(i) Using the Fourier transform analysis eq. (4.9) we obtain

\[ X(j\omega) = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{-\omega^2} - \frac{2e^{-j\omega} - 2}{j\omega^2}. \]

(j) \( x(t) \) is periodic with period 2. Therefore,

\[ X(j\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{X}(j\omega) \delta(\omega - 2\pi k), \]

where \( \hat{X}(j\omega) \) is the Fourier transform of one period of \( x(t) \). That is,

\[ \hat{X}(j\omega) = \frac{1}{1 - e^{-2}} \left[ \frac{1 - e^{-2(1+j\omega)}}{1 + j\omega} - \frac{e^{-2[1 - e^{-2(1+j\omega)}]}}{1 - j\omega} \right]. \]

4.22. (a) \( x(t) = \begin{cases} e^{2j\pi t}, & |t| < 3 \\ 0, & \text{otherwise} \end{cases} \)

(b) \( x(t) = \frac{1}{2} e^{-j\pi/3} \delta(t - 4) + \frac{1}{2} e^{j\pi/3} \delta(t + 4). \)

(c) The Fourier transform synthesis eq. (4.8) may be written as

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)| e^{j\alpha X(j\omega)} e^{j\omega t} d\omega. \]

From the given figure we have

\[ x(t) = \frac{1}{\pi} \left[ \frac{\sin(t - 3)}{t - 3} + \frac{\cos(t - 3) - 1}{(t - 3)^2} \right]. \]

(d) \( x(t) = \frac{2t}{\pi} \sin t + \frac{3}{\pi} \cos(2\pi t) \)

(e) Using the Fourier transform synthesis equation (4.8),

\[ x(t) = \frac{\cos 3t}{j\pi t} + \frac{\sin t - \sin 2t}{j\pi t^2}. \]

4.23. For the given signal \( x_0(t) \), we use the Fourier transform analysis eq. (4.8) to evaluate the corresponding Fourier transform

\[ X_0(j\omega) = \frac{1 - e^{-(1+j\omega)}}{1 + j\omega}. \]

(i) We know that

\[ x_1(t) = x_0(t) + x_0(-t). \]

Using the linearity and time reversal properties of the Fourier transform we have

\[ X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{2 - 2e^{-1} \cos \omega - 2we^{-1} \sin \omega}{1 + \omega^2}. \]
(ii) We know that
\[ x_2(t) = x_0(t) - x_0(-t). \]

Using the linearity and time reversal properties of the Fourier transform we have
\[ X_2(j\omega) = X_0(j\omega) - X_0(-j\omega) = j \left[ \frac{-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega}{1 + \omega^2} \right]. \]

(iii) We know that
\[ x_3(t) = x_0(t) + x_0(t + 1). \]

Using the linearity and time shifting properties of the Fourier transform we have
\[ X_3(j\omega) = X_0(j\omega) + e^{j\omega}X_0(-j\omega) = \frac{1 + e^{j\omega} - e^{-1}(1 + e^{-j\omega})}{1 + j\omega}. \]

(iv) We know that
\[ x_4(t) = tx_0(t). \]

Using the differentiation in frequency property
\[ X_4(j\omega) = \frac{d}{d\omega}X_0(j\omega). \]

Therefore,
\[ X_4(j\omega) = \frac{1 - 2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1 + j\omega)^2}. \]

4.24. (a) (i) For \(\Re\{X(j\omega)\}\) to be 0, the signal \(x(t)\) must be real and odd. Therefore, signals in figures (a) and (c) have this property.
(ii) For \(\Im\{X(j\omega)\}\) to be 0, the signal \(x(t)\) must be real and even. Therefore, signals in figures (c) and (f) have this property.
(iii) For there to exist a real \(\alpha\) such that \(e^{j\alpha \omega}X(j\omega)\) is real, we require that \(x(t + \alpha)\) be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.
(iv) For this condition to be true, \(x(0) = 0\). Therefore, signals in figures (a), (b), (c), (d), and (f) have this property.
(v) For this condition to be true the derivative of \(x(t)\) has to be zero at \(t = 0\). Therefore, signals in figures (b), (c), (e), and (f) have this property.
(vi) For this to be true, the signal \(x(t)\) has to be periodic. Only the signal in figure (a) has this property.
(b) For a signal to satisfy only properties (i), (iv), and (v), it must be real and odd, and
\[ x(t) = 0, \quad x'(0) = 0. \]

The signal shown below is an example of that.
Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.

![Figure S4.30](image)

Therefore,

$$x(t) = \frac{2\sin t}{\pi t}.$$  

(b) $X_1(j\omega)$ is as shown in Figure S4.30.

\[\boxed{4.31}\]

(a) We have

$$x(t) = \cos t \xrightarrow{FT} X(j\omega) = \pi[\delta(\omega + 1) + \delta(\omega - 1)].$$

(i) We have

$$h_1(t) = u(t) \xrightarrow{FT} H_1(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega + 1) - \delta(\omega - 1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(ii) We have

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \xrightarrow{FT} H_2(j\omega) = -2 + \frac{5}{2 + j\omega}.$$  

Therefore,

$$Y(j\omega) = X(j\omega)H_2(j\omega) = \frac{\pi}{j}[\delta(\omega + 1) - \delta(\omega - 1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$
(iii) We have
\[ h_3(t) = 2te^{-t}u(t) \xrightarrow{FT} H_2(j\omega) = \frac{2}{(1 + j\omega)^2}. \]
Therefore,
\[ Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega + 1) - \delta(\omega - 1)]. \]
Taking the inverse Fourier transform, we obtain
\[ y(t) = \sin(t). \]
(b) An LTI system with impulse response
\[ h_4(t) = \frac{1}{2}[h_1(t) + h_2(t)] \]
will have the same response to \( x(t) = \cos(t) \). We can find other such impulse responses by suitably scaling and linearly combining \( h_1(t) \), \( h_2(t) \), and \( h_3(t) \).

4.32. Note that \( h(t) = h_1(t - 1) \), where
\[ h_1(t) = \frac{\sin 4t}{\pi t}. \]
The Fourier transform \( H_1(j\omega) \) of \( h_1(t) \) is as shown in Figure S4.32.

From the above figure it is clear that \( h_1(t) \) is the impulse response of an ideal lowpass filter whose passband is in the range \( |\omega| < 4 \). Therefore, \( h(t) \) is the impulse response of an ideal lowpass filter shifted by one to the right. Using the shift property,
\[ H(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}. \]
(a) We have
\[ X_1(j\omega) = \pi e^{j\frac{\pi}{2}} \delta(\omega - 6) + \pi e^{j\frac{\pi}{2}} \delta(\omega + 6). \]
It is clear that
\[ Y_1(j\omega) = X_1(j\omega)H(j\omega) = 0 \Rightarrow y_1(t) = 0. \]
This result is equivalent to saying that \( X_1(j\omega) \) is zero in the passband of \( H(j\omega) \).
(b) We have
\[ X_2(j\omega) = \frac{\pi}{j} \left[ \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k \{ \delta(\omega - 3k) - \delta(\omega + 3k) \} \right]. \]
Therefore,
\[ Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{\pi}{j} \left[ (1/2) \{ \delta(\omega - 3) - \delta(\omega + 3) \} e^{-j\omega} \right]. \]
This implies that
\[ y_2(t) = \frac{1}{2} \sin(3t - 1). \]
We may have obtained the same result by noting that only the sinusoid with frequency 3 in \( X_2(j\omega) \) lies in the passband of \( H(j\omega) \).
Chapter 5 Answers

