# Mathematical proofs

The application of logic in the demonstration of mathematical statements

# On the nature of a mathematical proof

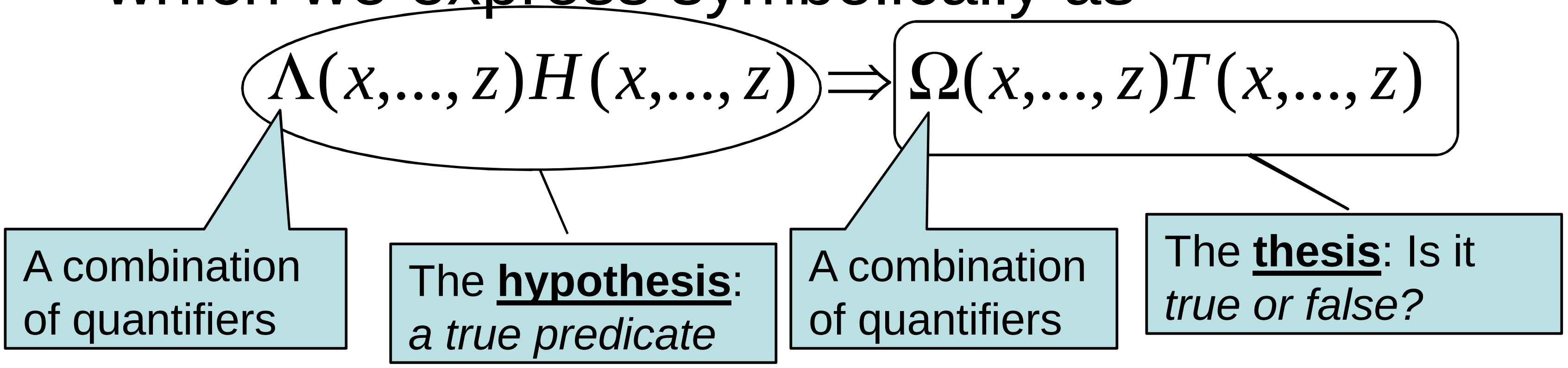
Mathematics is a "**purely deductive**" discipline. This means that mathematical statements are either axioms or theorems (i.e. lemmas, corollaries, propositions, theorems, etc)

- **Axioms** are statements declared to be true without proof
- **Theorems** are statements **derived** from true statements, using the rules of logic

# Derivation

The “pattern of thinking” of the **derivation of a statement from previously known statements** is captured by the logical operation of implication

Thus, the core of a **mathematical proof** is normally well represented as an **implication between predicates endowed with quantifiers** which we express symbolically as



$$\Lambda(x,\dots,z)H(x,\dots,z) \Rightarrow \Omega(x,\dots,z)T(x,\dots,z)$$

A combination of quantifiers

The **hypothesis**: *a true predicate*

A combination of quantifiers

The **thesis**: Is it *true or false?*

# Hidden implications

- Quite often, mathematical statements **do not show the implication explicitly**. In most of these cases, the hypothesis appears to be missing
  - Example: Consider the classical trigonometric relation

$$\sin^2\theta + \cos^2\theta = 1$$

  - The actual statement is

$$(\forall\theta)(0 \le \theta \le 2\pi \Rightarrow \sin^2\theta + \cos^2\theta = 1)$$

# Steps in reasoning about a proof

**Before starting the proof**:

- **Identify the actual logic structure of the theorem** (or given statement). **Rewrite** the theorem in terms of the identified logical structure
- **Examine and decide on possible "proving strategies" and general style of the proof**. This depends on:
  - Previously proven statements and axioms that may have an influence in the proof; and
  - The logic and mathematical tools that are available to compose the proof

# Main styles

- **Deductive proofs**
  - Direct proofs
  - Constructive proofs
  - Proofs by counterexample
  - Proofs by contrapositive
  - Proofs by reduction to absurdum
- **Inductive proofs**
  - Applies only to implications in which one of the predicate’s variable ranges over an infinite countable set

