Chapter 1

- **1.1** Let $x_1 = y$, $x_2 = y^{(1)}$, …, $x_n = y^{(n-1)}$.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= g(t, x_1, \ldots, x_n, u) \\
y &= x_1
\end{aligned}$$

- **1.2** Let $x_1 = y$, $x_2 = y^{(1)}$, …, $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)} - g_2\left(t, y, y^{(1)}, \ldots, y^{(n-2)}\right) u$.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} \\
\dot{x}_{n-1} &= y^{(n-1)} = x_n + g_2(t, x_1, x_2, \ldots, x_{n-1})u \\
\dot{x}_n &= y^{(n)} - g_2(t, x_1, \cdots, x_{n-1})\dot{u} - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1} \right) u \\
&= g_1(t, x_1, \ldots, x_{n-1}, x_n + g_2(\cdot)u, u) \\
&\quad - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}x_2 + \cdots + \frac{\partial g_2}{\partial x_{n-1}}(x_n + g_2(\cdot)u) \right) u \\
y &= x_1
\end{aligned}$$

- **1.3** Let $x_1 = y$, $x_2 = y^{(1)}$, …, $x_n = y^{(n-1)}$, $x_{n+1} = z$, …, $x_{n+m} = z^{(m-1)}$.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= g(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}, u) \\
\dot{x}_{n+1} &= x_{n+2} \\
&\vdots \\
\dot{x}_{n+m-1} &= x_{n+m} \\
\dot{x}_{n+m} &= u \\
y &= x_1
\end{aligned}$$

• **1.4** Let $x_1 = q$, $x_2 = \dot{q}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^{2m}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{q} = M^{-1}(x_1)[u - C(x_1, x_2)x_2 - Dx_2 - g(x_1)] \end{aligned}$$

• **1.5** Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u \end{aligned}$$

• **1.6** Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$, where $x_i \in R^m$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}(x_1)[h(x_1, x_2) + K(x_1 - x_3)] \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= J^{-1}K(x_1 - x_3) + J^{-1}u \end{aligned}$$

• **1.7** Let

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

be a state model of the linear system.

$$u = r - \psi(t, y) = r - \psi(t, Cx)$$

Hence

$$\dot{x} = Ax - B\psi(t, Cx) + Br, \qquad y = Cx$$

• **1.8** **(a)** Let $x_1 = \delta$, $x_2 = \dot{\delta}$, $x_3 = E_q$, and $u = E_{FD}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}x_3 \sin x_1 \\ \dot{x}_3 &= -\frac{\eta_2}{\tau}x_3 + \frac{\eta_3}{\tau}\cos x_1 + \frac{1}{\tau}u \end{aligned}$$

(b) The equilibrium points are the roots of the equations

$$\begin{aligned} 0 &= x_2 \\ 0 &= 0.815 - Dx_2 - 2.0x_3 \sin x_1 \\ 0 &= -2.7x_3 + 1.7\cos x_1 + 1.22 \end{aligned}$$

$$x_2 = 0 \Rightarrow x_3 = \frac{0.4075}{\sin x_1}$$

Substituting x_3 in the third equation yields

$$(1.22 + 1.7\cos x_1)\sin x_1 - 1.10025 = 0$$

The foregoing equation has two roots $x_1 = 0.4067$ and $x_1 = 1.6398$ in the interval $-\pi \le x_1 \le \pi$. Due to periodicity, $0.4067 + 2n\pi$ and $1.6398 + 2n\pi$ are also roots for $n = \pm 1, \pm 2, \ldots$. Each root $x_1 = x$ gives an equilibrium point $(x, 0, 0.4075/\sin x)$.

(c) With E_q = constant, the model reduces to

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}E_q \sin x_1\end{aligned}$$

which is a pendulum equation with an input torque.

- **1.9** **(a)** Let $x_1 = \phi_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= \dot{\phi}_L = v_L = v_C = x_2\\ \dot{x}_2 &= \dot{v}_C = \frac{1}{C}i_C = \frac{1}{C}\left[i_s - \frac{v_C}{R} - i_L\right]\\ &= \frac{1}{C}\left[i_s - I_0 \sin kx_1 - \frac{1}{R}x_2\right]\end{aligned}$$

(b) Let $x_1 = i_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= I_0 k \cos k\phi_L\ \dot{\phi}_L = k\sqrt{I_0^2 - i_L^2}\,v_C\ .\\ &= x_2 k\sqrt{I_0^2 - x_1^2}\\ \dot{x}_2 &= \frac{1}{C}\left[i_s - x_1 - \frac{1}{R}x_2\right]\end{aligned}$$

The model of (a) is more familiar since it is the pendulum equation.

- **1.10** **(a)** Let $x_1 = \phi_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= \dot{\phi}_L = v_L = v_C = x_2\\ \dot{x}_2 &= \dot{v}_C = \frac{1}{C}i_C = \frac{1}{C}\left[i_s - \frac{v_C}{R} - i_L\right]\\ &= \frac{1}{C}\left[i_s - Lx_1 - \mu x_1^3 - \frac{1}{R}x_2\right]\end{aligned}$$

(b) $x_2 = 0 \;\Rightarrow\; Lx_1 + \mu x_1^3 = 0 \;\Rightarrow\; x_1 = 0$. There is a unique equilibrium point at the origin.

- **1.11** **(a)**

$$\dot{z} = Az + Bu, \quad y = Cx, \quad u = \sin e$$
$$\dot{e} = \dot{\theta}_i - \dot{\theta}_o = -\dot{\theta}_o = -y = -Cz$$
$$\dot{z} = Az + B\sin e, \quad \dot{e} = -Cz$$

(b)

$$0 = Az + B\sin e \;\Rightarrow\; z = -A^{-1}B\sin e$$
$$0 = Cz \;\Rightarrow\; -CA^{-1}B\sin e = G(0)\sin e = 0$$
$$G(0) \neq 0 \;\Rightarrow\; \sin e = 0 \;\Rightarrow\; e = \pm n\pi,\ n = 0, 1, 2, \cdots \ \text{ and } \ z = 0$$

(c) For $G(s) = 1/(\tau s + 1)$, take $A = -1/\tau$, $B = 1/\tau$ and $C = 1$. Then

$$\dot{z} = -\frac{1}{\tau}z + \frac{1}{\tau}\sin e, \quad \dot{e} = -z$$

Let $x_1 = e$, $x_2 = -z$.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{1}{\tau}x_2 - \frac{1}{\tau}\sin x_1$$

• **1.12** The equation of motion is

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Let $x_1 = y$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| + g$$

• **1.13 (a)**

$$m\ddot{y} = -(k_1 + k_2)y - c\dot{y} + h(v_0 - \dot{y})$$

where $c > 0$ is the viscous friction coefficient.
(b) $h(v) \approx h(v_0) - h'(v_0)\dot{y}$.

$$m\ddot{y} = -(k_1 + k_2)y - [c + h'(v_0)]\dot{y} + h(v_0)$$

(c) To obtain negative friction, we want $c + h'(v_0) < 0$. This can be achieved with the friction characteristic of Figure 1.5(d) if v_0 is in the range where the slope is negative and the magnitude of the negative slope is greater than c.

• **1.14** The equation of motion is

$$M\dot{v} = F - Mg\sin\theta - k_1\text{sgn}(v) - k_2 v - k_3 v^2$$

where k_1, k_2, and k_3 are positive constants. Let $x = v$, $u = F$, and $w = \sin\theta$.

$$\dot{x} = \frac{1}{M}\left[-k_1\text{sgn}(x) - k_2 x - k_3 x^2 + u\right] - gw$$

• **1.15 (a)**

$$H = m\frac{d^2}{dt^2}(y + L\sin\theta) = m\frac{d}{dt}(\dot{y} + L\dot{\theta}\cos\theta) = m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

$$V = m\frac{d^2}{dt^2}(L\cos\theta) + mg = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) + mg = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta + mg$$

Substituting V and H in the $\ddot{\theta}$-equation yields

$$\begin{aligned}
I\ddot{\theta} &= VL\sin\theta - HL\cos\theta \\
&= -mL^2\ddot{\theta}(\sin\theta)^2 - mL^2\dot{\theta}^2\sin\theta\cos\theta + mgL\sin\theta \\
&\quad -mL\ddot{y}\cos\theta - mL^2\ddot{\theta}(\cos\theta)^2 + mL^2\dot{\theta}^2\sin\theta\cos\theta \\
&= -mL^2\ddot{\theta}[(\sin\theta)^2 + (\cos\theta)^2] + mgL\sin\theta - mL\ddot{y}\cos\theta \\
&= -mL^2\ddot{\theta} + mgL\sin\theta - mL\ddot{y}\cos\theta
\end{aligned}$$

Substituting H in the $\ddot{y}$-equation yields

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y}$$

(b)

$$D(\theta)\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

where

$$D(\theta) = \begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix}$$

$$\det(D(\theta)) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta = \Delta(\theta)$$

Hence,

$$D^{-1}(\theta) = \frac{1}{\Delta(\theta)} \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot\theta \\ \ddot y \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix} \begin{bmatrix} mgL\sin\theta \\ F + mL\dot\theta^2\sin\theta - k\dot y \end{bmatrix}$$

(c)

$$\begin{aligned}
\dot x_1 &= x_2 \\
\dot x_2 &= \ddot\theta = \frac{1}{\Delta(\theta)}\left[(m+M)mgL\sin\theta - mL\cos\theta(F + mL\dot\theta^2\sin\theta - k\dot y)\right] \\
&= \frac{1}{\Delta(x_1)}\left[(m+M)mgL\sin x_1 - mL\cos x_1(u + mLx_2^2\sin x_1 - kx_4)\right] \\
\dot x_3 &= x_4 \\
\dot x_4 &= \ddot y = \frac{1}{\Delta(\theta)}\left[-m^2L^2 g\sin\theta\cos\theta + (I+mL^2)(F + mL\dot\theta^2\sin\theta - k\dot y)\right] \\
&= \frac{1}{\Delta(x_1)}\left[-m^2L^2 g\sin x_1\cos x_1 + (I+mL^2)(u + mLx_2^2\sin x_1 - kx_4)\right]
\end{aligned}$$