5.1. (a) Let \( x[n] = (1/2)^{n-1}u[n - 1] \). Using the Fourier transform analysis equation (5.9), the Fourier transform \( X(e^{j\omega}) \) of this signal is

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \\
= \sum_{n=1}^{\infty} (1/2)^{n-1}e^{-jwn} \\
= \frac{1}{(1 - (1/2)e^{-j\omega})}
\]

(b) Let \( x[n] = (1/2)^{n-1} \). Using the Fourier transform analysis equation (5.9), the Fourier transform \( X(e^{j\omega}) \) of this signal is

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \\
= \sum_{n=0}^{\infty} (1/2)^{-n-1}e^{-jwn} + \sum_{n=1}^{\infty} (1/2)^{n-1}e^{-jwn}
\]

The second summation in the right-hand side of the above equation is exactly the same as the result of part (a). Now,

\[
\sum_{n=-\infty}^{0} (1/2)^{-n-1}e^{-jwn} = \sum_{n=0}^{\infty} (1/2)^{n+1}e^{jwn} = \left(\frac{1}{2}\right) \frac{1}{1 - (1/2)e^{j\omega}}.
\]

Therefore,

\[
X(e^{j\omega}) = \left(\frac{1}{2}\right) \frac{1}{1 - (1/2)e^{j\omega}} + e^{-j\omega} \frac{1}{1 - (1/2)e^{-j\omega}} = 0.75e^{-j\omega} + \frac{1.25 - \cos \omega}{2}
\]

5.2. (a) Let \( x[n] = \delta[n - 1] + \delta[n + 1] \). Using the Fourier transform analysis equation (5.9), the Fourier transform \( X(e^{j\omega}) \) of this signal is

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \\
= e^{-j\omega} + e^{j\omega} = 2 \cos \omega
\]
(b) Let $x[n] = \delta[n+2] - \delta[n-2]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = e^{2j\omega} - e^{-2j\omega} = 2j \sin(2\omega)$$

5.3. We note from Section 5.2 that a periodic signal $x[n]$ with Fourier series representation

$$x[n] = \sum_{k=-N}^{N} a_k e^{j(2\pi/N)n}$$

has a Fourier transform

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{5}n + \frac{\pi}{4})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 6$. The signal may be written as

$$x_1[n] = (1/2j)e^{j(\frac{\pi}{5}n + \frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{5}n + \frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}.$$

From this, we obtain the non-zero Fourier series coefficients $a_k$ of $x_1[n]$ in the range $-2 \leq k \leq 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}}, \quad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}.$$

Therefore, in the range $-\pi \leq \omega \leq \pi$, we obtain

$$X(e^{j\omega}) = 2\pi a_1 \delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1} \delta(\omega + \frac{2\pi}{6})$$

$$= \left(\frac{\pi}{j}\right)\left(e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)\right).$$

(b) Consider the signal $x_2[n] = 2 + \cos(\frac{\pi}{8}n + \frac{\pi}{5})$. We note that the fundamental period of the signal $x_2[n]$ is $N = 12$. The signal may be written as

$$x_2[n] = 2 + (1/2)e^{j(\frac{\pi}{8}n + \frac{\pi}{5})} + (1/2)e^{-j(\frac{\pi}{8}n + \frac{\pi}{5})} = 2 + (1/2)e^{j\frac{\pi}{8}}e^{j\frac{3\pi}{8}n} + (1/2)e^{-j\frac{\pi}{8}}e^{-j\frac{3\pi}{8}n}.$$

From this, we obtain the non-zero Fourier series coefficients $a_k$ of $x_2[n]$ in the range $-5 \leq k \leq 6$ as

$$a_0 = 2, \quad a_1 = (1/2)e^{j\frac{\pi}{8}}, \quad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}.$$

Therefore, in the range $-\pi \leq \omega \leq \pi$, we obtain

$$X(e^{j\omega}) = 2\pi a_0 \delta(\omega) + 2\pi a_1 \delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1} \delta(\omega + \frac{2\pi}{12})$$

$$= 4\pi \delta(\omega) + \pi\left(e^{j\pi/8}\delta(\omega - \pi/6) + e^{-j\pi/8}\delta(\omega + \pi/6)\right).$$
5.4. (a) Using the Fourier transform synthesis equation (5.8),

\[ x_1[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} [2\pi \delta(\omega) + \pi \delta(\omega - \pi/2) + \pi \delta(\omega + \pi/2)] e^{j\omega n} d\omega \]

\[ = e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n} \]

\[ = 1 + \cos(\pi n/2) \]

(b) Using the Fourier transform synthesis equation (5.8),

\[ x_2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = -(1/2\pi) \int_{-\pi}^{0} 2je^{j\omega n} d\omega + (1/2\pi) \int_{0}^{\pi} 2je^{j\omega n} d\omega \]

\[ = (j/\pi) \left[ -\frac{1 - e^{-jn\pi}}{jn} + \frac{e^{jn\pi} - 1}{jn} \right] \]

\[ = -(4/(n\pi)) \sin^2(n\pi/2) \]

5.5. From the given information,

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = (1/2\pi) \int_{-\pi}^{\pi} |X(e^{j\omega})| e^{j\delta(X(e^{j\omega}))} e^{j\omega n} d\omega \]

\[ = (1/2\pi) \int_{-\pi/4}^{\pi/4} e^{-\frac{\pi}{2} \omega} e^{j\omega n} d\omega \]

\[ = \frac{\sin(\frac{\pi}{4}(n - 3/2))}{\pi(n - 3/2)} \]

The signal \( x[n] \) is zero when \( \frac{\pi}{4}(n - 3/2) \) is a nonzero integer multiple of \( \pi \) or when \( |n| \to \infty \). The value of \( \frac{\pi}{4}(n - 3/2) \) can never be such that it is a nonzero integer multiple of \( \pi \). Therefore, \( x[n] = 0 \) only for \( n = \pm \infty \).

5.6. Throughout this problem, we assume that

\[ x[n] \leftrightarrow X_1(e^{j\omega}). \]

(a) Using the time reversal property (Sec. 5.3.6), we have

\[ x[-n] \leftrightarrow X(e^{-j\omega}). \]
5.20. (a) Since the LTI system is causal and stable, a single input-output pair is sufficient to determine the frequency response of the system. In this case, the input is \( x[n] = (4/5)^n u[n] \) and the output is \( y[n] = n(4/5)^n u[n] \). The frequency response is given by

\[
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}
\]

where \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) are the Fourier transforms of \( x[n] \) and \( y[n] \) respectively. Using Table 5.2, we have

\[
x[n] = \left( \frac{4}{5} \right)^n u[n] \xrightarrow{FT} X(e^{j\omega}) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}.
\]

Using the differentiation in frequency property (Table 5.1, Property 5.3.8), we have

\[
y[n] = n \left( \frac{4}{5} \right)^n u[n] \xrightarrow{FT} Y(e^{j\omega}) = \frac{dX(e^{j\omega})}{d\omega} = \frac{(4/5)e^{-j\omega}}{(1 - \frac{4}{5}e^{-j\omega})^2}.
\]

Therefore,

\[
H(e^{j\omega}) = \frac{(4/5)e^{-j\omega}}{1 - \frac{4}{5}e^{-j\omega}}.
\]

