Basic propositional and predicate logic

The fundamentals of deduction

Logic and its application

- Logic is the study of the **patterns of deduction**
- Logic plays two main roles in computation:
  - **Modeling** **:** **logical sentences** are the building blocks of computation models
  - **Analyzing**: **logic proofs** are fundamental for reasoning about, and inferring properties of computational systems' models

Basic elements in logic

- **Declarative sentences**

  also called **Propositions**

  Is any statement which can be **unambiguously** determined to be either **True** or **False**

- **Logical connectives**

  Connect logical statements. There are four logical connectives

NAME	Semantic	Symbol
Negation	**NO**	¬
Conjunction	**AND**	∧
Disjunction	**OR**	∨
Implication	**IF – THEN**	⇒

Atomic and compound sentences

- An **atomic** sentence is a proposition that does not contain any logical connectives
  - Example: A:“3 > 4”
  - This is clearly a **false** atomic sentence
- A **compound** sentence is a proposition composed by two or more sentences linked with logical connectives
  - Example: A:“$3 > 4 \Rightarrow 2 > 3$”
  - This is a compound proposition since it **has an implication**, but
  - **Is this proposition true or false?**

Proof system

- The previous question cannot be properly answer without a well-established **Proof System**
- A **proof system** is a set of rules and operations for determining the truth value of a compound sentence out of the truth value of its atomic constituents
- Examples of proof systems are:
  - **Boolean algebra** (to be reviewed soon)
  - **Natural deduction** (modus ponens, modus tollens, etc. **Not in the scope of this course**)

Variables

- In science, **variables** are **place holders of values**. As such, variables have:
  - **Meaning** – what the **symbol** represents
  - **Value** – any value in the **domain** of the variable
- Example: Let's represent the physical concept of **time** with a variable. Let
  - ***t*** be the **symbol** denoting the time
  - The natural **domain** of ***t*** is the set of all nonnegative real numbers.

  We write:
  - ***t*** : time when **defining** the **meaning of** symbol ***t***
  - ***t*** = 1.25 seconds when **evaluating** (*i.e.* choosing **a value in the domain of**) ***t***

Boolean Variables

- A **Boolean variable** is any variable whose domain is the set {0, 1} (*i.e.* can take only the values 0 or 1)
- Propositions are normally represented by Boolean variables. The values that such variable take are conventionally interpreted as
  - **0** if the proposition is **false** and
  - **1** if the proposition is **true**
- Example of a Boolean variable
  - A : "**3 is less than 3.2**" (meaning of symbol A)
  - A = **1** (value of A)

Propositional vs. predicate logic

- **Propositional logic** deals solely with propositions and logical connectives
  - Example: A: “3 < 4”
  - Clearly A=1
- **Predicate logic** adds **predicates** and **quantifiers**
- A **predicate** is a **logical statement that depends on one or more variables** (not necessarily Boolean variables)
  - Example: **B(x, y)** : “**x is less than y**”
    The definition and value of B(x, y) depend on x and y
    - If x = 3 and y = 4, then B(x, y) = B(3, 4) = 1
    - If x = 4 and y = 3, then B(x, y) = B(4, 3) = 0
    - If x = + and y = / , the B(x, y) is (in principle) undefined

Quantifiers

Quantifiers are used to **declare ranges** of a predicate's variables in which the predicate assumes **either 0 or 1, but no both.**

A predicate with no quantifiers is said to be a **free predicate**

There are two **quantifiers**

NAME	SEMANTIC	SYMBOL
Universal	FOR ALL	∀
Existential	THERE EXISTS	∃

Illustration: use of quantifiers

Consider the **free predicate**

$$A(x,y):"x<y \Rightarrow x<y^2"$$

where *x* and *y* are real numbers.