• **1.16 (a)**

$$F_x = m\frac{d^2}{dt^2}(x_c + L\sin\theta) = m\frac{d}{dt}(\dot x_c + L\dot\theta\cos\theta) = m(\ddot x_c + L\ddot\theta\cos\theta - L\dot\theta^2\sin\theta)$$

$$F_y = m\frac{d^2}{dt^2}(L\cos\theta) = m\frac{d}{dt}(-L\dot\theta\sin\theta) = -mL\ddot\theta\sin\theta - mL\dot\theta^2\cos\theta$$

Substituting F_x and F_y in the $\ddot\theta$-equation yields

$$\begin{aligned}
I\ddot\theta &= u + F_yL\sin\theta - F_xL\cos\theta \\
&= u - mL^2\ddot\theta(\sin\theta)^2 - mL^2\dot\theta^2\sin\theta\cos\theta \\
&\quad - mL\ddot x_c\cos\theta - mL^2\ddot\theta(\cos\theta)^2 + mL^2\dot\theta^2\sin\theta\cos\theta \\
&= u - mL^2\ddot\theta[(\sin\theta)^2 + (\cos\theta)^2] - mL\ddot x_c\cos\theta \\
&= u - mL^2\ddot\theta - mL\ddot x_c\cos\theta
\end{aligned}$$

Substituting F_x in the $\ddot x_c$-equation yields

$$M\ddot x_c = -m\ddot x_c - mL\ddot\theta\cos\theta + mL\dot\theta^2\sin\theta - kx_c$$

Thus,

$$D(\theta)\begin{bmatrix} \ddot\theta \\ \ddot x_c \end{bmatrix} = \begin{bmatrix} u \\ mL\dot\theta^2\sin\theta - kx_c \end{bmatrix}$$

where

$$D(\theta) = \begin{bmatrix} I+mL^2 & mL\cos\theta \\ mL\cos\theta & m+M \end{bmatrix}$$

(b)

$$\det(D(\theta)) = (I+mL^2)(m+M) - m^2L^2\cos^2\theta = \Delta(\theta)$$

Hence,

$$D^{-1}(\theta) = \frac{1}{\Delta(\theta)} \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix} \begin{bmatrix} u \\ mL\dot{\theta}^2 \sin\theta - kx_c \end{bmatrix}$$

(c)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{\theta} = \frac{1}{\Delta(\theta)} \left[(m+M)u - mL\cos\theta(mL\dot{\theta}^2 \sin\theta - kx_c)\right] \\ &= \frac{1}{\Delta(x_1)} \left[(m+M)u - mL\cos x_1(mLx_2^2 \sin x_1 - kx_3)\right] . \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \ddot{x}_c = \frac{1}{\Delta(\theta)} \left[-mLu\cos\theta + (I+mL^2)(mL\dot{\theta}^2 \sin\theta - kx_c)\right] \\ &= \frac{1}{\Delta(x_1)} \left[-mLu\cos x_1 + (I+mL^2)(mLx_2^2 \sin x_1 - kx_3)\right] \end{aligned}$$

(d) Take $u =$ constant. Setting the derivatives $\dot{x}_i = 0$, we obtain $x_2 = x_4 = 0$ and

$$\begin{aligned} 0 &= (m+M)u + mkLx_3 \cos x_1 \\ 0 &= -mLu\cos x_1 - k(I+mL^2)x_3 \end{aligned}$$

Eliminating x_3 between the two equations yields

$$u[(m+M)(I+mL^2) - m^2L^2\cos^2 x_1] = u\ \Delta(x_1) = 0$$

Since $\Delta(x_1) > 0$, equilibrium can be maintained only at $u = 0$. Then, $x_3 = 0$. Thus, the system has an equilibrium set $\{x_2 = x_3 = x_4 = 0\}$.

• **1.17** **(a)** Let $x_1 = i_f$, $x_2 = i_a$, and $x_3 = \omega$.

$$\begin{aligned} \dot{x}_1 &= -\frac{R_f}{L_f}x_1 + \frac{v_f}{L_f} \\ \dot{x}_2 &= -\frac{R_a}{L_a}x_2 - \frac{c_1}{L_a}x_1x_3 + \frac{v_a}{L_a} \\ \dot{x}_3 &= -\frac{c_3}{J}x_3 + \frac{c_2}{J}x_1x_2 \end{aligned}$$

(b) Take $v_a = V_a =$ constant and $v_f = u$.
(c) Take $v_f = V_f =$ constant and $v_a = u$. A constant field voltage implies that (at steady state) $i_f = V_f/R_f \stackrel{\text{def}}{=} I_f =$ constant. Hence, the model reduces to the second-oder linear model

$$\begin{aligned} \dot{x}_2 &= -\frac{R_a}{L_a}x_2 - \frac{c_1 I_f}{L_a}x_3 + \frac{u}{L_a} \\ \dot{x}_3 &= -\frac{c_3}{J}x_3 + \frac{c_2 I_f}{J}x_2 \end{aligned}$$

(d) Let $v = u$.

$$\begin{aligned} \dot{x}_1 &= -\frac{R_x + R_f}{L_f}x_1 + \frac{u}{L_f} \\ \dot{x}_2 &= -\frac{R_a}{L_a}x_2 - \frac{c_1}{L_a}x_1x_3 + \frac{u}{L_a} \\ \dot{x}_3 &= -\frac{c_3}{J}x_3 + \frac{c_2}{J}x_1x_2 \end{aligned}$$

• **1.18** **(a)** $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i$, $u = v$.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \ddot{y} = -\frac{k}{m}\dot{y} + g + \frac{1}{m}F(y,i) \\
&= g - \frac{k}{m}x_2 - \frac{1}{m}\cdot\frac{L_0 x_3^2}{2a(1+x_1/a)^2} \\
&= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2} \\
\dot{x}_3 &= \frac{di}{dt} = \frac{d}{dt}\left[\frac{\phi}{L(y)}\right] \\
&= \frac{\dot{\phi}}{L(y)} - \frac{\phi}{L^2(y)}\cdot\frac{dL}{dy}\dot{y} \\
&= \frac{1}{L(y)}(v - Ri) + \frac{\phi}{L^2(y)}\cdot\frac{L_0}{a(1+y/a)^2}\dot{y} \\
&= \frac{1}{L(x_1)}\left[-Rx_3 + \frac{L_0 a x_2 x_3}{(a+x_1)^2} + u\right]
\end{aligned}$$

(b) The equilibrium equations are

$$\begin{aligned}
0 &= \bar{x}_2 \\
0 &= g - \frac{k}{m}\bar{x}_2 - \frac{L_0 a \bar{x}_3^2}{2m(a+\bar{x}_1)^2} \\
0 &= -R\bar{x}_3 + \frac{L_0 a \bar{x}_2 \bar{x}_3}{(a+\bar{x}_1)^2} + \bar{u}
\end{aligned}$$

Set $\bar{x}_1 = r$, $\bar{x}_3 = I_{ss}$, and $\bar{u} = V_{ss}$. Then

$$I_{ss} = \left(\frac{2mg(a+r)^2}{L_0 a}\right)^{1/2}, \quad V_{ss} = RI_{ss}$$

• **1.19** **(a)**

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^h A(\lambda)\, d\lambda\right) &= w_i - k\sqrt{\rho g h} \\
A(h)\dot{h} &= u - k\sqrt{\rho g h}
\end{aligned}$$

Let $x = h$.

$$\dot{x} = \frac{1}{A(x)}[u - k\sqrt{\rho g x}], \quad y = x$$

(b) $x = p - p_a = \rho g h$.

$$\dot{x} = \frac{\rho g}{A(x/\rho g)}(u - k\sqrt{x}), \quad y = x/(\rho g)$$

(c) At equilibrium,

$$0 = u_{ss} - k\sqrt{\rho g x_{ss}}, \quad y_{ss} = x_{ss} = r$$

Hence, $u_{ss} = k\sqrt{\rho g r}$

• **1.20** **(a)** From the equations $\dot{v} = w_i - w_o$ and $p = p_a + (\rho g/A)v$, we have

$$\dot{p} = \frac{\rho g}{A}\dot{v} = \frac{\rho g}{A}(w_i - w_o) = \frac{\rho g}{A}[\phi^{-1}(\Delta p) - k\sqrt{\Delta p}]$$

Using $x = \Delta p$ as the state variable, we obtain

$$\dot{x} = \frac{\rho g}{A}[\phi^{-1}(x) - k\sqrt{x}]$$

(b) At equilibrium we have

$$\phi^{-1}(\bar{x}) = k\sqrt{\bar{x}}$$

Writing $\bar{x} = \phi(\bar{w}_i)$, we can rewrite the previous equation as

$$\bar{w}_i = k\sqrt{\phi(\bar{w}_i)}$$

Hence,

$$\left(\frac{\bar{w}_i}{k}\right)^2 = \phi(\bar{w}_i)$$

The solutions of this equation are given by the intersection of the curve $(\bar{w}_i/k)^2$ with the curve $\phi(\bar{w}_i)$, which is shown in Figure 1.29 of the text. From the figure, it is clear that there is only one intersection point.

• **1.21** **(a)** We have

$$\dot{v}_1 = w_p - w_1, \quad \dot{v}_2 = w_1 - w_2$$

$$\dot{p}_1 = \frac{\rho g}{A_1}\dot{v}_1, \quad \dot{p}_2 = \frac{\rho g}{A_2}\dot{v}_2$$

$$w_1 = k_1\sqrt{p_1 - p_2}, \quad w_2 = k_2\sqrt{p_2 - p_a}, \quad p_1 - p_a = \phi(w_p)$$

Let $x_1 = p_1 - p_a$ and $x_2 = p_2 - p_a$.

$$\begin{aligned} \dot{x}_1 &= \dot{p}_1 = \frac{\rho g}{A_1}(w_p - w_1) = \frac{\rho g}{A_1}\left[\phi^{-1}(x_1) - k_1\sqrt{x_1 - x_2}\right] \\ \dot{x}_2 &= \dot{p}_2 = \frac{\rho g}{A_2}(w_1 - w_2) = \frac{\rho g}{A_2}\left[k_1\sqrt{x_1 - x_2} - k_2\sqrt{x_2}\right] \end{aligned}$$

(b) The equilibrium equations are

$$\begin{aligned} \phi^{-1}(\bar{x}_1) &= k_1\sqrt{\bar{x}_1 - \bar{x}_2} \\ k_1\sqrt{\bar{x}_1 - \bar{x}_2} &= k_2\sqrt{\bar{x}_2} \end{aligned}$$

From the second equation, we have

$$\bar{x}_2 = \frac{k_1^2}{k_1^2 + k_2^2}\bar{x}_1$$

Substituting this expression in the first equilibrium equation yields

$$\phi^{-1}(\bar{x}_1) = k_{\rm eq}\sqrt{\bar{x}_1}, \quad \text{where } k_{\rm eq} = \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}}$$

Writing $\bar{x}_1 = \phi(\bar{w}_p)$, we can rewrite the previous equation as

$$\bar{w}_p = k_{\rm eq}\sqrt{\phi(\bar{w}_p)}$$

Hence,

$$\left(\frac{\bar{w}_p}{k_{\rm eq}}\right)^2 = \phi(\bar{w}_p)$$

The solutions of this equation are given by the intersection of the curve $(\bar{w}_p/k_{\rm eq})^2$ with the curve $\phi(\bar{w}_p)$, which is shown in Figure 1.29 of the text. From the figure, it is clear that there is only one intersection point.