(b) Since \( H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega}) \), we may write

\[
Y(e^{j\omega}) \left[ 1 - \frac{4}{5}e^{-j\omega} \right] = X(e^{j\omega}) \left( \frac{4}{5}e^{-j\omega} \right).
\]

Taking the inverse Fourier transform of both sides

\[
y[n] - \frac{4}{5}y[n-1] = \frac{4}{5}x[n].
\]

5.21 (a) The given signal is

\[
x[n] = u[n-2] - u[n-6] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5].
\]

Using the Fourier transform analysis eq. (5.9), we obtain

\[
X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}.
\]

(b) Using the Fourier transform analysis eq. (5.9), we obtain

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{1} \left( \frac{1}{2} \right)^{-n} e^{-j\omega n}
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{2} e^{j\omega} \right)^n
\]

\[
= \frac{e^{j\omega}}{2} \frac{1}{(1 - \frac{1}{2}e^{j\omega})}
\]

179
(c) Using the Fourier transform analysis eq. (5.9), we obtain

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{n=\infty} \left( \frac{1}{3} \right)^{-n} e^{-j\omega n}
\]

\[
= \sum_{n=2}^{\infty} \left( \frac{1}{3} e^{j\omega} \right)^n
\]

\[
= \frac{e^{2j\omega} - 1}{9 (1 - \frac{1}{3} e^{j\omega})}
\]

(d) Using the Fourier transform analysis eq. (5.9), we obtain

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{n=\infty} 2^n \sin(\pi n/4) e^{-j\omega n}
\]

\[
= -\sum_{n=0}^{\infty} 2^{-n} \sin(\pi n/4) e^{j\omega n}
\]

\[
= -\frac{1}{2j} \sum_{n=0}^{\infty} [(1/2)^n e^{j\pi/4} e^{j\omega n} - (1/2)^n e^{-j\pi/4} e^{j\omega n}]
\]

\[
= -\frac{1}{2j} \left[ \frac{1}{1 - (1/2)e^{j\pi/4} e^{j\omega}} - \frac{1}{1 - (1/2)e^{-j\pi/4} e^{j\omega}} \right]
\]

(e) Using the Fourier transform analysis eq. (5.9), we obtain

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{n=\infty} (1/2)^{n|n|} \cos[\pi (n-1)/8] e^{-j\omega n}
\]

\[
= \frac{1}{2} \left[ \frac{e^{-j\pi/8}}{1 - (1/2)e^{j\pi/8} e^{-j\omega}} + \frac{e^{j\pi/8}}{1 - (1/2)e^{-j\pi/8} e^{-j\omega}} \right]
\]

\[
+ \frac{1}{4} \left[ \frac{e^{j\pi/4} e^{j\omega}}{1 - (1/2)e^{j\pi/8} e^{j\omega}} + \frac{e^{-j\pi/4} e^{j\omega}}{1 - (1/2)e^{-j\pi/8} e^{j\omega}} \right]
\]

(f) The given signal is

\[x[n] = -3\delta[n+3] - 2\delta[n+2] - \delta[n+1] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3].\]

Using the Fourier transform analysis eq. (5.9), we obtain

\[X(e^{j\omega}) = -3e^{j3\omega} - 2e^{j2\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}.\]

(g) The given signal is

\[x[n] = \sin(\pi n/2) + \cos(n) = \frac{1}{2j}[e^{jn/2} - e^{-j\pi n/2}] + \frac{1}{2}[e^{jn} + e^{-jn}].\]

Therefore,

\[X(e^{j\omega}) = \frac{\pi}{j} [\delta(\omega - \pi/2) - \delta(\omega + \pi/2)] + \pi[\delta(\omega - 1) + \delta(\omega + 1)], \quad \text{in } 0 \leq |\omega| < \pi.
\]
(h) The given signal is
\[ x[n] = \sin(5\pi n/3) + \cos(7\pi n/3) \]
\[ = -\sin(\pi n/3) + \cos(\pi n/3) \]
\[ = -\frac{1}{2j}[e^{j\pi n/3} - e^{-j\pi n/3}] + \frac{1}{2}[e^{j\pi n/3} + e^{-j\pi n/3}] . \]

Therefore,
\[ X(e^{j\omega}) = -\frac{\pi}{j}[\delta(\omega - \pi/3) - \delta(\omega + \pi/3)] + \pi[\delta(\omega - \pi/3) + \delta(\omega + \pi/3)] , \quad 0 \leq |\omega| < \pi. \]

(i) \( x[n] \) is periodic with period 6. The Fourier series coefficients of \( x[n] \) are given by
\[ a_k = \frac{1}{6} \sum_{n=0}^{5} x[n] e^{-j(2\pi/6)kn} \]
\[ = \frac{1}{6} \sum_{n=0}^{4} e^{-j(2\pi/6)kn} \]
\[ = \frac{1}{6} \left[ \frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}} \right] \]

Therefore, from the results of Section 5.2
\[ X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi \left( \frac{1}{6} \right) \left[ \frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}} \right] \delta(\omega - \frac{2\pi}{6} - 2\pi l) . \]

(j) Using the Fourier transform analysis eq. (5.9) we obtain
\[ \left( \frac{1}{3} \right)^{|n|} \xrightarrow{FT} \frac{4}{5 - 3\cos \omega} . \]

Using the differentiation in frequency property of the Fourier transform,
\[ n \left( \frac{1}{3} \right)^{|n|} \xrightarrow{FT} -j \frac{12 \sin \omega}{(5 - 3 \cos \omega)^2} . \]

Therefore,
\[ x[n] = n \left( \frac{1}{3} \right)^{|n|} - \left( \frac{1}{3} \right)^{|n|} \xrightarrow{FT} \frac{4}{5 - 3\cos \omega} - j \frac{12 \sin \omega}{(5 - 3 \cos \omega)^2} . \]

(k) We have
\[ x_1[n] = \frac{\sin(\pi n/5)}{\pi n} \xrightarrow{FT} X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \leq |\omega| < \pi \end{cases} . \]
Also,

\[ x_2[n] = \cos(7\pi n/2) = \cos(\pi n/2) \xrightarrow{FT} X_2(e^{j\omega}) = \pi \{ \delta(\omega - \pi/2) + \delta(\omega + \pi/2) \}, \]

in the range \(0 \leq |\omega| < \pi\). Therefore, if \( x[n] = x_1[n]x_2[n] \), then

\[ X(e^{j\omega}) = \text{Periodic convolution of } X_1(e^{j\omega}) \text{ and } X_2(e^{j\omega}). \]

Using the mechanics of periodic convolution demonstrated in Example 5.15, we obtain in the range \(0 \leq |\omega| < \pi\),

\[ X(e^{j\omega}) = \begin{cases} 
1, & \frac{3\pi}{10} < |\omega| < \frac{7\pi}{10} \\
0, & \text{otherwise}
\end{cases} \]

(5.22) (a) Using the Fourier transform synthesis eq. (5.8), we obtain

\[ x[n] = \frac{1}{2\pi} \int_{-\pi/4}^\pi e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega \\
= \frac{1}{\pi n} [\sin(3\pi n/4) - \sin(\pi n/4)] \]

(b) Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

\[ x[n] = \delta[n] + 3\delta[n - 1] + 2\delta[n - 2] - 4\delta[n - 3] + \delta[n - 10]. \]

(c) Using the Fourier transform synthesis eq. (5.8), we obtain

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^\pi e^{-j\omega n} d\omega \\
= \frac{(-1)^{n+1}}{\pi(n - \frac{1}{2})} \]