Some values for this predicate are:

$$A(.5,.9)=1$$

$$A(.5,.6)=0$$

$$A(-.6,-.5)=1$$

$$A(.9,.93)=0$$

$$A(.9,.96)=1$$

By imposing quantifiers we may produce the following **true** predicates:

$$B(x,y):"(\forall x)(x \geq 1)A(x,y)"$$

$$C(x,y):"(\forall x,y)(0 \leq x \wedge y \leq \sqrt[2]{x} < 1)\neg A(x,y)"$$

$$D(x,y)^{(}:"(\forall x)(x<0)A(x,y)"$$

$$E(x,y):"(\forall x,y)(0 \leq x \wedge \sqrt[2]{x} < y < 1)A(x,y)"$$

Remark: A predicate that is **"always true or always false"** can be treated as a **proposition** that is true or false, respectively. This is, it can be evaluated and analyzed with the tools of **propositional logic**

Propositional logic

- A logical theory that involves **only propositions** (*i.e.* it has no predicates) is called **Propositional Logic**
- There are two main versions of Propositional Logic:
  - **Semantic propositional logic**
  - **Propositional calculus**
- These two versions **differ** only in their **proof rules**
  - Proofs in semantic propositional logic are based on Boolean algebra
  - Proofs in Propositional calculus are based on the so-called rules of natural deduction

Truth tables and valuations

- A **valuation** of a sentence is an assignment (any assignment) of truth values to the statements that compose the sentence
- If we combine two statements with a logical connective, the value of the resulting compound statement is determined by two things:
  - **The chosen valuation**
  - **The semantic (meaning) of the logical connective**
- The meaning of each connective under all possible valuations is declared in a **truth table**

The truth table of negation

- The simplest case of truth value is negation. In this case, there is only one sentence, and therefore, only two possible valuations.
- The semantic is also very natural: if the sentence is true, its negation is false, and vice versa.
- Thus, the truth table of **negation** is simply

A	¬ A
0	1
1	0

Disjunction and conjunction truth tables

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

There are **four possible valuations** in each of these cases

Implication's truth table: Hypothesis->Thesis

A	B	$A \Rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

In implications, the variable A represents a **hypothesis statement** while the variable B, a **thesis statement**

The first two rows in this table indicate that when the hypothesis is false, the implication will not discriminate between a true or a false thesis. It will always return true. These are called **vacuously true implications**

The second two rows tell a different story. In this case (as we always hope in practice) the hypothesis is true. Then, **the truth value of the thesis is faithfully reflected in the value of the implication.** If the thesis is false, so is the implication, and vice versa.

Computing with truth tables

We are now in position to answer the question

Is A : ”3 < 4 $\Longrightarrow$ 2 < 3” true or false?

Answer: Identify the atomic sentences that compose A. Let

B : “3 < 4”, and C : “2 < 3”

Now, we see that A has the form

B $\Longrightarrow$ C

Since B = 0 and C = 0, **from the very first row of the table of implication** we conclude that **A is True** (although vacuously!)

Is there a valuation making the statement true?

P	**Q**	**¬P**	**P∧ (¬P∧ Q)**
0	0	1	0
0	**1**	**1**	**1**
1	0	0	0
1	1	0	0

Yes: P=0, Q=1

Equivalence between propositions

- Two propositions *A* and *B* are said to be **equivalent** if their truth values are the same in all possible valuations. Equivalence is denoted

$$A \Leftrightarrow B$$

- Equivalence is not the same as equality: two propositions may be equivalent even if they differ in their meaning
- Equivalence is a shorthand for

$$(A \Rightarrow B) \wedge (B \Rightarrow A)$$

Important equivalences

- **Conjunctive form of implication**

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

- **Contrapositive form of implication**

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

- **Idempotency rules**

$$\neg(\neg P) \Leftrightarrow P$$
$$P \wedge P \Leftrightarrow P$$
$$P \vee P \Leftrightarrow P$$

- **De Morgan's rules**

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$
$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

- **Distributive rules**

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

- **Associative rules**

$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$$
$$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$$

