Implication and quantifiers

Lecture 2
ICOM 4075
Implications

An **implication** is a logical sentence of the form of “If **A** then **B**” where **A** and **B** are Boolean variables representing logical statements.

In an implication **A** is called **hypothesis** and **B**, **thesis**.

**Examples:**

- If **Earth** is a planet then **the Moon** is a satellite.
- If **water** is a chemical element then **water** is a combination of elements in the periodic table.
- If **it rains** and I am not covered then I get wet.
Implication’s variants

In natural speech, implication may appear in many alternative forms. Among them are:

– A is a sufficient condition for B
– B is a necessary condition for A
– B is true under the assumption A
– B holds provided that A is true
– Etc...

Each of them is equivalent to “if A then B”
Conditional branching

Conditional branching are computer program’s segments of the form of:

If ( A = 0)
   Then (Print “Hello World”)

Here

“A = 0” is the hypothesis

But

“Print “Hello World”” is the thesis in the sense that the action is executed
More examples

- An integer is even if it is divisible by two

  **Usual form:**
  
  If an integer is divisible by two then it is even

- Every nonnegative real number has a square root

  **Usual form:**
  
  If a real number is nonnegative then it has a square root
What does *implication* models?

An implication **models a process** for deriving the truth value of the thesis given the truth value of the hypothesis.

In this regard, an implication is true if the process for deriving the truth value of the thesis is correct according to some set of accepted rules.

Thus, the implication is often visualized with an arrow. This is, “if A then B” is often expressed as

\[ A \Rightarrow B \]

The implication is true if the process is correct.
Illustration

A : “Earth is a planet”
B : “Moon is a satellite”

Implication: If A then B

Is this implication correct? Let’s propose a process that may be used to conclude B from A:

A process:
1. A satellite is a celestial body that orbit around a planet
2. The moon orbits around the Earth
3. Since by hypothesis, Earth is a planet, Moon is a satellite
Evaluating the truth of an implication

The truth table for the implication is:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A implies B</th>
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</thead>
<tbody>
<tr>
<td>True</td>
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</tbody>
</table>
Some further analysis

The first line asserts that “If A true hypothesis leads to a true thesis, the process is correct”. The second line asserts “If a true hypothesis leads to a false thesis the process is wrong”. Both assertions agree with our intuition

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A implies B</th>
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</thead>
<tbody>
<tr>
<td>True</td>
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<td>False</td>
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</tbody>
</table>
Some further analysis

The last two lines assert that: “If a false hypothesis leads to a true thesis” and “if a false hypothesis leads to a false thesis” the process is, in both cases, correct. These assertions are far less intuitive.

<table>
<thead>
<tr>
<th></th>
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<th>A implies B</th>
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</thead>
<tbody>
<tr>
<td>True</td>
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</tbody>
</table>
Some further analysis

In these cases, the implication is said to be **vacuously true** meaning that a false hypothesis can lead to a true or to a false statement, **independent of the process**. Thus, **we can always select or assume a correct process**.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A implies B</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
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</tbody>
</table>
Illustration of a true implication with a false hypothesis

Let’s illustrate first the case

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<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A implies B</td>
</tr>
<tr>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
</tbody>
</table>

with the IMPLICATION: If $3 < 2$ then $4 < 3$

Derivation Process:
1. By hypothesis $3 < 2$
2. Inequalities are preserved by adding a constant to both sides of the inequality
3. By adding 1 to both sides of the hypothesis: $3 + 1 < 2 + 1$
4. Thus, by adding, $4 < 3$.

The process is correct since it conforms with the rules of algebra
True implication with a false hypothesis

The case:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A implies B</th>
</tr>
</thead>
<tbody>
<tr>
<td>False</td>
<td>True</td>
<td>True</td>
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</tbody>
</table>

**IMPLICATION:** If $0 = 2$ then $0 = 0$

**Derivation process:**

1. By hypothesis $0 = 2$
2. Equalities are preserved by multiplication by a constant
3. By multiplying both sides by 0, we get $0 \times 0 = 2 \times 0$
4. Since multiplication by 0 gives 0, $2 \times 0 = 0$ and $0 \times 0 = 0$,
5. We conclude that $0 = 0$.