(d) The given Fourier transform is

\[ X(e^{j\omega}) = \cos^2 \omega + \sin^2(3\omega) \\
= \frac{1 + \cos(2\omega)}{2} + \frac{1 - \cos(3\omega)}{2} \\
= 1 + \frac{1}{4} e^{2j\omega} + \frac{1}{4} e^{-2j\omega} + \frac{1}{4} e^{3j\omega} - \frac{1}{4} e^{-3j\omega} \]

Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

\[ x[n] = \delta[n] + \frac{1}{4} \delta[n - 2] + \frac{1}{4} \delta[n + 2] - \frac{1}{4} \delta[n - 3] - \frac{1}{4} \delta[n + 3]. \]
(e) This is the Fourier transform of a periodic signal with fundamental frequency $\pi/2$. Therefore, its fundamental period is 4. Also, the Fourier series coefficients of this signal are $a_k = (-1)^k$. Therefore, the signal is given by

$$x[n] = \sum_{k=0}^{3} (-1)^k e^{jk\pi n/2} = 1 - e^{j\pi n/2} + e^{j\pi n} - e^{j3\pi n/2}.$$

(f) The given Fourier transform may be written as

$$X(e^{j\omega}) = e^{-j\omega} \sum_{n=0}^{\infty} (1/5)^n e^{-j\omega n} - (1/5) \sum_{n=0}^{\infty} (1/5)^n e^{-j\omega n}$$

$$= 5 \sum_{n=1}^{\infty} (1/5)^n e^{-j\omega n} - (1/5) \sum_{n=0}^{\infty} (1/5)^n e^{-j\omega n}.$$

Comparing each of the two terms in the right-hand side of the above equation with the Fourier transform analysis eq. (5.9) we obtain

$$x[n] = (\frac{1}{5})^{n-1} u[n-1] - (\frac{1}{5})^{n+1} u[n].$$

(g) The given Fourier transform may be written as

$$X(e^{j\omega}) = \frac{2/9}{1 - \frac{1}{5} e^{-j\omega}} + \frac{7/9}{1 + \frac{1}{4} e^{-j\omega}}.$$

Therefore,

$$x[n] = \frac{2}{9} \left(\frac{1}{2}\right)^n u[n] + \frac{7}{9} \left(-\frac{1}{4}\right)^n u[n].$$

(h) The given Fourier transform may be written as

$$X(e^{j\omega}) = 1 + \frac{1}{3} e^{-j\omega} + \frac{1}{32} e^{-j2\omega} + \frac{1}{34} e^{-j3\omega} + \frac{1}{34} e^{-j4\omega} + \frac{1}{35} e^{-j5\omega}.$$

Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + \frac{1}{3} \delta[n-1] + \frac{1}{9} \delta[n-2] + \frac{1}{27} \delta[n-3] + \frac{1}{81} \delta[n-4] + \frac{1}{243} \delta[n-5].$$

5.23. (a) We have from eq. (5.9)

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n] = 6.$$

(b) Note that $y[n] = x[n + 2]$ is an even signal. Therefore, $Y(e^{j\omega})$ is real and even. This implies that $\omega Y(e^{j\omega}) = 0$. Furthermore, from the time shifting property of the Fourier transform we have $Y(e^{j\omega}) = e^{j\omega} X(e^{j\omega})$. Therefore, $\omega X(e^{j\omega}) = e^{j\omega}$. 

183
Chapter 6 Answers

6.1 The signal \( x(t) \) may be broken up into a sum of the two complex exponentials \( x_1(t) = (1/2)e^{j\omega_0 t + \phi_0} \) and \( x_2(t) = (1/2)e^{-j\omega_0 t - \phi_0} \). Since complex exponentials are Eigen functions of LTI systems, we know that when \( x_1(t) \) passes through the LTI system, the output is

\[
y_1(t) = x_1(t)H(j\omega_0) = x_1(t)|H(j\omega_0)|e^{j\angle H(j\omega_0)} = (1/2)|H(j\omega_0)|e^{j\omega_0 t + \phi_0 + j\angle H(j\omega_0)}
\]

Similarly, when the input is \( x_2(t) \), the output is

\[
y_2(t) = (1/2)|H(-j\omega_0)|e^{-j(\omega_0 t + \phi_0 - j\angle H(-j\omega_0))}.
\]

But since \( h[n] \) is given to be real, \( |H(j\omega_0)| = |H(-j\omega_0)| \) and \( \angle H(j\omega_0) = -\angle H(-j\omega_0) \). Therefore,

\[
y_2(t) = (1/2)|H(j\omega_0)|e^{-j(\omega_0 t + \phi_0 + j\angle H(j\omega_0))}.
\]

Using linearity we may argue that when the input to the LTI system is \( x(t) = x_1(t) + x_2(t) \), the output will be \( y(t) = y_1(t) + y_2(t) \). Therefore,

\[
y(t) = |H(j\omega_0)|\cos(\omega_0 t + \phi_0 + j\angle H(j\omega_0)) = |H(j\omega_0)|\cos\left(\omega_0(t - \frac{-\angle H(j\omega_0)}{\omega_0} + \phi_0\right)
\]

(a) From \( y(t) \), we have \( A = |H(j\omega_0)| \).

(b) From \( y(t) \), we have \( t_0 = -\frac{-\angle H(j\omega_0)}{\omega_0} \).

6.2 The signal \( x[n] \) may be broken up into a sum of the two complex exponentials \( x_1[n] = (1/2j)e^{j\omega_0 n + \phi_0} \) and \( x_2[n] = (-1/2j)e^{-j\omega_0 n - \phi_0} \). Since complex exponentials are Eigen functions of LTI systems, we know that when \( x_1[n] \) passes through the LTI system, the output is

\[
y_1[n] = x_1[n]H(e^{j\omega_0}) = x_1[n]|H(e^{j\omega_0})|e^{j\angle H(e^{j\omega_0})} = (1/2j)|H(e^{j\omega_0})|e^{j(\omega_0 n + \phi_0 + j\angle H(e^{j\omega_0}))}
\]

Similarly, when the input is \( x_2[n] \), the output is

\[
y_2[n] = (-1/2j)|H(e^{-j\omega_0})|e^{-j(\omega_0 n + \phi_0 - j\angle H(e^{-j\omega_0}))}.
\]

But since \( h(t) \) is given to be real, \( |H(e^{j\omega_0})| = |H(e^{-j\omega_0})| \) and \( \angle H(e^{j\omega_0}) = -\angle H(e^{-j\omega_0}) \). Therefore,

\[
y_2[n] = (-1/2j)|H(e^{j\omega_0})|e^{-j(\omega_0 t + \phi_0 + j\angle H(e^{j\omega_0}))}.
\]

Using linearity we may argue that when the input to the LTI system is \( x[n] = x_1[n] + x_2[n] \), the output will be \( y[n] = y_1[n] + y_2[n] \). Therefore,

\[
y[n] = |H(e^{j\omega_0})|\sin(\omega_0 n + \phi_0 + \angle H(e^{j\omega_0})) = |H(e^{j\omega_0})|\sin\left(\omega_0(n - \frac{-\angle H(e^{j\omega_0})}{\omega_0} + \phi_0\right)
\]

Now note that if we require that \( y[n] = |H(e^{j\omega_0})|x[n] - n_0 \), then \( n_0 = -\frac{-\angle H(e^{j\omega_0})}{\omega_0} \) has to be an integer. Therefore, \( \angle H(e^{j\omega_0}) = -n_0\omega_0 \). Now also, note that if we add an integer multiple of \( 2\pi \) to this \( \angle H(e^{j\omega_0}) \), it does not make any difference. Therefore, we require in general that \( \angle H(e^{j\omega_0}) = -n_0(\omega_0 + 2k\pi) \).
6.3. (a) We have

\[ |H(j\omega)| = \frac{|1 - j\omega|}{|1 + j\omega|} = \frac{\sqrt{1 + \omega^2}}{\sqrt{1 + \omega^2}} = 1. \]

Therefore, \( A = 1 \).