- **1.22** (a)

$$\begin{aligned}\dot{x}_1 &= dx_{1f} - dx_1 + r_1 \\ \dot{x}_2 &= dx_{2f} - dx_2 - r_2\end{aligned}$$

The assumptions $r_1 = \mu x_1$, $r_2 = r_1/Y = \mu x_1/Y$, and $x_{1f} = 0$ yield

$$\begin{aligned}\dot{x}_1 &= (\mu - d)x_1 \\ \dot{x}_2 &= d(x_{2f} - x_2) - \mu x_1/Y\end{aligned}$$

(b) When $\mu = \mu_m x_2/(k_m + x_2)$, the equilibrium equations are

$$\begin{aligned}0 &= \left(\frac{\mu_m \bar{x}_2}{k_m + \bar{x}_2} - d\right)\bar{x}_1 \\ 0 &= d(x_{2f} - \bar{x}_2) - \frac{\mu_m \bar{x}_1 \bar{x}_2}{Y(k_m + \bar{x}_2)}\end{aligned}$$

from the first equation,

$$\bar{x}_1 = 0 \quad \text{or} \quad \frac{\mu_m \bar{x}_2}{k_m + \bar{x}_2} = d \;\Rightarrow\; \bar{x}_2 = \frac{k_m d}{\mu_m - d}$$

Substituting $\bar{x}_1 = 0$ in the second equilibrium equation yields $\bar{x}_2 = x_{2f}$. Substituting $\bar{x}_2 = k_m d/(\mu_m - d)$ in the second equilibrium equation yields

$$\bar{x}_1 = Y\left(x_{2f} - \frac{k_m d}{\mu_m - d}\right)$$

Hence, there are two equilibrium points at

$$\left(Y\left(x_{2f} - \frac{k_m d}{\mu_m - d}\right), \frac{k_m d}{\mu_m - d}\right) \quad \text{and} \quad (0, x_{2f})$$

(c) When $\mu = \mu_m x_2/(k_m + x_2 + k_1 x_2^2)$, the equilibrium equations are

$$\begin{aligned}0 &= \left(\frac{\mu_m \bar{x}_2}{k_m + \bar{x}_2 + k_1 \bar{x}_2^2} - d\right)\bar{x}_1 \\ 0 &= d(x_{2f} - \bar{x}_2) - \frac{\mu_m \bar{x}_1 \bar{x}_2}{Y(k_m + \bar{x}_2 + k_1 \bar{x}_2^2)}\end{aligned}$$

from the first equation, $\bar{x}_1 = 0$ or $\bar{x}_2$ is the root of $d = \mu(\bar{x}_2)$. Since $d < \max_{x_2 \geq 0}\{\mu(x_2)\}$, the equation $d = \mu(\bar{x}_2)$ has two roots. Denote these roots by $\bar{x}_{2a}$ and $\bar{x}_{2b}$. Substituting $\bar{x}_1 = 0$ in the second equilibrium equation yields $\bar{x}_2 = x_{2f}$. Substituting $\bar{x}_2 = \bar{x}_{2a}$ in the second equilibrium equation yields

$$d(x_{2f} - \bar{x}_{2a}) - \mu(\bar{x}_{2a})\bar{x}_1/Y = 0 \;\Rightarrow\; \bar{x}_1 = Y(x_{2f} - \bar{x}_{2a})$$

since $\mu(\bar{x}_{2a}) = d$. Similarly, substituting $\bar{x}_2 = \bar{x}_{2b}$ in the second equilibrium equation yields $\bar{x}_1 = Y(x_{2f} - \bar{x}_{2b})$. Thus, there are three equilibrium points at

$$(Y(x_{2f} - \bar{x}_{2a}), \bar{x}_{2a}), \quad (Y(x_{2f} - \bar{x}_{2b}), \bar{x}_{2b}), \quad \text{and} \quad (0, x_{2f})$$

Chapter 2

- **2.1** (1)

$$0 = -x_1 + 2x_1^3 + x_2, \qquad 0 = -x_1 - x_2$$

$$x_2 = -x_1 \;\Rightarrow\; 0 = 2x_1(x_1^2 - 1) \;\Rightarrow\; x_1 = 0,\ 1,\ \text{or}\ -1$$

There are three equilibrium points at $(0,0)$, $(1,-1)$, and $(-1,1)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1+6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1 \pm j \;\Rightarrow\; (0,0) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = 2 \pm \sqrt{8} \;\Rightarrow\; (1,-1) \text{ is a saddle}$$

Similarly, $(-1,1)$ is a saddle.

(2)

$$0 = x_1(1+x_2), \qquad 0 = -x_2 + x_2^2 + x_1x_2 - x_1^3$$

$$0 = x_1(1+x_2) \;\Rightarrow\; x_1 = 0 \text{ or } x_2 = -1$$

$$x_1 = 0 \;\Rightarrow 0 = -x_2 + x_2^2 \;\Rightarrow\; x_2 = 0 \text{ or } x_2 = 1$$

$$x_2 = -1 \;\Rightarrow\; 0 = 2 - x_1 - x_1^3 \;\Rightarrow\; x_1 = 1$$

There are three equilibrium points at $(0,0)$, $(0,1)$, and $(1,-1)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1+x_2 & x_1 \\ x_2 - 3x_1^2 & -1+2x_2+x_1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = 1,\ -1 \;\Rightarrow\; (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,1)} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = 2,\ 1 \;\Rightarrow\; (0,1) \text{ is unstable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1 \pm j\sqrt{3} \;\Rightarrow\; (1,-1) \text{ is a stable focus}$$

(3)

$$0 = (1-x_1)x_1 - \frac{2x_1x_2}{1+x_1}, \qquad 0 = \left(2 - \frac{x_2}{1+x_1}\right)x_2$$

From the second equation, $x_2 = 0$ or $x_2 = 2(1 + x_1)$.

$$x_2 = 0 \;\Rightarrow\; x_1 = 0 \text{ or } x_1 = 1$$

$$x_2 = 2(1 + x_1) \;\Rightarrow 0 = (x_1 + 3)x_1 \;\Rightarrow\; x_1 = 0 \text{ or } x_1 = -3$$

There are four equilibrium points at $(0,0)$, $(1,0)$, $(0,2)$, and $(-3,-4)$. Notice that we have assumed $1 + x_1 \neq 0$; otherwise the equation would not be well defined.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 - 2x_1 - \frac{2x_2}{(1+x_1)^2} & -2\frac{x_1}{(1+x_1)} \\ \frac{x_2^2}{(1+x_1)^2} & 2 - \frac{2x_2}{(1+x_1)} \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \text{ Eigenvalues : } 1, 2 \Rightarrow (0,0) \text{ is unstable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(1,0)} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}; \text{ Eigenvalues : } -1, 2 \Rightarrow (1,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,2)} = \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}; \text{ Eigenvalues : } -3, -2 \Rightarrow (0,2) \text{ is a stable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(-3,-4)} = \begin{bmatrix} 9 & -3 \\ 4 & -2 \end{bmatrix}; \text{ Eigenvalues : } 7.722, -0.772 \Rightarrow (-3,-4) \text{ is a saddle}$$

(4)

$$0 = x_2, \qquad 0 = -x_1 + x_2(1 - x_1^2 + 0.1x_1^4)$$

There is a unique equilibrium point at $(0,0)$. Determine its type using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 + 0.4x_1^3x_2 & 1 - x_1^2 + 0.1x_1^4 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = (1/2) \pm j\sqrt{3}/2 \;\Rightarrow\; (0,0) \text{ is unstable focus}$$

(5)

$$0 = (x_1 - x_2)(1 - x_1^2 - x_2^2), \qquad 0 = (x_1 + x_2)(1 - x_1^2 - x_2^2)$$

$\{x_1^2 + x_2^2 = 1\}$ is an equilibrium set and $(0,0)$ is an isolated equilibrium point.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \left.\begin{bmatrix} 1 - 3x_1^2 - x_2^2 + 2x_1x_2 & -2x_1x_2 - 1 + x_1^2 + 3x_2^2 \\ 1 - 3x_1^2 - x_2^2 - 2x_1x_2 & -2x_1x_2 + 1 - x_1^2 - 3x_2^2 \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues are $1 \pm j$; hence, $(0,0)$ is unstable focus.

(6)

$$0 = -x_1^3 + x_2, \qquad 0 = x_1 - x_2^3$$

$$x_2 = x_1^3 \;\Rightarrow\; x_1(1 - x_1^8) = 0 \;\Rightarrow\; x_1 = 0 \text{ or } x_1^8 = 1$$

The equation $x_1^8 = 1$ has two real roots at $x_1 = \pm 1$. Thus, there are three equilibrium points at $(0,0)$, $(1,1)$, $(-1,-1)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \text{Eigenvalues} : 1, -1 \;\Rightarrow\; (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(1,1)} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}; \quad \text{Eigenvalues} : -2, -4 \;\Rightarrow\; (1,1) \text{ is a stable node}$$

Similarly, $(-1,-1)$ is a stable node.

- **2.2 (1)**

$$0 = x_2, \qquad 0 = -x_1 + (1/16)x_1^5 - x_2$$

$$x_2 = 0 \;\Rightarrow\; 0 = x_1(x_1^4 - 16) \;\Rightarrow\; x_1 = 0,\ 2,\ \text{or} \ -2$$

There are three equilibrium points at $(0,0)$, $(2,0)$, and $(-2,0)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + (5/16)x_1^4 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{3}/2 \;\Rightarrow\; (0,0) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(2,0)} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm \sqrt{17}/2 \;\Rightarrow\; (2,0) \text{ is a saddle}$$

Similarly, $(-2,0)$ is a saddle.