- **Commutative rules**

$$P \vee Q \Leftrightarrow Q \vee P$$
$$P \wedge Q \Leftrightarrow Q \wedge P$$

Some applications

Demonstrate that

$$\neg(p \vee \neg(r \wedge \neg s))$$

is equivalent to

$$\neg(p \vee s) \wedge r$$

Proof:

$$\neg(p \vee \neg(r \wedge \neg s)$$

$$\Leftrightarrow \neg p \wedge (r \wedge \neg s)$$

$$\Leftrightarrow \neg p \wedge (\neg s \wedge r)$$

$$\Leftrightarrow (\neg p \wedge \neg s) \wedge r$$

$$\Leftrightarrow \neg(p \vee s) \wedge r$$

Demonstrate that

$$p \Rightarrow (q \Rightarrow (r \Rightarrow s))$$

is equivalent to

$$(p \wedge q \wedge r) \Rightarrow s$$

Proof:

$$p \Rightarrow (q \Rightarrow (r \Rightarrow s))$$

$$\Leftrightarrow \neg p \vee (q \Rightarrow (r \Rightarrow s))$$

$$\Leftrightarrow \neg p \vee (\neg q \vee (r \Rightarrow s))$$

$$\Leftrightarrow \neg p \vee (\neg q \vee (\neg r \vee s))$$

$$\Leftrightarrow (\neg p \vee \neg q \vee \neg r) \vee s$$

$$\Leftrightarrow \neg(p \wedge q \wedge r) \vee s$$

$$\Leftrightarrow (p \wedge q \wedge r) \Rightarrow s$$

The negation of a predicate

- Negating a predicate that contains quantifiers demands, in addition, the use of the semantic table
- These negation rules must be applied together with the previously discussed equivalences, when treating predicates as propositions

QUANTIFIER	**NEGATION**
Universal	Existential
Existential	Universal

Yet another important equivalence

- The next equivalences describe the actual negation operation for elemental predicates with quantifiers
- Let $P(x, y, ..., z)$ be a predicate. Then,

$$\neg(\forall x, y, ..., z)P(x, y, ..., z) \Leftrightarrow (\exists x \vee \exists y \vee ... \vee \exists z)\neg P(x, y, ..., z)$$

$$\neg(\exists x, y, ..., z)P(x, y, ..., z) \Leftrightarrow (\forall x \vee \forall y \vee ... \vee \forall z)\neg P(x, y, ..., z)$$

Illustrations

Let’s consider the negation of

$$B:"(\forall x)(1\leq x)A(x,y)"$$

where

$$A(x,y):"x<y\Rightarrow x^2<y^2"$$

Answer:

$$\neg B\Leftrightarrow(\exists x)(1\leq x)\neg A(x,y)$$

- **Remark**: this is, as expected, a false statement

The informal expression $(\forall x,y)$ is a short hand for $(\forall x)\wedge(\forall y)$

In general, the **symbol** “, ” is a to be taken as a **short hand** for “and”

Thus, $\neg\forall(x,y)\Leftrightarrow(\exists x)\vee(\exists y)$

and

$$\neg((\forall x,y)(0\leq x\wedge y\leq 1)\neg A(x,y))$$

$$\Leftrightarrow(\exists x\vee\exists y)(0\leq x\wedge y\leq 1)A(x,y)$$

Composed quantifiers

Also common are predicates whose quantifier is a **composition** of the universal and the existential quantifiers

- For each x there is a y such that *P(x, y)* is true

$$(\forall x)(\exists y)P(x, y)$$

- There is an x such that for all y *P(x, y)* is true

$$(\exists x)(\forall y)P(x, y)$$

- Their negations are

$$(\exists x)(\forall y)\neg P(x, y)$$, and

$$(\forall x)(\exists y)\neg P(x, y)$$

Respectively

Boolean formulas

- A **Boolean formula** is a **“well-formed”** combination of Boolean variables and logical connectives
  - The combination $p \Rightarrow (\neg q \wedge (s \wedge \neg r))$

    Is a Boolean formula
  - The combination $\Rightarrow \; p \neg) qsr \; (\vee \neg \; \vee$

    Is not a Boolean formula
- Any representation of a proposition in terms of Boolean variables and logical connectives yields a Boolean formula.

Two important problems

- Given a combination of symbols and logical connectives: Is it a Boolean formula?
  - Domain: A set of symbols and logical connectives
  - Instance: A **string** of symbols and logical connectives
  - Question: Is the string a Boolean formula?
- Given a Boolean formula: can we find a valuation that makes it true?
  - Domain: All Boolean formulas
  - Instance: A Boolean formula
  - Question: Is there a valuation that makes the formula true?