(b) We have

\[ \angle H(j\omega) = \tan^{-1}(-\omega) - \tan^{-1}(\omega) = 2\tan^{-1}(\omega). \]

Therefore, the group delay is

\[ \tau(\omega) = -\frac{d}{d\omega} \angle H(j\omega) = -\frac{2}{1 + \omega^2}. \]

Clearly, \( \tau(\omega) > 0 \) for \( \omega > 0 \). Therefore, statement 2 is true.

6.4 (a) The signal \( \cos(\pi n/2) \) can be broken up into a sum of two complex exponentials \( x_1[n] = (1/2)e^{j\pi n/2} \) and \( x_2[n] = (1/2)e^{-j\pi n/2} \). From the given information, we know that when \( x_1[n] \) passes through the given LTI system, it experiences a delay of 2 samples. Since the system has a real impulse response, it has an even group delay function. Therefore, the complex exponential \( x_2[n] \) with frequency \(-\omega_0\) also experiences a group delay of 2 samples. The output \( y[n] \) of the LTI system when the input is \( x[n] = x_1[n] + x_2[n] \) is therefore

\[ y[n] = 2x_1[n - 2] + 2x_2[n - 2] = 2\cos\left(\frac{\pi}{2}(n - 2)\right) = 2\cos\left(\frac{\pi}{2}n - \pi\right) \]

(b) The signal \( x[n] = \sin(7\pi n/4 + \pi/4) \) is the same as \(-\sin(7\pi n - 7\pi/4)\). This signal may once again be broken up into complex exponentials of frequency \( \pi/2 \) and \(-\pi/2 \). We may then use an argument similar to the one used in part (a) to argue that the output \( y[n] \) is

\[ y[n] = 2x[n - 2] = 2\sin\left(\frac{7\pi}{4}(n - 2) + \frac{\pi}{4}\right) = 2\sin\left(\frac{7\pi}{4}n - 7\pi + \frac{\pi}{4}\right) = 2\sin\left(\frac{7\pi}{4}n - \pi + \frac{\pi}{4}\right) = 2\sin\left(\frac{7\pi}{4}n - 3\pi/4\right) \]

6.5 (a) The frequency response \( H(j\omega) \) is as shown in Figure S6.5.

Consider the signal \( h_1(t) = \sin(\omega_c t)/\pi t \). Its Fourier transform \( H_1(j\omega) \) is as shown in Figure S6.5.

Clearly,

\[ H(j\omega) = H_1(j(\omega - 2\omega_c)) + H_1(j(\omega + 2\omega_c)). \]

214
Taking the inverse Fourier transform, we have
\[ h(t) = h_1(t) e^{j2\omega t} + h_1(t) e^{-j2\omega t} = 2h_1(t) \cos(2\omega t) \]

Therefore, \( g(t) = \cos(2\omega_c t) \).

(b) The impulse response \( h_1(t) \) is as shown in Figure S6.5. As \( \omega_c \) increases, it is clear that the significant central lobe of \( h_1(t) \) becomes more concentrated around the origin. Consequently \( h(t) = 2h_1(t) \cos(2\omega_c t) \) also becomes more concentrated about the origin.

6.6 The frequency response \( H(e^{j\omega}) \) is as shown in Figure S6.6.
(a) Consider the signal \( h_1[n] = \sin(\omega_c n)/(\pi n) \). Its Fourier transform \( H_1(e^{j\omega}) \) is as shown in the figure below.

Clearly,
\[ H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}) \]

Taking the inverse Fourier transform, we have
\[ h[n] = h_1[n] e^{j\pi n} = h_1[n](-1)^n \]

Therefore, \( g[n] = (-1)^n \).
(b) The impulse response \( h_1[n] \) is as shown in Figure S6.6. As \( \omega_c \) increases, it is clear that the significant central lobe of \( h_1[n] \) becomes more concentrated around the origin. Consequently \( h[n] = h_1[n](-1)^n \) also becomes more concentrated about the origin.

6.7. The frequency response magnitude \( |H(j\omega)| \) is as shown in Figure S6.7. The frequency response of the bandpass filter \( G(j\omega) \) will be given by

\[
G(j\omega) = FT\{2h(t)\cos(4000\pi t)\} = H(j(\omega - 4000\pi)) + H(j(\omega + 4000\pi))
\]

This is as shown in Figure S6.7

(a) From the figure, it is obvious that the passband edges are at 2000\( \pi \) rad/sec and 6000\( \pi \) rad/sec. This translates to 1000 Hz and 3000 Hz, respectively.

(b) From the figure, it is obvious that the stopband edges are at 1600\( \pi \) rad/sec and 6400\( \pi \) rad/sec. This translates to 800 Hz and 3200 Hz, respectively.

6.8. Taking the Fourier transform of both sides of the first difference equation and simplifying, we obtain the frequency response \( H(e^{j\omega}) \) of the first filter.

\[
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 - \sum_{k=1}^{N} a_k e^{-j\omega k}}.
\]

Taking the Fourier transform of both sides of the second difference equation and simplifying, we obtain the frequency response \( H_1(e^{j\omega}) \) of the second filter.

\[
H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 - \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}}.
\]

216
6.13. Using an approach similar to the one used in the previous problem, we obtain

\[ H(j\omega) = \frac{320}{(j\omega + 2)(j\omega + 80)}. \]

(a) Let us assume that we desire to construct this system by cascading two systems with frequency responses \( H_1(j\omega) \) and \( H_2(j\omega) \), respectively. We require that

\[ H(j\omega) = H_1(j\omega)H_2(j\omega). \]

We see that \( H_1(j\omega) \) and \( H_2(j\omega) \) may be defined in different ways to obtain \( H(j\omega) \). For instance

\[ H_1(j\omega) = \frac{40}{(j\omega + 2)} \quad \text{and} \quad H_2(j\omega) = \frac{8}{(j\omega + 80)} \]

and

\[ H_1(j\omega) = \frac{32}{(j\omega + 2)} \quad \text{and} \quad H_2(j\omega) = \frac{10}{(j\omega + 80)} \]

are both valid combinations.

(b) Let us assume that we desire to construct this system by connecting two systems with frequency responses \( H_1(j\omega) \) and \( H_2(j\omega) \) in parallel. We require that

\[ H(j\omega) = H_1(j\omega) + H_2(j\omega). \]

Using partial fraction expansion on \( H(j\omega) \), we obtain

\[ H(j\omega) = \frac{160/39}{(j\omega + 2)} - \frac{160/39}{(j\omega + 80)} \]

From the above expression it is clear that we can define \( H_1(j\omega) \) and \( H_2(j\omega) \) in only one way.