(2)

$$0 = 2x_1 - x_1x_2, \qquad 0 = 2x_1^2 - x_2$$

$$x_1(2 - x_2) = 0 \;\Rightarrow\; x_1 = 0 \text{ or } x_2 = 2$$

$$x_1 = 0 \;\Rightarrow\; x_2 = 0, \qquad x_2 = 2 \;\Rightarrow\; x_1^2 = 1 \;\Rightarrow\; x_1 = 1 \text{ or } -1$$

There are three equilibrium points at $(0,0)$, $(1,2)$, and $(-1,2)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = 2, -1 \;\Rightarrow\; (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{15}/2 \;\Rightarrow\; (1,2) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{15}/2 \;\Rightarrow\; (-1,2) \text{ is a stable focus}$$

(3)

$$0 = x_2, \qquad 0 = -x_2 - \psi(x_1 - x_2)$$

$$x_2 = 0 \;\Rightarrow\; \psi(x_1) = 0 \;\Rightarrow\; x_1 = 0$$

There is a unique equilibrium point at $(0,0)$. Determine its type by linearization.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -3(x_1 - x_2)^2 - 0.5 & -1 + 3(x_1 - x_2)^2 + 0.5 \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix}$$

The eigenvalues are $-(1/4) \pm j\sqrt{7}/4$. Hence, $(0,0)$ is stable focus.

• **2.3** **(1)** The system has three equilibrium points: $(0,0)$ is a stable focus, $(1,-1)$ and $(-1,1)$ are saddle points. The phase portrait is shown in Figure 2.1. The stable trajectories of the saddle form a lobe around the stable focus. All trajectories inside the lobe converge to the stable focus. All trajectories outside it diverge to infinity.

(2) The system has three equilibrium points: $(0,0)$ is a saddle, $(0,1)$ is unstable node, and $(1,-1)$ is a stable focus. The phase portrait is shown in Figure 2.2. The x_2-axis is a trajectory itself since $x_1(t) \equiv 0 \Rightarrow \dot{x}_1(t) \equiv 0$. The x_2-axis is a separatrix. All trajectories in the right half converge to the stable focus. All trajectories in the left have diverge to infinity. On the x_2-axis itself, trajectories starting at $x_2 < 1$ converge to the origin, while trajectories starting at $x_2 > 1$ diverge to infinity.

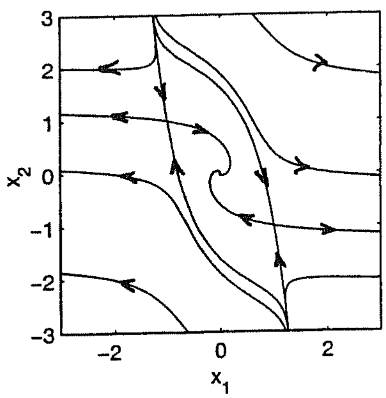

Figure 2.1: Exercise 2.3(1).

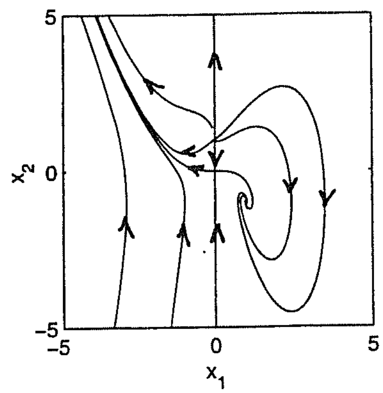

Figure 2.2: Exercise 2.3(2).

(3) The system has four equilibrium points: $(0,0)$ is unstable node, $(1,0)$ is a saddle, $(0,2)$ is a stable node, and $(-3,-4)$ is a saddle. To avoid the condition $x_1 + 1 = 0$, we limit our analysis to the right half of the plane, that is, $\{x_1 \geq 0\}$. This makes sense in view of the fact that the x_2-axis is a trajectory since $x_1(t) \equiv 0 \Rightarrow \dot{x}_1(t) \equiv 0$. Hence, trajectories starting in $\{x_1 \geq 0\}$ stay there for all time. The phase portrait is shown in Figure 2.3. Notice that the x_1-axis is a trajectory since $x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0$. It is a separatrix that divides the half plane $\{x_1 \geq 0\}$ into two quarters. All trajectories starting in the quarter $\{x_1 > 0,\ x_2 > 0\}$ converge to the stable node $(0,2)$. All trajectories starting in the quarter $\{x_1 > 0,\ x_2 < 0\}$ diverge to infinity. Trajectories starting on the x_2-axis approach the stable node $(0,2)$ if $x_2(0) > 0$ and diverge to infinity if $x_2(0) < 0$. Trajectories starting on the x_1-axis with $x_1(0) > 0$ approach the saddle $(1,0)$.

(4) There is a unique equilibrium point at the origin, which is unstable focus. The phase portrait is shown in Figure 2.4. There are two limit cycles. The inner cycle is stable while the outer one is unstable. Al trajectories starting inside the stable limit cycle, except the origin, approach it as t tends to infinity. Trajectories starting in the region between the two limit cycles approach the stable limit cycle. Trajectories starting outside the unstable limit cycle diverge to infinity.

(5) The system has an equilibrium set at the unit circle and unstable focus at the origin. The phase portrait is shown in Figure 2.5. All trajectories, except the origin, approach the unit circle at t tends to infinity.

(6) The system has three equilibrium points: a saddle at $(0,0)$ and stable nodes at $(1,1)$ and $(-1,-1)$. The phase portrait is shown in Figure 2.6. The stable trajectories of the saddle lie on the line $x_1 + x_2 = 0$. All trajectories to the right of this line converge to the stable node $(1,1)$ and all trajectories to its left converge to the stable node $(-1,-1)$. Trajectories on the line $x_1 + x_2 = 0$ itself converge to the origin.

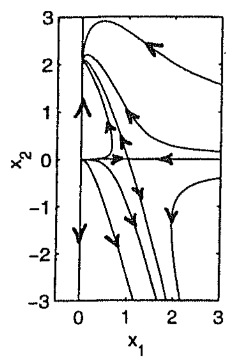

Figure 2.3: Exercise 2.3(3).

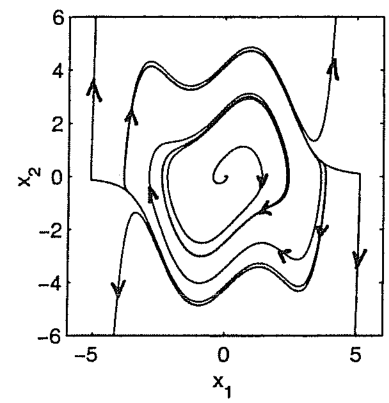

Figure 2.4: Exercise 2.3(4).

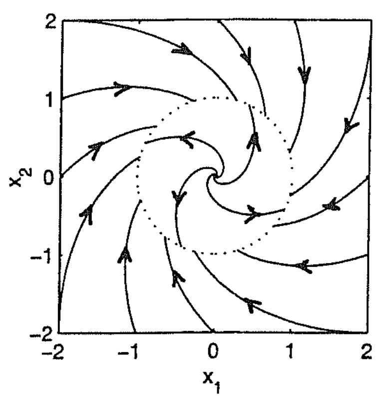

Figure 2.5: Exercise 2.3(5).

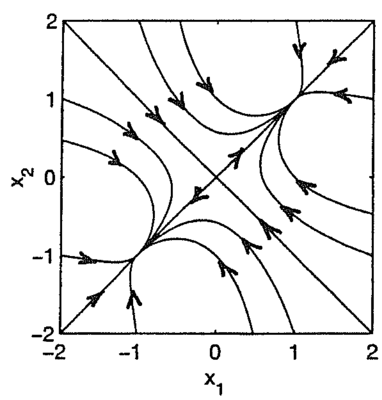

Figure 2.6: Exercise 2.3(6).

• **2.4 (1)** The system has three equilibrium points at $(0,0)$, $(a,0)$, and $(-a,0)$, where a is the root of

$$a = \tan(a/2) \quad \Rightarrow \quad a \approx 2.3311$$

The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \left[\begin{array}{cc} 0 & 1 \\ 1 - 2/[1+(x_1+x_2)^2] & -2/[1+(x_1+x_2)^2] \end{array} \right]$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ -1 & -2 \end{array} \right] \quad \Rightarrow \quad \lambda_{1,2} = -1, \; -1$$

Although we have multiple eigenvalues, we can conclude that the origin is a stable node because $f(x)$ is an analytic function of x in the neighborhood of the origin.

$$\left.\frac{\partial f}{\partial x}\right|_{(2.3311,0)} = \left[\begin{array}{cc} 0 & 1 \\ 0.6892 & -0.3108 \end{array} \right] \quad \Rightarrow \quad \lambda_{1,2} = 0.6892, \; -1 \quad \Rightarrow \quad (2.3311, 0) \text{ is a saddle}$$

Similarly, $(-2.3311, 0)$ is a saddle. The phase portrait is shown in Figure 2.7 with the arrowheads. The stable trajectories of the two saddle points forms two separatrices, which divide the plane into three regions. All trajectories in the middle region converge to the origin as t tends to infinity. All trajectories in the outer regions diverge to infinity.

(2)

$$0 = x_1(2 - x_2), \qquad 0 = 2x_1^2 - x_2$$

From the first equation, $x_1 = 0$ or $x_2 = 2$.

$$x_1 = 0 \;\Rightarrow\; x_2 = 0$$

$$x_2 = 2 \;\Rightarrow\; x_1^2 = 1 \;\Rightarrow\; x_1 = \pm 1$$

There are three equilibrium points at $(0,0)$, $(1,2)$, and $(-1,2)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = 2,\ -1 \;\Rightarrow\; (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{15}/2 \;\Rightarrow\; (1,2) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{15}/2 \;\Rightarrow\; (-1,2) \text{ is a stable focus}$$

The phase portrait is shown in Figure 2.7 with the arrowheads. The stable trajectories of the saddle lie on the x_2-axis. They form a separatrix that divides the plane in two halves. Trajectories in the right half converge to the stable focus $(1,2)$ and those in the half converge to the stable focus $(-1,2)$.

(3) There is a unique equilibrium point at the origin.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = (1/2) \pm j\sqrt{3}/2 \;\Rightarrow\; (-1,2) \text{ is unstable focus}$$

The phase portrait is shown in Figure 2.7 with the arrowheads. There is a stable limit cycle around the origin. All trajectories, except the origin, approach the limit cycle as t tends to infinity.