# Proof by construction

- Applies to **existential** properties only

$$(\exists y)(\Lambda(x,....,z)H(x,...,y,....,z) \Rightarrow$$
$$\Omega(x,...,z)T(x,...,y,...,z))$$

- Consist of two major steps:
  - – **Construction of a mathematical object**
  - – **Proof that the constructed object satisfies the corresponding existential property**

# Example

*Theorem*: The set of all odd numbers is infinite countable

- **Rewrite as a logical statement:**

If $Odd = \{n \in N : \exists m \in N \wedge n = 2m+1\}$ then, there exists a function

$$f : N \rightarrow Odd$$

which is a bijection

- **Proving strategy:**

By construction:

  - Build $f$
  - Demonstrate that the $f$ that has been built is a bijection

# The demonstration

Definition of the object

***Proof:*** Let *f(n)=2n+1*, where *n* is a natural number. To show that *f* is a bijection:

Is the object appropriated?

- *f* is injective:

  Let $p \neq q,\ p \in N \wedge q \in N$

  Assumption

  Then, $2p+1 \neq 2q+1$

  Rules of arithmetic

  Therefore, $f(p) \neq f(q)$

  Definition of *f*

- *f* is onto:

  $(\forall m \in Odd)\exists n \in N \wedge m = 2n+1$

  Definition of *Odd*

  $(\forall m \in Odd)\exists n \in N \wedge m = f(n)$

  Definition of *f*

# Counterexamples

In practice one is often faced with the question: Is this statement true or false? Counterexamples are **constructions** used to **demonstrate** that a **universal statement is false**

- A proof by counterexample is similar in spirit to a proof by construction and as such, it consists of the same two major steps:
  - **Construction of a mathematical object**
  - **Demonstration that the constructed object makes the statement false**

# Example

*Theorem*: It is not the case that

$$(\forall x, y)(0 < x < y \Rightarrow x < y^2)$$

- **Rewrite as a logical statement**: Unnecessary
- **Proving strategy:**

  Counterexample. Find values for the variables that render the statement false

# Proof

*Proof*:

Let $x = .5$ and $y = .6$

Then

$$y^2 = .36 < .5 = x$$

# Direct proofs

By a **direct proof** it is normally understood a proof in which the implication is not replaced with its equivalent contrapositive form.

We distinguish two main cases:

- **Proof by exhaustion**
  - This is a proof in which all values of the predicate's variables are verified with a direct **calculation**
- **Proof based on general arguments**
  - This is a proof in which general logical and/or mathematical arguments are used to demonstrate the implication

# Example 1

*Theorem*:

$$(P \wedge Q) \Rightarrow (P \vee Q)$$

**Rewrite as logical statement:** unnecessary

**Proving strategy:** By exhaustion

(**it works!!:** there are only **four** valuations)

*Proof:*

| P | Q | $P \wedge Q$ | $P \vee Q$ | $() \Rightarrow ()$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

Always true !

# Example 2

*Theorem:* The product of two odd numbers is always an odd number

**Rewrite as a logical statement:**

$$(\forall x, y)(x \in Odd \wedge y \in Odd) \Rightarrow xy \in Odd$$

**Proving strategy:** Direct. The rules of arithmetic will be used to show that the general form of an odd number is preserved under multiplication

# Proof

***Proof:***

Let $x \in Odd \wedge y \in Odd$

Hypothesis

Definition of *Odd*

Then, $(\exists n \in N) x = 2n + 1$ and $(\exists m \in N) y = 2m + 1$

Now, $xy = (2n+1)(2m+1)$

$$= (4nm + 2(n+m) + 1)$$

Rules of arithmetic

$$= 2(2nm + n + m) + 1$$

Rules of arithmetic

Since $k = 2nm + n + m \in N$

$$xy = 2k + 1 \in Odd$$

Definition of *Odd*

# Contrapositve forms

Replace the implication with its equivalent **contrapositive** form

- For a universal property:

$$(\forall y)(\neg T(x,\ldots, y,\ldots, z) \Rightarrow \neg H(x,\ldots, y,\ldots, z))$$