6.14. Using an approach similar to the one used in Problem 6.12, we have

\[ H(j\omega) = \frac{50000(j\omega + 0.2)^2}{(j\omega + 50)(j\omega + 10)}. \]

The inverse to this system has a frequency response

\[ H_1(j\omega) = \frac{1}{H(j\omega)} = \frac{0.2 \times 10^{-4}(j\omega + 50)(j\omega + 10)}{(j\omega + 0.2)^2}. \]

6.15. We will use the results from Section 6.5 in this problem.

(a) We may write the frequency response of the system described by the given differential equation as

\[ H_1(j\omega) = \frac{1}{(j\omega)^2 + 4j\omega + 4}. \]
6.19. Let us first find the differential equation governing the input and output of this circuit. Current through resistor and inductor = Current through capacitor = \( C \frac{dx(t)}{dt} \).
Voltage across resistor = \( RC \frac{dy(t)}{dt} \).
Voltage across inductor = \( LC \frac{d^2y(t)}{dt^2} \).
Total input voltage = Voltage across inductor + Voltage across resistor + Voltage across capacitor.
Therefore,
\[
x(t) = LC \frac{d^2y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t).
\]
The frequency response of this circuit is therefore
\[
H(j\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + 1}.
\]
We may rewrite this to be
\[
H(j\omega) = \frac{1}{\left(\frac{j\omega}{\sqrt{1/\sqrt{LC}}}\right)^2 + 2(R/2)\sqrt{C/L} \frac{j\omega}{\sqrt{1/\sqrt{LC}}} + 1}.
\]
Therefore, the damping constant \( \zeta = (R/2)\sqrt{C/L} \). In order for the step response to have no oscillations, we must have \( \zeta \geq 1 \). Therefore, we require
\[
R \geq 2\sqrt{\frac{L}{C}}.
\]

6.20. Let us call the given impulse response \( h[n] \). It is easily observed that the signal \( h_1[n] = h[n + 2] \) is real and even. Therefore, (using properties of the Fourier transform) we know that the Fourier transform \( H_1(e^{j\omega}) \) of \( h_1[n] \) is real and even. Therefore \( H_1(e^{j\omega}) \) has zero phase. We also know that the Fourier transform \( H(e^{j\omega}) = H_1(e^{j\omega})e^{-2j\omega} \). Since \( H_1(e^{j\omega}) \) is zero phase, we have
\[
\angle H(e^{j\omega}) = -2\omega.
\]
Therefore, the group delay is
\[
\tau(\omega) = \frac{d}{d\omega} \angle H(e^{j\omega}) = 2.
\]

6.21. Note that in all parts of this problem \( Y(j\omega) = H(j\omega)X(j\omega) = -2j\omega X(j\omega) \). Therefore,
\[
y(t) = -2dx(t)/dt.
\]
(a) Here, \( x(t) = e^{jt} \). Therefore, \( y(t) = -2dx(t)/dt = -2je^{jt} \). This part could also have been solved by noting that complex exponentials are Eigen functions of LTI systems. Then, when \( x(t) = e^{jt}, y(t) = y(t) = H(jt)e^{jt} = -2je^{jt} \).
(b) Here, \( x(t) = \sin(\omega_0t)u(t) \). Then, \( dx(t)/dt = \omega_0 \cos(\omega_0t)u(t) + \sin(\omega_0t)\delta(t) = \omega_0 \cos(\omega_0t)u(t) \).
Therefore, \( y(t) = -2dx(t)/dt = -2\omega_0 \cos(\omega_0t)u(t) \).
(c) Here, \( Y(j\omega) = X(j\omega)H(j\omega) = -2/(6 + j\omega) \). Taking the inverse Fourier transform we obtain \( y(t) = -2e^{-6t}u(t) \).
(d) Here, \( X(j\omega) = 1/(2 + j\omega) \). From this we obtain \( x(t) = e^{-2t}u(t) \). Therefore, \( y(t) = -2dx(t)/dt = 4e^{-2t}u(t) - 2\delta(t) \).

Note that
\[
H(j\omega) = \begin{cases} \frac{j\omega}{2\pi}, & -3\pi \leq \omega \leq 3\pi \\ 0, & \text{otherwise} \end{cases}
\]

(a) Since \( x(t) = \cos(2\pi t + \theta) \), \( X(j\omega) = e^{j\theta}\pi \delta(\omega - 2\pi) + e^{-j\theta}\pi \delta(\omega + 2\pi) \). This is zero outside the region \(-3\pi < \omega < 3\pi\). Thus, \( Y(j\omega) = H(j\omega)X(j\omega) = (j\omega/3\pi)X(j\omega) \). This implies that \( y(t) = (1/3\pi)dx(t)/dt = (2/3)\sin(2\pi t + \theta) \).

(b) Since \( x(t) = \cos(4\pi t + \theta) \), \( X(j\omega) = e^{j\theta}\pi \delta(\omega - 4\pi) + e^{-j\theta}\pi \delta(\omega + 4\pi) \). Therefore, the nonzero portions of \( X(j\omega) \) lie outside the range \(-3\pi < \omega < 3\pi\). This implies that \( Y(j\omega) = X(j\omega)H(j\omega) = 0 \). Therefore, \( y(t) = 0 \).

(c) The Fourier series coefficients of the signal \( x(t) \) are given by
\[
a_k = \frac{1}{T_0} \int_{<T_0>} x(t)e^{-jkw_0 t},
\]
where \( T_0 = 1 \) and \( w_0 = 2\pi/T_0 = 2\pi \). Also,

\[
X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - kw_0).
\]

The only impulses of \( X(j\omega) \) which lie in the region \(-3\pi < \omega < 3\pi\) are at \( \omega = 0, 2\pi, \) and \( 2\pi \). Defining the signal \( x_{ip}(t) = a_0 + a_1e^{j2\pi t} + a_{-1}e^{-j2\pi t} \), we note that \( y(t) = (1/3\pi)dx_{ip}(t)/dt \). We can also easily show that \( a_0 = 1/\pi, \ a_1 = a_{-1} = -1/(4j) \). Putting these into the expression for \( x_{ip}(t) \) we obtain \( x_{ip}(t) = (1/\pi) + (1/2)\sin(2\pi t) \). Finally, \( y(t) = (1/3\pi)dx_{ip}(t)/dt = (1/3)\cos(2\pi t) \).

6.23. (a) From the given information, we have
\[
H_a(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}
\]

Using Table 4.2, we get
\[
h_a(t) = \frac{\sin(\omega_c t)}{\pi t}.
\]

(b) Here,
\[
H_b(j\omega) = H_a(j\omega)e^{j\omega T}.
\]

Using Table 4.1, we get
\[
h_b(t) = h_a(t + T).
\]

Therefore,
\[
h_b(t) = \frac{\sin[\omega_c (t + T)]}{\pi (t + T)}.
\]
(c) Let us consider a frequency response \( H_0(j\omega) \) given by

\[
H_0(j\omega) = \begin{cases} 
1, & |\omega| \leq \omega_c/2 \\
0, & \text{otherwise}
\end{cases}
\]

Clearly,

\[
H_c(j\omega) = \frac{1}{2\pi}[H_0(j\omega) \ast W(j\omega)],
\]

where

\[
W(j\omega) = j2\pi\delta(\omega - \omega_c/2) - j2\pi\delta(\omega + \omega_c/2).
\]

Therefore, from Table 4.1

\[
h_c(t) = h_0(t)w(t) = \left[ \frac{\sin(\omega_c t/2)}{\pi t} \right] [-2\sin(\omega_c t/2)].
\]

6.24 If \( \tau(\omega) = k_1 \), where \( k_1 \) is a constant, then

\[
\angle H(j\omega) = -k_1\omega + k_2 \tag{S6.24-1}
\]

where \( k_2 \) is another constant.