(4) The equilibrium points are given by the real roots of the equation

$$0 = y^4 - 2y^2 + y$$

where $x_1 = y^2$ and $x_2 = 1 - y$. It can be seen that the equation has four roots at $y = 0, 1, (-1 \pm \sqrt{5})/2$. Hence, there are four equilibrium points at $(0,1)$, $(1,0)$, $((3-\sqrt{5})/2, (3-\sqrt{5})/2)$, and $((3+\sqrt{5})/2, (3+\sqrt{5})/2)$. The following table shows the Jacobian matrix and the type of each point.

Point	Jacobian matrix	Eigenvalues	Type
$(0,1)$	$\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$	$-2.4142,\ 0.4142$	saddle
$(1,0)$	$\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$	$0.4142,\ -2.4142$	saddle
$((3-\sqrt{5})/2, (3-\sqrt{5})/2)$	$\begin{bmatrix} -1.2361 & -1 \\ -1 & -1.2361 \end{bmatrix}$	$-0.2361,\ -2.2361$	stable node
$((3+\sqrt{5})/2, (3+\sqrt{5})/2)$	$\begin{bmatrix} 3.2361 & -1 \\ -1 & 3.2361 \end{bmatrix}$	$4.2361,\ 2.2361$	unstable node

The phase portrait is shown in Figure 2.7 with the arrowheads. The stable trajectories of the saddle points divide the plane into two regions. The region that contains the stable focus has the feature that all trajectories inside it converge to the stable focus. All trajectories in the other region diverge to infinity.

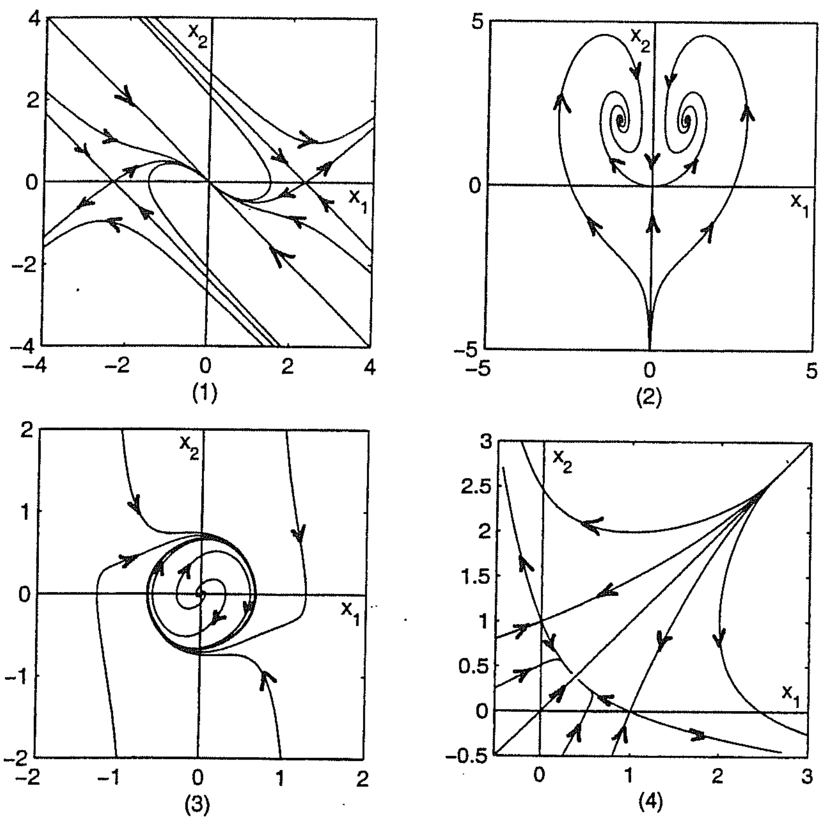

Figure 2.7: Phase portraits of Exercise 2.4.

• **2.5** (a)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + x_1 x_2 \alpha(x) & -\beta(x) + x_2^2 \alpha(x) \\ \beta(x) - x_1^2 \alpha(x) & -1 - x_1 x_2 \alpha(x) \end{bmatrix}$$

where

$$\alpha(x) = \frac{1}{(x_1^2 + x_2^2)(\ln \sqrt{x_1^2 + x_2^2})^2}, \quad \beta(x) = \frac{1}{\ln \sqrt{x_1^2 + x_2^2}}$$

Noting that $\lim_{x \to 0} x_i x_j \alpha(x) = 0$ for $i, j = 1, 2$ and $\lim_{x \to 0} \beta(x) = 0$, it can be seen that

$$\left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{the origin is a stable node}$$

(b) Transform the state equation into the polar coordinates

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

to obtain

$$\dot{r} = -r \Rightarrow r(t) = r_0 e^{-t}$$

and, for $0 < r_0 < 1$,

$$\dot{\theta} = \frac{1}{\ln r} = \frac{1}{\ln r_0 - t} \Rightarrow \theta(t) = \theta_0 - \ln(|\ln r_0| + t) + \ln(|\ln r_0|)$$

Hence, for $0 < r_0 < 1$, $r(t)$ and $\theta(t)$ are strictly decreasing and $\lim_{t\to\infty} r(t) = 0$, $\lim_{t\to\infty} \theta(t) = -\infty$. Thus, the trajectory spirals clockwise toward the origin.
(c) $f(x)$ is continuously differentiable, but not analytic, in the neighborhood of $x = 0$. See the discussion on page 54 of the text.

• **2.6** **(a)** The equilibrium points are the real roots of

$$0 = -x_1 + ax_2 - bx_1x_2 + x_2^2, \qquad 0 = -(a+b)x_1 + bx_1^2 - x_1x_2$$

From the second equation we have

$$x_1[-(a+b) + bx_1 - x_2] = 0 \Rightarrow x_1 = 0 \text{ or } x_2 = -(a+b) + bx_1$$

Substitution of $x_1 = 0$ in the first equation yields

$$x_2(x_2 + a) = 0 \Rightarrow x_2 = 0 \text{ or } x_2 = -a$$

Thus, there are equilibrium points at $(0,0)$ and $(0,-a)$. Substitution of $x_2 = -(a+b) + bx_1$ in the first equation yields

$$0 = b(a+b) - (1+b^2)x_1 = 0 \Rightarrow x_1 = \frac{b(a+b)}{1+b^2} \Rightarrow x_2 = \frac{-(a+b)}{1+b^2}$$

Hence, there is an equilibrium point at $\left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2}\right)$.

(b)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 - bx_2 & a - bx_1 + 2x_2 \\ -(a+b) + 2bx_1 - x_2 & -x_1 \end{bmatrix}$$

1. $x = (0,0)$

$$A = \begin{bmatrix} -1 & a \\ -(a+b) & 0 \end{bmatrix}$$

The eigenvalues of A are

$$\lambda = \frac{-1 \pm \sqrt{1 - 4a(a+b)}}{2}$$

The equilibrium point $(0,0)$ is a stable focus if $4a(a+b) > 1$, a stable node if $0 < 4a(a+b) < 1$, and a saddle if $a(a+b) < 0$.

2. $x = (0,-a)$

$$A = \begin{bmatrix} -1 + ab & -a \\ -b & 0 \end{bmatrix}$$

The eigenvalues of A are $\lambda = ab$ and $\lambda = -1$. The equilibrium point $(0,-a)$ is a saddle if $b > 0$ and a stable node if $b < 0$.

3. $x = \left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2}\right)$

$$A = \frac{1}{1+b^2}\begin{bmatrix} -1 + ab & -b^3 - a - 2b \\ (a+b)b^2 & -b(a+b) \end{bmatrix}$$

The eigenvalues of A are

$$\lambda = \frac{-1 \pm \sqrt{1-4b(a+b)}}{2}$$

The equilibrium point $\left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2}\right)$ is a stable focus if $4b(a+b) > 1$, a stable node if $0 < 4b(a+b) < 1$, and a saddle if $b(a+b) < 0$.

The various cases are summarized in the following table.

	$(0,0)$	$(0,-a)$	$\left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2}\right)$
$b>0,\ 4a(a+b)>1,\ 4b(a+b)>1$	stable focus	saddle	stable focus
$b>0,\ 4a(a+b)>1,\ 4b(a+b)<1$	stable focus	saddle	stable node
$b>0,\ 4a(a+b)<1,\ 4b(a+b)>1$	stable node	saddle	stable focus
$b>0,\ 4a(a+b)<1,\ 4b(a+b)<1$	stable node	saddle	stable node
$b<0,\ a+b>0,\ 4a(a+b)>1$	stable focus	stable node	saddle
$b<0,\ a+b>0,\ 4a(a+b)<1$	stable node	stable node	saddle
$b<0,\ a+b<0,\ 4b(a+b)>1$	saddle	stable node	stable focus
$b<0,\ a+b<0,\ 4a(a+b)<1$	saddle	stable node	stable node

If any one of the above conditions holds with equality rather than inequality, we end up with multiple eigenvalues or eigenvalues with zero real parts, in which case linearization fails to determine the type of the equilibrium point of the nonlinear system.

(c) The phase portraits of the three cases are shown in Figures 2.8 through 2.10.

i $a=b=1$. The equilibrium points are

$(0,0)$	stable focus
$(0,-1)$	saddle
$(1,-1)$	stable focus

The linearization at the saddle is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. The stable eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the unstable eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. They are used to generate the stable and unstable trajectories of the saddle. The stable trajectories form a separatrix that divides the plane into two halves, with all trajectories in the right half approaching $(1,-1)$ and all trajectories in the left half approaching $(0,0)$.

ii $a=1$, $b=-\frac{1}{2}$. The equilibrium points are

$(0,0)$	stable focus
$(0,-1)$	stable node
$(\frac{-1}{5}, \frac{-2}{5})$	saddle

The linearization at the saddle is A, where $(1+b^2)A = \begin{bmatrix} -(3/2) & (1/8) \\ (1/8) & (1/4) \end{bmatrix}$. The stable eigenvector is $\begin{bmatrix} 0.9975 \\ -0.0709 \end{bmatrix}$ and the unstable eigenvector is $\begin{bmatrix} 0.0709 \\ 0.9975 \end{bmatrix}$. They are used to generate the stable and unstable trajectories of the saddle. The stable trajectories form a separatrix in the form of a lobe. All trajectories outside the lobe approach $(0,-1)$; all trajectories inside the lobe approach $(0,0)$.

iii $a=1$, $b=-2$. The equilibrium points are

$(0,0)$	saddle
$(0,-1)$	stable node
$(\frac{2}{5}, \frac{1}{5})$	stable focus

The linearization at the saddle is $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. The stable eigenvector is $\begin{bmatrix} 0.8507 \\ -0.5257 \end{bmatrix}$ and the unstable eigenvector is $\begin{bmatrix} 0.5257 \\ 0.8507 \end{bmatrix}$. They are used to generate the stable and unstable trajectories of the saddle. The stable trajectories form a separatrix in the form of a lobe. All trajectories outside the lobe approach $(0, -1)$; all trajectories inside the lobe approach $\left(\frac{2}{5}, \frac{1}{5}\right)$.

