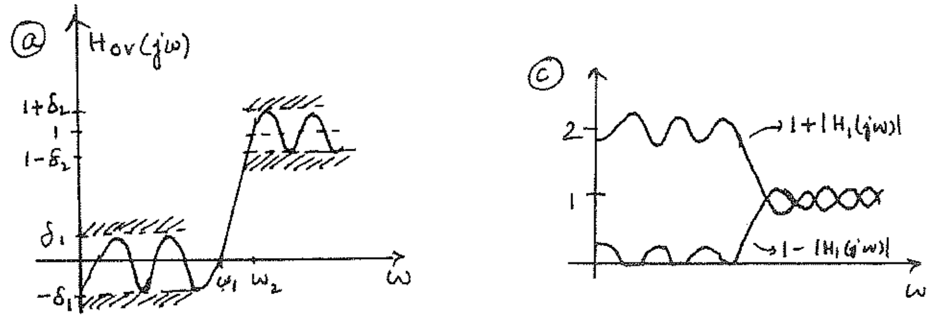

Figure S6.34

(d) From the tolerances derived in the previous part, it is clear that $H_{ov}(j\omega)$ is not necessarily highpass.

6.35. Since $x[n] = \cos(\omega_0 n + \theta)$, we have

$$X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} [e^{j\theta}\delta(\omega - \omega_0 - 2\pi l) + e^{-j\theta}\delta(\omega + \omega_0 - 2\pi l)].$$

Let ω_0' be the principal value of ω_0 in $[-\pi, \pi]$. Then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} [e^{j\theta} j\omega_0' \delta(\omega - \omega_0 - 2\pi l) - e^{-j\theta} j\omega_0' \delta(\omega + \omega_0 - 2\pi l)].$$

It follows that

$$y[n] = -\omega_0' \sin(\omega_0 n + \theta).$$

If $-\pi \leq \omega_0 \leq \pi$, then

$$y[n] = -\omega_0 \sin(\omega_0 n + \theta).$$

6.36. Let $H_1(e^{j\omega}) = |H(e^{j\omega})|$. Then from Table 5.2 we know that

$$h_1[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

If $\tau(\omega) = -\frac{d}{d\omega}\sphericalangle H(e^{j\omega}) = k$ (where k is a constant), then $\sphericalangle H(e^{j\omega}) = -k\omega + k_1$, where k_1 is a constant. If $h[n]$ is real, then $\sphericalangle H(e^{j\omega})$ is an odd function, and therefore we may conclude that $k_1 = 0$. Therefore,

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\sphericalangle H(e^{j\omega})} = H_1(e^{j\omega})e^{-jk\omega}.$$

Taking the inverse Fourier transform we obtain

$$h[n] = h_1[n-k] = \frac{\sin[\pi(n-k)/2]}{\pi(n-k)}.$$

(a) If $\tau(\omega) = 5$, then from the above result,

$$h[n] = \frac{\sin[\pi(n-5)/2]}{\pi(n-5)}.$$

(b) If $\tau(\omega) = 5/2$, then from the result derived at the beginning of this problem

$$h[n] = \frac{\sin[\pi(n-5/2)/2]}{\pi(n-5/2)}.$$

(c) If $\tau(\omega) = -5/2$, then from the result derived at the beginning of this problem

$$h[n] = \frac{\sin[\pi(n+5/2)/2]}{\pi(n+5/2)}.$$

The results of all the parts of this problem are sketched in Figure S6.36.

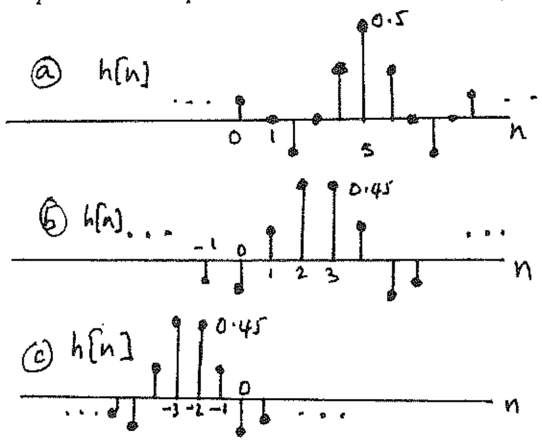

Figure S6.36

6.37. (a) We have

$$|H(e^{j\omega})| = \frac{|1-\frac{1}{2}e^{j\omega}|}{|1-\frac{1}{2}e^{-j\omega}|} = 1.$$

(b) We have

$$\begin{aligned}
\sphericalangle H(e^{j\omega}) &= \sphericalangle[e^{-j\omega}] + \sphericalangle\left[1-\frac{1}{2}e^{j\omega}\right] - \sphericalangle\left[1-\frac{1}{2}e^{-j\omega}\right] \\
&= \sphericalangle[e^{-j\omega}] + \sphericalangle\left[1-\frac{1}{2}\cos(\omega)-\frac{j}{2}\sin(\omega)\right] - \sphericalangle\left[1-\frac{1}{2}\cos(\omega)+\frac{j}{2}\sin(\omega)\right] \\
&= -\omega - 2\tan^{-1}\left[\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right]
\end{aligned}$$