(a) Note that if \( h(t) \) is real, the phase of the Fourier transform \( \angle H(j\omega) \) has to be an odd function. Therefore, the value of \( k_2 \) in eq. (S6.24-1) will be zero.

Also, let us define \( H_0(j\omega) = |H(j\omega)| \). Then,

\[
h_0(t) = \frac{\sin(200\pi t)}{\pi t}.
\]

(i) Here \( k_1 = 5 \). Hence, \( \angle H(j\omega) = -5\omega \). Then,

\[
H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} = H_0(j\omega)e^{-j5\omega}.
\]

Therefore,

\[
h(t) = h_0(t - 5) = \frac{\sin[200\pi(t - 5)]}{\pi(t - 5)}.
\]

(ii) Here \( k_1 = 5/2 \). Hence, \( \angle H(j\omega) = -(5/2)\omega \). Then,

\[
H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} = H_0(j\omega)e^{-j(5/2)\omega}.
\]

Therefore,

\[
h(t) = h_0(t - 5/2) = \frac{\sin[200\pi(t - 5/2)]}{\pi(t - 5/2)}.
\]

(iii) Here \( k_1 = -5/2 \). Hence, \( \angle H(j\omega) = (5/2)\omega \). Then,

\[
H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} = H_0(j\omega)e^{j(5/2)\omega}.
\]

Therefore,

\[
h(t) = h_0(t + 5/2) = \frac{\sin[200\pi(t + 5/2)]}{\pi(t + 5/2)}.
\]
(b) If \( h(t) \) is not specified to be real, then \( \angle H(j\omega) \) does not have to be an odd function. Therefore, the value of \( k_2 \) in eq. (S6.24-1) does not have to be zero. Given only \(|H(j\omega)|\) and \( \tau(\omega) \), \( k_2 \) cannot be determined uniquely. Therefore, \( h(t) \) cannot be determined uniquely.

6.25. (a) We may write \( H_a(j\omega) \) as

\[
H_a(j\omega) = \frac{(1 - j\omega)}{(1 + j\omega)(1 - j\omega)} = \frac{1 - j\omega}{2}.
\]

Therefore,

\[
\angle H_a(j\omega) = \tan^{-1}[-\omega].
\]

and

\[
\tau_a(\omega) = -\frac{d\angle H_a(j\omega)}{d\omega} = \frac{1}{1 + \omega^2}.
\]

Since \( \tau_a(0) = 1 \neq 2 = \tau_a(1) \), \( \tau_a(\omega) \) is not a constant for all \( \omega \). Therefore, the frequency response has nonlinear phase.

(b) In this case, \( H_b(j\omega) \) is the frequency response of a system which is a cascade combination of two systems, each of which has a frequency response \( H_a(j\omega) \). Therefore,

\[
\angle H_b(j\omega) = \angle H_a(j\omega) + \angle H_a(j\omega)
\]

and

\[
\tau_b(\omega) = -2\frac{d\angle H_a(j\omega)}{d\omega} = \frac{2}{1 + \omega^2}.
\]

Since \( \tau_b(0) = 2 \neq 4 = \tau_b(1) \), \( \tau_b(\omega) \) is not a constant for all \( \omega \). Therefore, the frequency response has nonlinear phase.

(c) In this case, \( H_c(j\omega) \) is again the frequency response of a system which is a cascade combination of two systems. The first system has a frequency response \( H_a(j\omega) \), while the second system has a frequency response \( H_0(j\omega) = 1/(2 + j\omega) \). Therefore,

\[
\angle H_b(j\omega) = \angle H_a(j\omega) + \angle H_0(j\omega)
\]

and

\[
\tau_c(\omega) = -\frac{d\angle H_a(j\omega)}{d\omega} - \frac{d\angle H_0(j\omega)}{d\omega} = \frac{1}{1 + \omega^2} + \frac{2}{4 + \omega^2}.
\]

Since \( \tau_c(0) = (3/2) \neq (3/5) = \tau_c(1) \), \( \tau_b(\omega) \) is not a constant for all \( \omega \). Therefore, the frequency response has nonlinear phase.

6.26. (a) Note that \( H(j\omega) = 1 - H_0(j\omega) \), where \( H_0(j\omega) \) is

\[
H_0(j\omega) = \begin{cases} 
1, & 0 \leq |\omega| \leq \omega_c \\
0, & \text{otherwise}
\end{cases}
\]

Therefore,

\[
h(t) = \delta(t) - h_0(t).
\]
(d) From the tolerances derived in the previous part, it is clear that $H_{oo}(j\omega)$ is not necessarily highpass.

6.35. Since $x[n] = \cos(\omega_0 n + \theta)$, we have

$$X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} [e^{j\theta}(\omega - \omega_0 - 2\pi l) + e^{-j\theta}(\omega + \omega_0 - 2\pi l)].$$

Let $\omega_0'$ be the principal value of $\omega_0$ in $[-\pi, \pi]$. Then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} [e^{j\theta} j\omega_0' \delta(\omega - \omega_0 - 2\pi l) - e^{-j\theta} j\omega_0' \delta(\omega + \omega_0 - 2\pi l)].$$

It follows that

$$y[n] = -\omega_0' \sin(\omega_0 n + \theta).$$

If $-\pi \leq \omega_0 \leq \pi$, then

$$y[n] = -\omega_0 \sin(\omega_0 n + \theta).$$

6.36. Let $H_1(e^{j\omega}) = |H(e^{j\omega})|$. Then from Table 5.2 we know that

$$h_1[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

If $\tau(\omega) = -\frac{d}{d\omega} \angle H(e^{j\omega}) = k$ (where $k$ is a constant), then $\angle H(e^{j\omega}) = -k\omega + k_1$, where $k_1$ is a constant. If $h[n]$ is real, then $\angle H(e^{j\omega})$ is an odd function, and therefore we may conclude that $k_1 = 0$. Therefore,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} = H_1(e^{j\omega})e^{-j\omega k}.$$ 

Taking the inverse Fourier transform we obtain

$$h[n] = h_1[n - k] = \frac{\sin[\pi(n - k)/2]}{\pi(n - k)}.$$
(a) If \( \tau(\omega) = 5 \), then from the above result,
\[
h[n] = \frac{-\sin[\pi(n - 5)/2]}{\pi(n - 5)}.
\]

(b) If \( \tau(\omega) = 5/2 \), then from the result derived at the beginning of this problem
\[
h[n] = \frac{-\sin[\pi(n - 5/2)/2]}{\pi(n - 5/2)}.
\]

(c) If \( \tau(\omega) = -5/2 \), then from the result derived at the beginning of this problem
\[
h[n] = \frac{-\sin[\pi(n + 5/2)/2]}{\pi(n + 5/2)}.
\]

The results of all the parts of this problem are sketched in Figure S6.36.