The process is correct since it conforms with all the rules of algebra
Contrapositive form of the implication

Statement:

“$A$ implies $B$” is equivalent to

“Not $B$ implies Not $A$”

Verification:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>Not $B$</th>
<th>Not $A$</th>
<th>Not $B$ implies Not $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>False</td>
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</tbody>
</table>
Examples

Original form:

If Earth is a planet then the Moon is a satellite

If water is a chemical element then water is a combination of elements in the periodic table

If it rains and I am not covered then I get wet

Equivalent form:

If the Moon is not a satellite then Earth is not a planet

If water is not a combination of elements in the periodic table then water is not a chemical element

If I do not get wet then it does not rain or I am covered
Disjunctive form of implication

Statement:

“A implies B” is equivalent to “Not A or B”

Verification:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Not A</th>
<th>Not A or B</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>False</td>
<td><strong>True</strong></td>
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<td><strong>True</strong></td>
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</table>

Same values as in the last column of “A implies B”
<table>
<thead>
<tr>
<th><strong>Examples</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Original form:</strong></td>
</tr>
<tr>
<td>If Earth is a planet <strong>then</strong> Moon is a satellite</td>
</tr>
<tr>
<td>If water is a chemical element <strong>then</strong> water is a combination of elements in the periodic table</td>
</tr>
<tr>
<td>If it rains <strong>and</strong> I am not covered <strong>then</strong> I get wet</td>
</tr>
<tr>
<td><strong>Equivalent form:</strong></td>
</tr>
<tr>
<td>Earth <strong>is not</strong> a planet <strong>or</strong> the Moon is a satellite</td>
</tr>
<tr>
<td>Water <strong>is not</strong> a chemical elements <strong>or</strong> water is a combination of elements in the periodic table</td>
</tr>
<tr>
<td>It <strong>does not</strong> rain or I am covered, <strong>or</strong> I get wet</td>
</tr>
</tbody>
</table>
If and only if

The compounded statement:

\[(\text{If } A \text{ then } B) \text{ and (if } B \text{ then } A)\]

is said to be an **if and only if** statement, and denoted

“\(A \text{ iff } B\)” or \(A \iff B\)

Its truth table is easily derived from the tables of implication and conjunction:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(A \text{ implies } B)</th>
<th>(B \text{ implies } A)</th>
<th>(A \text{ iff } B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
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</table>
If and only if

It is worth pointing out that, according to the iff table:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A iff B</th>
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<tbody>
<tr>
<td>True</td>
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*A iff B is true if A and B have the same truth value*

*A iff B is true if A and B are equivalent*

*and*

*A iff B is false if A and B are not equivalent*
Domain of a variable

**Definition:** The domain of a variable is the collection of all values that the variable may take.

**Examples:**
- If \( t \) is time, then the domain of \( t \) is the collection of all nonnegative real numbers, since there is no such a thing as negative time.
- If \( A \) is a Boolean variable, its values are either True or False. So, its domain is \( \{ \text{True}, \text{False} \} \). Often, this domain is represented as \( \{1, 0\} \).
Quantifiers

Whenever a logical statement involves variables, a quantifier is used to indicate for how many values of the variable the statement is made.

There are two main quantifiers:

- The **universal quantifier**: For all
- The **existential quantifier**: There is (at least one)
Standard notation

The standard notation for a sentence \( A \) about a variable \( x \) is \( A(x) \). The standard requires also that all quantifiers be declared at the beginning of the sentence. This is, either

– (For all \( x \)) \( A(x) \) means that \( A(x) \) is about all possible values of the variable \( x \)

– (There is \( x \)) \( A(x) \) means that \( A(x) \) is about at least one value of \( x \)
Illustration

Consider the logical sentence: “The students that registered in ICOM 4075 in the second semester of 08/09 are all enrolled in the baccalaureate in Computer Engineering”

Analysis:

– **Variable**: students
– **Domain**: sections of ICOM 4075 in the 2\textsuperscript{nd} semester of 08/09
– **Quantifier**: universal (the sentence holds for all students in the domain)
Rewriting in standard form

The previous sentence is standardized as follows:

Let

- \( s \) : variable denoting the students,
- \( I \) : variable denoting the sections of ICOM 4075 in the 2\(^{nd}\) semester of 08/09, and
- \( C \) : variable denoting the baccalaureate in Computer Engineering

The logical sentence is:

\[ A(s) : \text{“If } s \text{ in } I, \text{ then } s \text{ in } C” \]

and the standard expression with quantifiers is:

\[(\text{For all } s) (\text{If } s \text{ in } I \text{ then } s \text{ in } C) \text{ or simply} \]

\[(\text{For all } s) A(s) \]
Negation in the presence of one quantifier

The negation of **For all** is **There is** in the following sense:

\[
\neg ((\forall s) \ A(s))
\]

is equivalent to

\[
(\exists s) \ \neg A(s)
\]

And vice versa, the negation of **There is**, is **For all**:

\[
\neg ((\exists s) \ A(s))
\]

is equivalent to

\[
(\forall s) \ \neg A(s)
\]
Illustration

The negation of the sentence in our example is:

“The students that registered in ICOM 4075 in the second semester of 08/09 are not all enrolled in the baccalaureate in Computer Engineering”

This means that: “There is (at least) one student registered in ICOM 4075 in the second semester of 08/08 that is not enrolled in the baccalaureate in Computer Engineering”
Multiple quantifiers

Sentences may have more than one quantifier. For example:

“Between any pair of different even numbers there always is an odd number”

Analysis: $P(a, b)$ : “a and b is a pair of different even numbers” $O(c, a, b)$ : “c is an odd number between a and b”

Standard sentence:

(For all $a, b$) $P(a, b)$ (There is $c$) $O(c, a, b)$
Negation of sentences with multiple quantifiers

The general negation rule is as follows:

– Change all universal quantifiers by existential quantifiers and vice versa
– Negate only the last sentence in the statement
Negation of sentences with multiple quantifiers

The general negation rule is as follows:

– Change all universal quantifiers by existential quantifiers and vice versa
– Negate only the last sentence in the statement

In the previous example:

\[
\text{Not } ((\text{For all } a, b) \ P(a, b) \ (\text{There is } c) \ O(c, a, b))
\]

is equivalent to

\[
(\text{There is } a, b) \ P(a, b) \ (\text{For all } c) \ \text{Not} \ O(c, a, b)
\]
Summary

• Notions of implication, hypothesis and thesis
• Implication as a model of a derivation process
• Truth table of the implication
• Vacuously true implications
• Alternative forms of an implication
• Quantifiers and their standard notation
• Negation of sentences with one quantifier
• Negation of sentences with multiple quantifiers
Exercises

1. In each of the following statements:
   (a) identify hypothesis and thesis
   (b) write the statement as an implication of the form “If A then B”,
   (c) rewrite them under the form “If not B then not A”, and
   (d) rewrite them under the form “Not A or B”:
      – “The sun shines, today will be a warm day”
      – “Whenever a request occur the process acknowledges or stalls”
      – “If you study the lecture notes and do the exercises, you’ll be in good shape for the exam”
      – “Cancer cannot be cured unless its cause is determined and a powerful drug is developed”
Exercises

2. Evaluate the following statements identifying all the values of the hypothesis under which the thesis is true:
   - “(A and B) implies (A or B)”
   - “(A or B) does not imply (A and B)”
   - “If (A implies B) then (B implies C)”
   - “(A or B or C) implies (not A and B or C)”
Exercises

3. Write in standard form the following sentences with quantifiers
   a) Professional basketball players are form in university teams with very few exceptions
   b) Non–leap years have 365 days
   c) Every USA citizen has a unique Social Security Number
   d) There are two senators for representing all the citizen of each state of USA

4. Write the negation of each of the sentences in 3 both, in standard and non–standard form