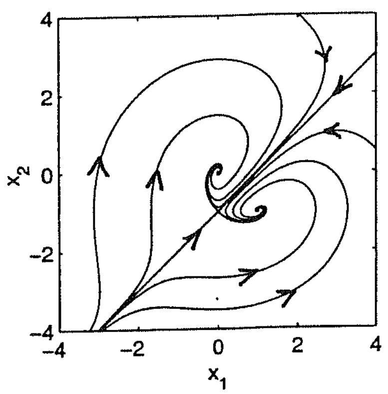

Figure 2.8: Exercise 2.6(i).

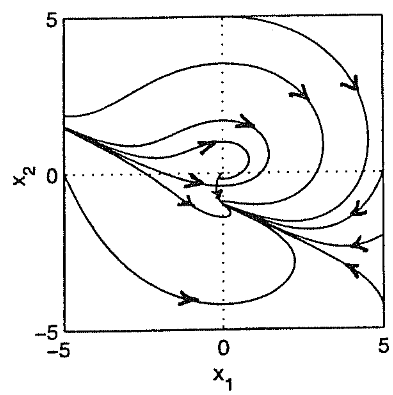

Figure 2.9: Exercise 2.6(ii).

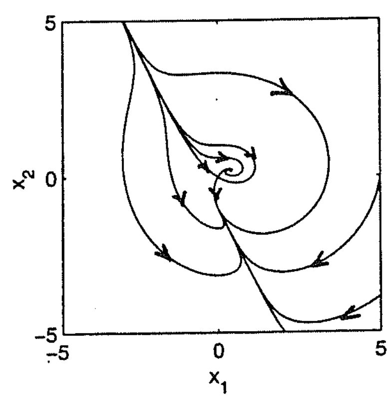

Figure 2.10: Exercise 2.6(iii).

• **2.7** The system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2\left(-1 + 3x_1^2 - x_1^4 + \tfrac{1}{15}x_1^6\right)$$

has a unique equilibrium point at the origin. Linearization at the origin yields $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ whose eigenvalues are $0.5 \pm 0.866j$. Hence the origin is unstable focus. The phase portrait is shown in Figure 2.11. There are three limit cycles. The inner limit cycle is stable, the middle one is unstable, and the outer one is stable. All trajectories starting inside the middle limit cycle, other than the origin, approach the inner limit cycle as t tends to infinity. All trajectories starting outside the middle limit cycle approach the outer limit cycle as t tends to infinity. Trajectories starting at the unstable focus or on the unstable limit cycle remain there.

• **2.8** **(a)** The equilibrium points are the real roots of

$$0 = x_2, \qquad 0 = -x_1 + \tfrac{1}{16}x_1^5 - x_2$$

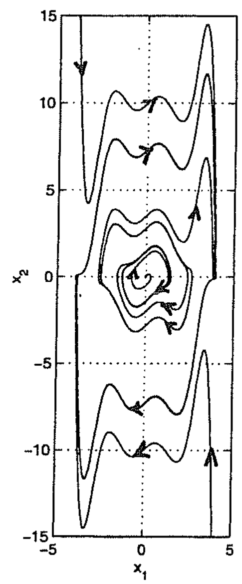

Figure 2.11: Exercise 2.7.

Hence

$$x_1(16 - x_1^4) = 0 \Rightarrow x_1 = 0, \pm 2$$

There are three equilibrium points at $(0,0)$, $(2,0)$, and $(-2,0)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{5}{16}x_1^4 & -1 \end{bmatrix}$$

$$x = (0,0) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm j\sqrt{3}}{2}$$

$(0,0)$ is a stable focus.

$$x = (2,0) \text{ or } (-2,0) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm \sqrt{17}}{2}$$

$(2,0)$ and $(-2,0)$ are saddle points.

(b) The phase portrait can be sketched by constructing a vector field diagram and using the information about the equilibrium points, especially the directions of the stable and unstable trajectories at the saddle points. The stable and unstable eigenvectors of the linearization at the saddle points are

$$v_{stable} = \begin{bmatrix} 1 \\ \frac{-1-\sqrt{17}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.5616 \end{bmatrix}, \quad v_{unstable} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{17}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5616 \end{bmatrix}$$

Find the directions of the vector fields on the two axes. On $x_1 = 0$, $f = \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix}$. Hence the vector field makes an angle -45 deg with the x_1 axis and its magnitude increases with $|x_2|$. On $x_2 = 0$, $f =$

$\begin{bmatrix} 0 \\ -x_1 + \frac{x_1^4}{16} \end{bmatrix}$. Hence the vector field is parallel to the x_2-axis. The sketch can be improved by finding the vector field at other points.

• **2.9** **(a)**

$$\dot{x}_1 = v_d - x_2, \qquad \dot{x}_2 = \frac{1}{m}\left[K_I x_1 + K_P(v_d - x_2) - K_c \,\text{sgn}(x_2) - K_f x_2 - K_a x_2^2\right]$$

(b) At the equilibrium points, we have

$$0 = v_d - x_2, \qquad 0 = K_I x_1 + K_P(v_d - x_2) - K_c \,\text{sgn}(x_2) - K_f x_2 - K_a x_2^2$$

From the first equation, $x_2 = v_d$, and from the second one, $x_1 = (K_c + K_f v_d + K_a v_d^2)/K_I$. This is the only equilibrium point. Linearization at the equilibrium point yields

$$A = \begin{bmatrix} 0 & -1 \\ K_I/m & -(K_f + 2K_a v_d + K_P)/m \end{bmatrix}$$

whose eigenvalues are

$$\lambda = \frac{-(K_f + 2K_a v_d + K_P)/m \pm \sqrt{(K_f + 2K_a v_d + K_P)^2/m^2 - 4K_I/m}}{2}$$

If $(K_f + 2K_a v_d + K_P)^2 > 4mK_I$, the equilibrium is a stable node, and if $(K_f + 2K_a v_d + K_P)^2 < 4mK_I$, the equilibrium is a stable focus.

(c) For the given numerical values, the eigenvalues of the linearization are -0.0289 and -0.3461. Hence, the equilibrium point is a stable node. The phase portrait is shown in Figure 2.12. All trajectories approach the stable node along the slow eigenvector of the node, which has a small slope. Starting from different initial speeds, the trajectory reaches the desired speed with no (or very little) overshoot.

(d) The eigenvalues of the linearization are $-0.1875 \pm 0.2546j$; hence the equilibrium point is a stable focus. The phase portrait is shown in Figure 2.13. All trajectories approach the stable focus. Notice the increased overshoot compared with the previous case. For example, starting at the initial state ($x_1 = 15,\ x_2 = 10$), the speed reaches about 36 m/sec before approaching the steady-state of 30 m/sec.

(e) The phase portrait is shown in Figure 2.14. The local behavior near the equilibrium point is not affected since saturation will not be effective. However, far from the equilibrium point we can see that the state of the integrator, x_1, takes large values during saturation, resulting in an increased overshoot.

• **2.10** **(a)** Using the same scaling as in Example 2.1, the state equation is given by

$$\dot{x}_1 = 0.5[-h(x_1) + x_2], \qquad \dot{x}_2 = 0.2(-x_1 - 0.2x_2 + 0.2)$$

where $h(x_1)$ is given in Example 2.1. The equilibrium points are the intersection points of the curves $x_2 = h(x_1)$ and $x_2 = 1 - 5x_1$. Figure 2.15 shows that there is a unique equilibrium point. Using the "roots" command of MATLAB, the equilibrium point was determined to be $\bar{x} = (0.057, 0.7151)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.04 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} -1.0461 & 0.5 \\ -0.2 & -0.04 \end{bmatrix} \Rightarrow \text{Eigenvalues} = -0.9343, -0.1518 \Rightarrow \text{stable node}$$

(b) The phase portrait is shown in Figure 2.16. All trajectories approach the stable node. This circuit is known as "monostable" because it has one steady-state operating point.

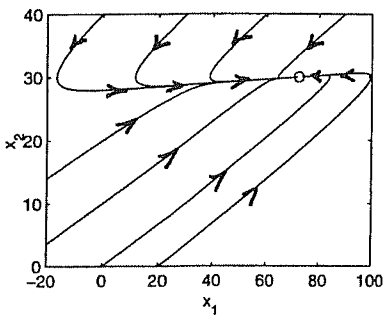

Figure 2.12: Exercise 2.9(c).

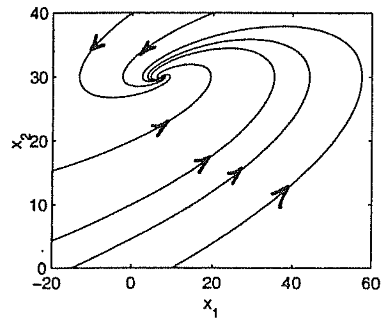

Figure 2.13: Exercise 2.9(d).

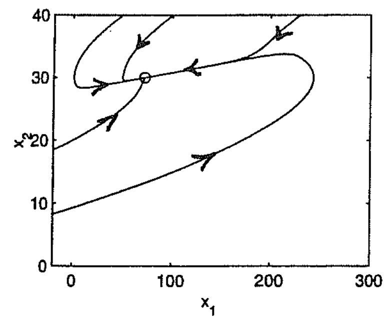

Figure 2.14: Exercise 2.9(e).