(c) Using the result of the previous part, we can show with some algebraic manipulation that

$$\tau(\omega) = -\frac{d\sphericalangle H(e^{j\omega})}{d\omega} = \frac{\frac{3}{4}}{\frac{5}{4} - \cos\omega}.$$

This is as sketched below

(d) Let $x[n] = \cos(\pi n/3)$. We may write this as $x[n] = e^{j\pi n/3}/2 + e^{-j\pi n/3}/2$. From the result of part (c), we know that the delay suffered by a complex exponential of frequency $\pi/3$ is

$$\frac{\frac{3}{4}}{\frac{5}{4} - \cos(\pi/3)} = 1.$$

Similarly, we know that the delay suffered by a complex exponential of frequency $-\pi/3$ is also 1. Therefore, the output of the system is $y[n] = e^{j\pi(n-1)/3}/2 + e^{-j\pi(n-1)/3}/2 = \cos(\pi(n-1)/3)$.

6.38. We may express $H(e^{j\omega})$ as

$$H(e^{j\omega}) = \frac{1}{2\pi}\left[H_1(e^{j\omega}) * \{2\pi\delta(\omega - \pi/2) + 2\pi\delta(\omega + \pi/2)\}\right],$$

and

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}.$$

Using the properties of the Fourier transform, we obtain

$$h[n] = h_1[n]\,[2\cos(\pi n/2)],$$

where

$$h_1[n] = \frac{\sin(\omega_c n)}{\pi n}.$$

(a) When $\omega_c = \pi/5$, $h[n] = 2\frac{\sin(\pi n/5)}{\pi n}\cos(\pi n/2)$. This is as shown in Figure S6.38.

(b) When $\omega_c = \pi/4$, $h[n] = 2\frac{\sin(\pi n/4)}{\pi n}\cos(\pi n/2)$. This is as shown in Figure S6.38.

(c) When $\omega_c = \pi/3$, $h[n] = 2\frac{\sin(\pi n/3)}{\pi n}\cos(\pi n/2)$. This is as shown in Figure S6.38. As ω_c increases, $h[n]$ becomes more concentrated about the origin.

6.39. The plots are as shown in Figure S6.39.

6.40. We may write $h_1[n]$ as

$$\begin{aligned} H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h_1[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h_1[2n]e^{-j2\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j2\omega n} \\ &= H(e^{j2\omega}) \end{aligned}$$

Therefore, $H_1(e^{j\omega})$ is $H(e^{j\omega})$ compressed by a factor of two. This is as shown in Figure S6.40.

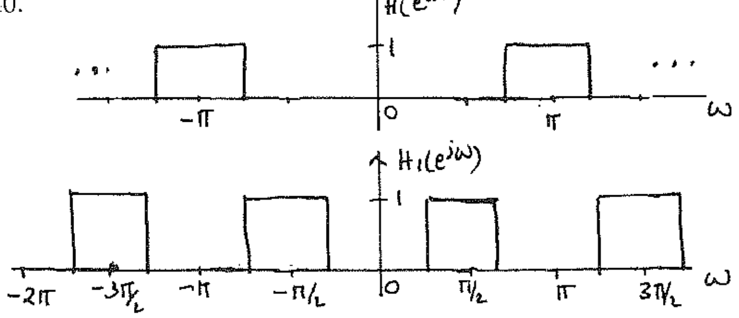

Figure S6.40

Therefore, $H_1(e^{j\omega})$ corresponds to a band-stop filter.

6.41. **(a)** Taking the Fourier transform of both sides of the given difference equation, we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - e^{-j\omega}}{1 - \frac{1}{\sqrt{2}}e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}.$$

Taking the inverse Fourier transform of $H(e^{j\omega})$ we obtain

$$h[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/4)u[n] - (2\sqrt{2} - 1)\left(\frac{1}{2}\right)^n \sin(\pi n/4)u[n].$$

(b) The log-magnitude and phase of the frequency response are as shown in Figure S6.41.