- For an existential property:

$$(\exists y)(\neg T(x,\ldots, y,\ldots, z) \Rightarrow \neg H(x,\ldots, y,\ldots, z))$$

Keep in mind that a **contrapositive proof** **has nothing to do** with the **negation** of the implication

# Example

*Theorem:* If the square of a number is even, then so is the number

**Rewrite in logical terms:**

Let $Even = \{x \in N : (\exists n \in N) x = 2n\}$

$$(\forall x)(x^2 \in Even \Rightarrow x \in Even)$$

**Proving strategy:** A direct proof will involve the taking of a square root, which is not rich enough in arithmetic properties. The contrapositive replaces the square root with a multiplication

# Proof

*Proof:*

$$\text{The contrapositive is:}$$

$$(\forall x)(x \notin Even \Rightarrow x^2 \notin Even)$$

$$\text{Now, } x \notin Even \Rightarrow x \in Odd$$

$$\text{Therefore, } (\exists n \in N)x = 2n + 1$$

$$\begin{aligned}\text{But then, } x^2 = xx &= (2n+1)(2n+1) \\ &= (4n^2 + 4n + 1) \\ &= 2(2n^2 + 2n) + 1\end{aligned}$$

$$\text{Since } k = 2n^2 + 2n \in N,\ \ x^2 = 2k + 1 \in Odd$$

$$\text{Thus, } x^2 \notin Even$$

# Reduction to the absurdum

- It works by demonstrating that the **negation of the implication** implies, in turn, a contradiction with an already accepted truth
- For proving $(H(x) \Rightarrow T(x))$ by contradiction
  - **Prove** $(H(x) \wedge \neg T(x)) \Rightarrow A(x)$
  - **Demonstrate that** $A(x)$ **is false** (contradicts an established fact)

# Example

*Theorem:* Let *U* be an infinite set, *S* be a finite subset of *U*, and *T* the complement of *S* with respect to *U*. Then, *T* is infinite.

**Rewrite in logical terms:**

$$(\forall U)(U \text{ infinite}) \wedge (\forall S)(S \subseteq U \wedge \text{finite})$$

$$T = U - S \Rightarrow T \text{ infinite}$$

**Proving strategy:** Reduction to absurdum

# Proof

*Proof:*

By contradiction: assume

$T = U - S \wedge \neg(T \text{ infinite})$

Then, $T$ is finite

Since $U = T \cup S$, $U$ is the union of two finite sets

Contradiction:

The union of two finite sets is always a finite set

and $U$ is infinite

# Induction

- Recall that a set is **infinite countable** if it can be put in a one-to-one and onto correspondence with the set of natural numbers
- **Induction applies solely to predicates** of the form of

  $$(\forall n)P(n)$$

  where $n$ ranges over a countable domain
- A **proof by induction** consists of two steps:
  - **Prove that** $(\exists n_0)P(n_0) = 1$
  - **Prove that** $(\forall n \geq n_0)(P(n) \Rightarrow P(n+1))$

# Example

*Theorem* : $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

**Rewrite in logical terms:**

$$(\forall n)(n \in N \Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6})$$

**Proving strategy:** Induction on *n*

# Proof

*Proof:*

1.- Let $n_0 = 1$. Then, the statement becomes

$$\sum_{i=1}^{1} i^2 = \frac{1(2)(3)}{6} = 1$$

which is true.

2.- To show that:

$$(\forall \mathrm{n} \geq 1)(\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \Rightarrow \sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6})$$

Proof : (next page)

# Proof (continuation)

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n^2 + 2n + 1)$$

$$= \frac{2n^3 + 3n^2 + n}{6} + \frac{6n^2 + 12n + 6}{6}$$

$$= \frac{2n^3 + 9n^2 + 13n + 6}{6}$$

$$= \frac{(\mathrm{n}+1)(\mathrm{n}+2)(2\mathrm{n}+3)}{6}$$