• **2.11** **(a)** Using the same scaling as in Example 2.1, the state equation is given by

$$\dot{x}_1 = 0.5[-h(x_1) + x_2], \qquad \dot{x}_2 = 0.2(-x_1 - 0.2x_2 + 0.4)$$

where $h(x_1)$ is given in Example 2.1. The equilibrium points are the intersection points of the curves $x_2 = h(x_1)$ and $x_2 = 2 - 5x_1$. Figure 2.17 shows that there is a unique equilibrium point. Using the "roots" command of MATLAB, the equilibrium point was determined to be $\bar{x} = (0.2582, 0.7091)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 1.8173 & 0.5 \\ -0.2 & -0.04 \end{bmatrix} \Rightarrow \text{Eigenvalues} = 1.7618, 0.0155 \Rightarrow \text{unstable node}$$

(b) The phase portrait is shown in Figure 2.18. The circuit has a stable limit cycle. All trajectories, except the constant solution at the equilibrium point, approach the limit cycle. This circuit is known as "astable."

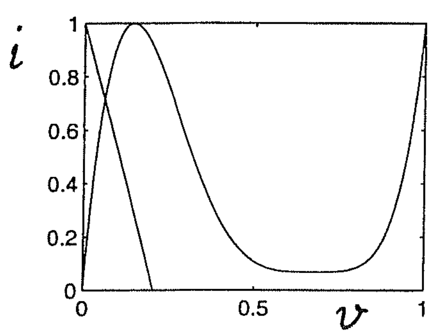

Figure 2.15: Exercise 2.10: equilibrium point.

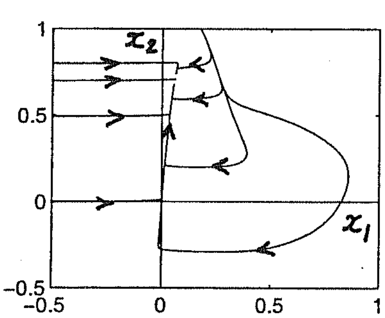

Figure 2.16: Exercise 2.10: phase portrait.

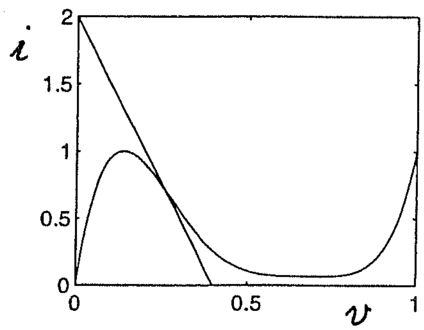

Figure 2.17: Exercise 2.11: equilibrium point.

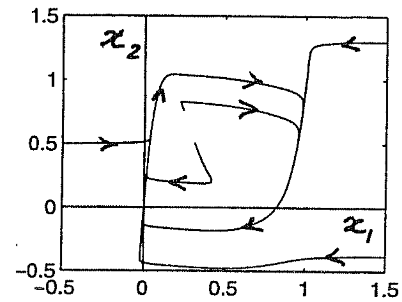

Figure 2.18: Exercise 2.11: phase portrait.

• **2.12** **(a)** Note that $T_{12} = T_{21} = 1 \Rightarrow \frac{1}{R_{12}} = \frac{1}{R_{21}} = 1$ and $T_{11} = T_{22} = 0 \Rightarrow \frac{1}{R_{11}} = \frac{1}{R_{22}} = 0$. Hence the state equation is given by

$$\dot{x}_1 = h(x_1)[x_2 - 2\eta(x_1)] \stackrel{\text{def}}{=} f_1(x), \qquad \dot{x}_2 = h(x_2)[x_1 - 2\eta(x_2)] \stackrel{\text{def}}{=} f_2(x)$$

where $h(x) = \lambda\cos^2(\pi x/2)$ and $\eta(x) = g^{-1}(x) = (2/\pi\lambda)\tan(\pi x/2)$. Equilibrium points are the intersection points of the curves $x_2 = 2\eta(x_1)$ and $x_1 = 2\eta(x_2)$. Note that $\eta'(0) = 1/\lambda$ and $\eta'(x) = (1/\lambda)\sec^2(\pi x/2) \geq 1/\lambda$. Therefore, for $\lambda \leq 2$, the two curves intersect only at the origin $(0,0)$. For $\lambda > 2$, there are three intersection points at $(0,0)$, (a,a) and $(-a,-a)$ where $0 < a < 1$ depends on λ. This fact can be seen be sketching the curves and using symmetry; see Figure 2.19. The partial derivatives of f_1 and f_2 are given by

$$\frac{\partial f_1}{\partial x_1} = h'(x_1)[x_2 - 2\eta(x_1)] - 2h(x_1)\eta'(x_1), \qquad \frac{\partial f_1}{\partial x_2} = h(x_1)$$

$$\frac{\partial f_2}{\partial x_1} = h(x_2), \qquad \frac{\partial f_2}{\partial x_2} = h'(x_2)[x_1 - 2\eta(x_2)] - 2h(x_2)\eta'(x_2)$$

At equilibrium points, $[x_2 - 2\eta(x_1)] = 0$ and $[x_1 - 2\eta(x_2)] = 0$. Therefore, the Jacobian matrix reduces to

$$\left.\frac{\partial f}{\partial x}\right|_{x=(b,b)} = h(b)\begin{bmatrix} -2\eta'(b) & 1 \\ 1 & -2\eta'(b) \end{bmatrix}$$

where $b = 0$, a, or $-a$, depending on the equilibrium point.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} -2 & \lambda \\ \lambda & -2 \end{bmatrix} \Rightarrow \text{Eigenvalues} = -2 \pm \lambda$$

For $\lambda < 2$, the unique equilibrium point at $(0,0)$ is a stable node. For $\lambda > 2$, the equilibrium point $(0,0)$ is a saddle. For $\lambda > 2$ there are two other equilibrium points at (a,a) and $(-a,-a)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(a,a)} = h(a)\begin{bmatrix} -2\eta'(a) & 1 \\ 1 & -2\eta'(a) \end{bmatrix} \Rightarrow \text{Eigenvalues} = h(a)[-2\eta'(a) \pm 1]$$

It is not hard to see from the sketch of the curves $x_2 = 2\eta(x_1)$ and $x_1 = 2\eta(x_2)$ that at the intersection point (a,a), the slope $2\eta'(a) > 1$. Hence, (a,a) is a stable node. Similarly, it can be shown that $(-a,-a)$ is a stable node.

(b) The phase portrait is shown in Figure 2.20. The stable trajectories of the saddle point at the origin form a separatrix that divides the plane into two regions. Trajectories in each region approach the stable node in that region.

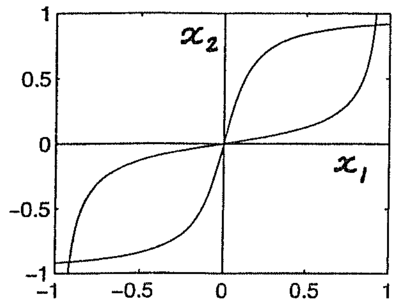

Figure 2.19: Exercise 2.12: equilibrium point.

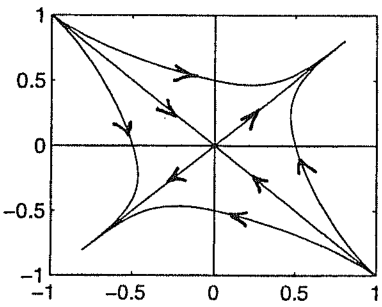

Figure 2.20: Exercise 2.12: phase portrait.

• **2.13** **(a)** Using Kirchhoff's Voltage law, we obtain

$$\dot{x}_1 = \frac{1}{C_1R_1}[v_2 - v_1 - g(v_2)]$$

Using Kirchhoff's Current law, we obtain

$$\dot{x}_2 = \frac{1}{C_2}\left\{-\frac{v_2}{R_2} - \frac{1}{R_1}[v_2 - v_1 - g(v_2)]\right\}$$

Thus, the state equation is

$$\begin{aligned}\dot{x}_1 &= \frac{1}{C_1R_1}[-x_1 + x_2 - g(x_2)]\\ \dot{x}_2 &= \frac{1}{C_2R_1}x_1 - \frac{1}{C_2R_2}x_2 - \frac{1}{C_2R_1}x_2 + \frac{1}{C_2R_1}g(x_2)\end{aligned}$$

(b) For the given data, the state equation is given by

$$\dot{x}_1 = -x_1 + x_2 - g(x_2), \quad \dot{x}_2 = x_1 - 2x_2 + g(x_2)$$

$$g(x_2) = 3.234x_2 - 2.195x_2^3 + 0.666x_2^5$$

The system has a unique equilibrium point at the origin. The Jacobian at the origin is given by

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} -1 & -2.234 \\ 1 & 1.234 \end{bmatrix} \Rightarrow \text{Eigenvalues } = 0.117 \pm 0.993j$$

Hence the origin is an unstable focus. The phase portrait is shown in Figures 2.21 and 2.22 using two different scales. The system has two limit cycles. The inner limit cycle is stable, while the outer one is unstable. All trajectories starting inside the outer limit cycle, except the origin, approach the inner one. All trajectories starting outside the outer limit cycle diverge to infinity.

• **2.14** The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -kx_1 - cx_2 - \eta(x_1, x_2)\end{aligned}$$

where

$$\eta(x_1, x_2) = \begin{cases} \mu_k mg \ \text{sign}(x_2), & \text{for } |x_2| > 0 \\ -kx_1, & \text{for } x_2 = 0 \& |x_1| \le \mu_s mg/k \\ -\mu_s mg \ \text{sign}(x_1), & \text{for } x_2 = 0 \& |x_1| > \mu_s mg/k \end{cases}$$

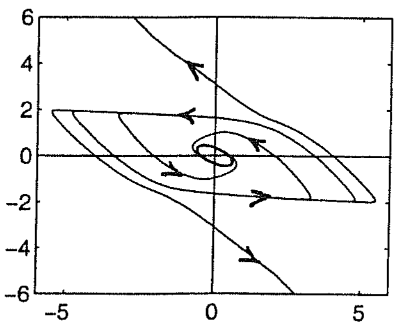

Figure 2.21: Exercise 2.13.

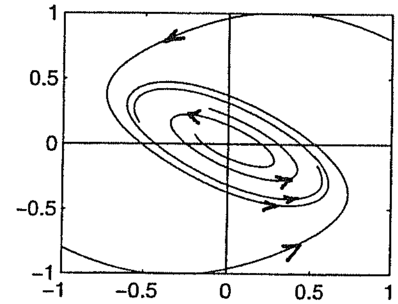

Figure 2.22: Exercise 2.13.