6.42. **(a)** We get

$$|H_1(e^{j\omega})| = |H_2(e^{j\omega})| = \frac{5/4 + \cos\omega}{17/6 + (1/2)\cos\omega}$$

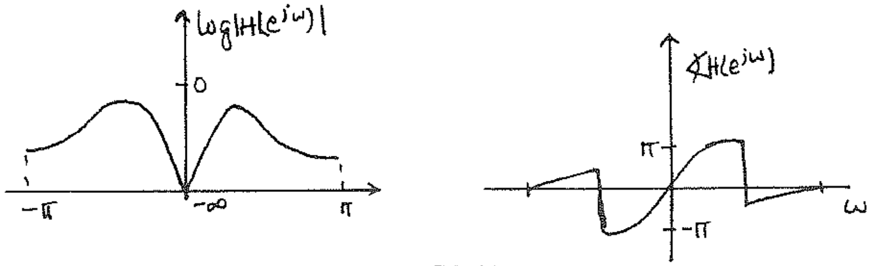

Figure S6.41

and

$$\sphericalangle H_1(e^{j\omega}) = \tan^{-1}\left(\frac{(1/2)\sin\omega}{1+(1/2)\cos(\omega)}\right) \quad \text{and} \quad \sphericalangle H_2(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{1+(1/2)\cos(\omega)}\right).$$

Comparing tangents of these angle in the range $0 \le \omega \le \pi$, we get

$$\sphericalangle H_2(e^{j\omega}) > \sphericalangle H_1(e^{j\omega}).$$

(b) We get

$$h_1[n] = \left(-\frac{1}{4}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{4}\right)^{n-1} u[n-1]$$

and

$$h_2[n] = \frac{1}{2}\left(-\frac{1}{4}\right)^n u[n] + \left(-\frac{1}{4}\right)^{n-1} u[n-1].$$

This is as sketched in Figure S6.42.

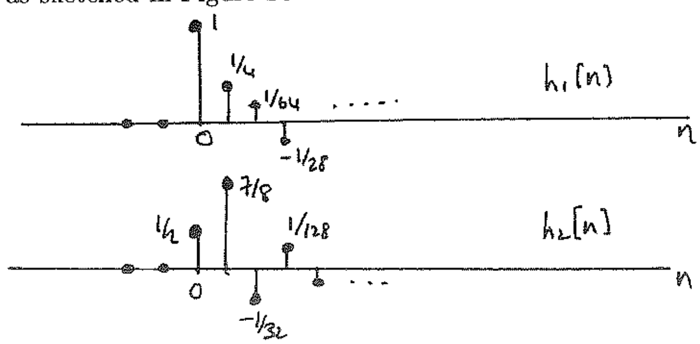

Figure S6.42

(c) We get

$$H_2(e^{j\omega}) = \left(\frac{1/2 + e^{-j\omega}}{1+(1/2)e^{-j\omega}}\right) H_1(e^{j\omega}).$$

Therefore,

$$G(e^{j\omega}) = \left(\frac{1/2 + e^{-j\omega}}{1 + (1/2)e^{-j\omega}} \right)$$

and

$$|G(e^{j\omega})| = \frac{(5/4) + \cos\omega}{(5/4) + \cos\omega} = 1.$$

6.43. **(a)** If $h_{hp}[n] = (-1)^n h_{lp}[n] = e^{j\pi n} h_{lp}[n]$, then

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}).$$

Therefore, $H_{hp}(e^{j\omega})$ is as shown in Figure S6.43. Clearly, it corresponds to a highpass filter.

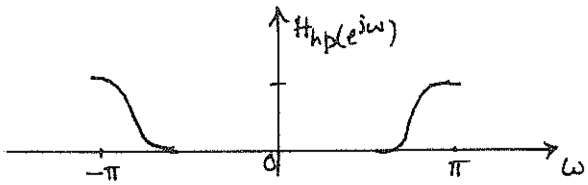

Figure S6.43

(b) Now let us define $h[n] = (-1)^n h_{hp}[n]$, where $h_{hp}[n]$ is the impulse response of a highpass filter. Then

$$H(e^{j\omega}) = H_{hp}(e^{j(\omega-\pi)}).$$

Therefore, if $H_{hp}(e^{j\omega})$ is as shown in Figure S6.43, then $H(e^{j\omega})$ is lowpass.

6.44. **(a)** Note that $(-1)^n = e^{j\pi n}$. From the figure we have

$$y[n] = \left(x[n]e^{j\pi n} * h_{lp}[n]\right) e^{j\pi n}.$$

We may write this as

$$y[n] = a[n]e^{j\pi n},$$

where $a[n] = \left(x[n]e^{j\pi n} * h_{lp}[n]\right)$. Taking the Fourier transform of $a[n]$, we obtain

$$A(e^{j\omega}) = X(e^{j(\omega-\pi)})H_{lp}(e^{j\omega}).$$

Suppose that the input to the system is now $x[n-n_0]$. Let the corresponding output be $y_1[n]$. Then we may write

$$y_1[n] = b[n]e^{j\pi n},$$

where $b[n] = \left(x[n-n_0]e^{j\pi n} * h_{lp}[n]\right)$. Taking the Fourier transform of $b[n]$, we obtain

$$B(e^{j\omega}) = X(e^{j(\omega-\pi)})H_{lp}(e^{j\omega})e^{-j\omega n_0} = A(e^{j\omega})e^{-j\omega n_0}.$$

Therefore,

$$b[n] = a[n-n_0].$$

Consequently, $y_1[n] = y[n-n_0]$. Therefore, the system in time invariant.