![Figure S6.36](image)

6.37. (a) We have
\[
|H(e^{j\omega})| = \left|\frac{1 - \frac{1}{2}e^{j\omega}}{1 - \frac{1}{2}e^{-j\omega}}\right| = 1.
\]

(b) We have
\[
\angle H(e^{j\omega}) = \angle [e^{-j\omega}] + \angle \left[1 - \frac{1}{2}e^{j\omega}\right] - \angle \left[1 - \frac{1}{2}e^{-j\omega}\right]
\]
\[= \angle [e^{-j\omega}] + \angle \left[1 - \frac{1}{2}\cos(\omega) - \frac{j}{2}\sin(\omega)\right] - \angle \left[1 - \frac{1}{2}\cos(\omega) + \frac{j}{2}\sin(\omega)\right]
\]
\[= \angle [e^{-j\omega}] - \frac{1}{2}\tan^{-1}\left(\frac{\sin(\omega)}{1 - \frac{1}{2}\cos(\omega)}\right)
\]
(c) Using the result of the previous part, we can show with some algebraic manipulation that

\[ \tau(\omega) = -\frac{d}{d\omega}H(e^{j\omega}) = \frac{3}{4} \left( \frac{3}{4} - \cos \omega \right). \]

This is as sketched below.

(d) Let \( x[n] = \cos(\pi n/3) \). We may write this as \( x[n] = e^{j\pi n/3}/2 + e^{-j\pi n/3}/2 \). From the result of part (c), we know that the delay suffered by a complex exponential of frequency \( \pi/3 \) is

\[ \frac{3}{4} \left( \frac{3}{4} - \cos(\pi/3) \right) = 1. \]

Similarly, we know that the delay suffered by a complex exponential of frequency \(-\pi/3\) is also 1. Therefore, the output of the system is \( y[n] = e^{j\pi(n-1)/3}/2 + e^{-j\pi(n-1)/3}/2 = \cos(\pi(n-1)/3) \).

6.38. We may express \( H(e^{j\omega}) \) as

\[ H(e^{j\omega}) = \frac{1}{2\pi} \left[ H_1(e^{j\omega}) \ast \{2\pi \delta(\omega - \pi/2) + 2\pi \delta(\omega + \pi/2)\} \right], \]

and

\[ H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases} \]

Using the properties of the Fourier transform, we obtain

\[ h[n] = h_1[n] \left[ 2 \cos(\pi n/2) \right], \]

where

\[ h_1[n] = \frac{\sin(\omega_c n)}{\pi n}. \]

(a) When \( \omega_c = \pi/5 \), \( h[n] = 2^{\sin(\pi n/5)} \cos(\pi n/2) \). This is as shown in Figure S6.38.

(b) When \( \omega_c = \pi/4 \), \( h[n] = 2^{\sin(\pi n/4)} \cos(\pi n/2) \). This is as shown in Figure S6.38.

(c) When \( \omega_c = \pi/3 \), \( h[n] = 2^{\sin(\pi n/3)} \cos(\pi n/2) \). This is as shown in Figure S6.38.

As \( \omega_c \) increases, \( h[n] \) becomes more concentrated about the origin.

6.39. The plots are as shown in Figure S6.39.
6.40. We may write $h_1[n]$ as

$$H_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_1[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h_1[2n]e^{-j2\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h[n]e^{-j2\omega n}$$

$$= H(e^{j2\omega})$$

Therefore, $H(e^{j\omega})$ is $H(e^{j\omega})$ compressed by a factor of two. This is as shown in Figure S6.40.

![Figure S6.40](image)

Therefore, $H_1(e^{j\omega})$ corresponds to a band-stop filter.

6.41. (a) Taking the Fourier transform of both sides of the given difference equation, we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - e^{-j\omega}}{1 - \frac{1}{\sqrt{2}}e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}.$$  

Taking the inverse Fourier transform of $H(e^{j\omega})$ we obtain

$$h[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/4)u[n] - (2\sqrt{2} - 1) \left(\frac{1}{2}\right)^n \sin(\pi n/4)u[n].$$

(b) The log-magnitude and phase of the frequency response are as shown in Figure S6.41.

6.42. (a) We get

$$|H_1(e^{j\omega})| = |H_2(e^{j\omega})| = \frac{5/4 + \cos \omega}{17/6 + (1/2) \cos \omega}$$
and

\[ \angle H_1(e^{j\omega}) = \tan^{-1}\left(\frac{(1/2) \sin \omega}{1 + (1/2) \cos(\omega)}\right) \quad \text{and} \quad \angle H_2(e^{j\omega}) = \tan^{-1}\left(\frac{\sin \omega}{1 + (1/2) \cos(\omega)}\right). \]

Comparing tangents of these angle in the range \(0 \leq \omega \leq \pi\), we get

\[ \angle H_2(e^{j\omega}) > \angle H_1(e^{j\omega}). \]

(b) We get

\[ h_1[n] = \left(\frac{-1}{4}\right)^n u[n] + \frac{1}{2} \left(\frac{-1}{4}\right)^{n-1} u[n - 1] \]

and

\[ h_2[n] = \frac{1}{2} \left(\frac{-1}{4}\right)^n u[n] + \left(\frac{-1}{4}\right)^{n-1} u[n - 1]. \]

This is as sketched in Figure S6.42.

(c) We get

\[ H_2(e^{j\omega}) = \frac{1/2 + e^{-j\omega}}{1 + (1/2)e^{-j\omega}} H_1(e^{j\omega}). \]
Therefore,

\[ G(e^{j\omega}) = \left( \frac{1/2 + e^{-j\omega}}{1 + (1/2)e^{-j\omega}} \right) \]

and

\[ |G(e^{j\omega})| = \frac{(5/4) + \cos \omega}{(5/4) + \cos \omega} = 1. \]

6.43. (a) If \( h_{hp}[n] = (-1)^{n} h_{ip}[n] = e^{j\pi n} h_{ip}[n] \), then

\[ H_{hp}(e^{j\omega}) = H_{ip}(e^{j(\omega - \pi)}). \]

Therefore, \( H_{hp}(e^{j\omega}) \) is as shown in Figure 6.43. Clearly, it corresponds to a highpass filter.

![Figure S6.43](image)

(b) Now let us define \( h[n] = (-1)^{n} h_{hp}[n] \), where \( h_{hp}[n] \) is the impulse response of a highpass filter. Then

\[ H(e^{j\omega}) = H_{hp}(e^{j(\omega - \pi)}). \]

Therefore, if \( H_{hp}(e^{j\omega}) \) is as shown in Figure 6.43, then \( H(e^{j\omega}) \) is lowpass.

6.44. (a) Note that \((-1)^{n} = e^{j\pi n}\). From the figure we have

\[ y[n] = (x[n]e^{j\pi n} \ast h_{hp}[n])e^{j\pi n}. \]

We may write this as

\[ y[n] = a[n]e^{j\pi n}, \]

where \( a[n] = (x[n]e^{j\pi n} \ast h_{hp}[n]) \). Taking the Fourier transform of \( a[n] \), we obtain

\[ A(e^{j\omega}) = X(e^{j(\omega - \pi)})H_{hp}(e^{j\omega}). \]

Suppose that the input to the system is now \( x[n - n_0] \). Let the corresponding output be \( y_1[n] \). Then we may write

\[ y_1[n] = b[n]e^{j\pi n}, \]

where \( b[n] = (x[n - n_0]e^{j\pi n} \ast h_{hp}[n]) \). Taking the Fourier transform of \( b[n] \), we obtain

\[ B(e^{j\omega}) = X(e^{j(\omega - \pi)})H_{hp}(e^{j\omega})e^{-j\omega n_0} = A(e^{j\omega})e^{-j\omega n_0}. \]

Therefore,

\[ b[n] = a[n - n_0]. \]

Consequently, \( y_1[n] = y[n - n_0] \). Therefore, the system is time invariant.