For $x_2 > 0$, the state equation is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -kx_1 - cx_2 - \mu_k mg\end{aligned}$$

while, for $x_2 < 0$, it is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -kx_1 - cx_2 + \mu_k mg\end{aligned}$$

In each half, we can determine the trajectories by studying the respective linear equation. Let us start with $x_2 > 0$. The linear state equation has an equilibrium point at $(-\mu_k mg/k, 0)$. Shifting the equilibrium to the origin, we obtain a linear state equation with the matrix $\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}$, whose characteristic equation is $\lambda^2 + c\lambda + k = 0$, where k and c are positive constants. The equilibrium point is a stable focus when $4k > c^2$ and a stable node when $4k \leq c^2$. We shall continue our discussion assuming $4k > c^2$. Trajectories would tend to spiral toward the equilibrium point $(-\mu_k mg/k, 0)$. It will not actually spiral toward the point because the equation is valid only for $x_2 > 0$. Thus, for any point in the upper half, we can solve the linear equation to find the trajectory that should spiral toward the equilibrium point, but follow the trajectory only until it hits the x_1-axis. For $x_2 < 0$, we have a similar situation except that trajectories tend to spiral toward the point $(\mu_k mg/k, 0)$. On the x_1-axis itself, we should distinguish between two regions. If a trajectory hits the x_1-axis within the interval $[-\mu_s mg/k, \mu_s mg/k]$, it will rest at equilibrium. If it hits outside this interval, it will have $\dot{x}_2 \neq 0$ and will continue motion. Notice that trajectories reaching the x_1-axis in the interval $x_1 > \mu_s mg/k$ will be coming from the upper half of the plane and will continue their motion into the lower half. By symmetry, trajectories reaching the x_1-axis in the interval $x_1 < -\mu_s mg/k$ will be coming from the lower half of the plane and will continue their motion into the upper half. Thus, a trajectory starting far from the origin, will spiral toward the origin, until it hits the x_1-axis within the interval $[-\mu_s mg/k, \mu_s mg/k]$. The phase portrait is sketched in Figure 2.23.

• **2.15** The solution of the state equation

$$\begin{aligned}\dot{x}_1 &= x_2, \quad x_1(0) = x_{10}\\ \dot{x}_2 &= k, \quad x_2(0) = x_{20}\end{aligned}$$

where $k = \pm 1$, is given by

$$\begin{aligned}x_2(t) &= kt + x_{20}\\ x_1(t) &= \tfrac{1}{2}kt^2 + x_{20}t + x_{10}\end{aligned}$$

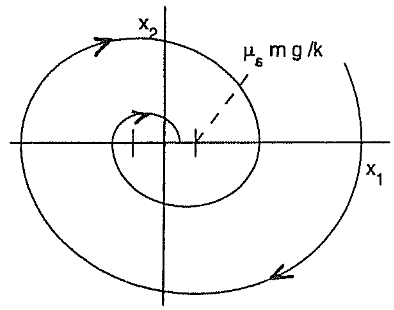

Figure 2.23: Exercise 2.14.

Eliminating t between the two equations, we obtain

$$x_1 = \frac{1}{2k}x_2^2 + c$$

where $c = x_{10} - x_{20}^2/(2k)$. This is the equation of the trajectories in the x_1–x_2 plane. Different trajectories correspond to different values of c. Figures 2.24 and 2.25 show the phase portraits for $u = 1$ and $u = -1$, respectively. The two portraits are superimposed in Figure 2.26. From Figure 2.26 we see that trajectories can reach the origin through only two curves, which are highlighted. The curve in the lower half corresponds to $u = 1$ and the curve in the upper half corresponds to $u = -1$. We will refer to these curves as the switching curves. To move any point in the plane to the origin, we can switch between ± 1. For example, to move the point A to the origin, we apply $u = -1$ until the trajectory hits the switching curve, then we switch to $u = 1$. Similarly, to move the point B to the origin, we apply $u = 1$ until the trajectory hits the switching curve, then we switch to $u = -1$. When the trajectory reaches the origin we can keep it there by switching to $u = 0$ which makes the origin an equilibrium point.

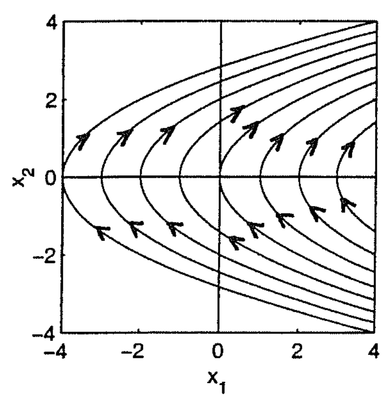

Figure 2.24: Exercise 2.15.

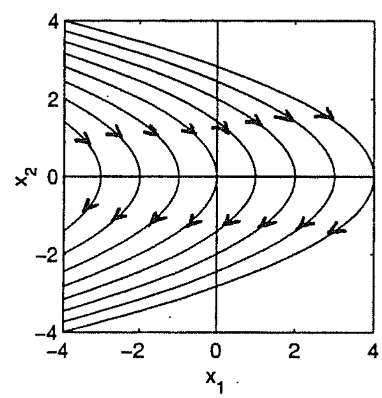

Figure 2.25: Exercise 2.15.

• **2.16** **(a)** The equilibrium points are the roots of

$$0 = x_1(1 - x_1 - ax_2), \qquad 0 = bx_2(x_1 - x_2)$$

From the first equation, we have $x_1 = 0$ or $x_1 = 1 - ax_2$. Substitution of $x_1 = 0$ in the second equation results in $x_2 = 0$. Substitution of $x_1 = 1 - ax_2$ in the second equation results in $x_2(1 - ax_2 - x_2) = 0$ which yields

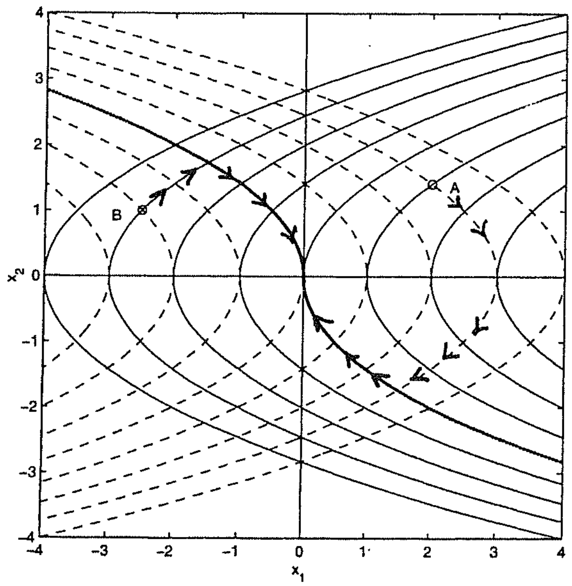

Figure 2.26: Exercise 2.15.

$x_2 = 0$ or $x_2 = 1/(1+a)$. Thus, there are three equilibrium points at $(0,0)$, $(1,0)$, and $(1/(1+a),\ 1/(1+a))$. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1-2x_1-ax_2 & -ax_1 \\ bx_2 & bx_1-2bx_2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,0)} = \begin{bmatrix} -1 & -a \\ 0 & b \end{bmatrix}, \quad \left.\frac{\partial f}{\partial x}\right|_{\left(\frac{1}{1+a},\frac{1}{1+a}\right)} = \frac{1}{1+a}\begin{bmatrix} -1 & -a \\ b & -b \end{bmatrix}$$

At the equilibrium point $(0,0)$ the matrix has a zero eigenvalue; hence linearization fails to determine the type of the equilibrium point. At $(1,0)$, the equilibrium point is a saddle. At $\left(\frac{1}{1+a},\frac{1}{1+a}\right)$, the eigenvalues are

$$\lambda_{1,2} = \frac{-(1+b) \pm \sqrt{1-2b+b^2-4ab}}{2(1+a)}$$

Hence, $\left(\frac{1}{1+a},\frac{1}{1+a}\right)$ is a stable node if $1-2b+b^2-4ab > 0$ and a stable focus if $1-2b+b^2-4ab < 0$.

The phase portrait is shown in Figure 2.27. The equilibrium point $(\frac{1}{2},\frac{1}{2})$ is a stable focus that attracts all trajectories in the first quadrant except trajectories on the x_1-axis or the x_2 axis. Trajectories on the x_1-axis move on it approaching the saddle point at $(1,0)$. Motion on the x_1-axis corresponds to the case when there are no predators, in which case the prey population settles at $x_1 = 1$. Motion on the x_2 axis corresponds to the case when there are no preys, in which the case the predator population settles at $x_2 = 0$; i.e., the predators vanish. In the presence of both preys and predators, their populations reach a balance at the equilibrium point $(\frac{1}{2},\frac{1}{2})$.

• **2.17** **(1)** Assume $\varepsilon > 0$ and let $x_1 = y$, $x_2 = \dot{y}$, and $V(x) = x_1^2 + x_2^2$.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2)$$

$$f(x) \cdot \nabla V(x) = 2\varepsilon x_2^2(1 - x_1^2 - x_2^2) = 2\varepsilon x_2^2(1 - V)$$

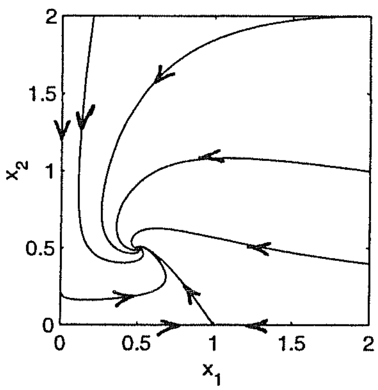

Figure 2.27: Exercise 2.16.

Hence, $f(x) \cdot \nabla V(x) \leq 0$ for $V(x) \geq 1$. In particular, all trajectories starting in $M = \{V(x) \leq 1\}$ stay in M for all future time. M contains only one equilibrium point at the origin. Linearization at the origin yields the matrix $\begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$. Hence, the origin is unstable node or unstable focus. By the Poincaré-Bendixson criterion, there is a periodic orbit in M.

(2) Let $V(x) = x_1^2 + x_2^2$.

$$f(x) \cdot \nabla V(x) = 2x_2^2(2 - 3x_1^2 - 2x_2^2) = 4x_2^2(1 - x_1^2 - x_2^2) - 2x_1^2x_2^2 \leq 4x_2^2(1 - x_1^2 - x_2^2)$$

Hence, $f(x) \cdot \nabla V(x) \leq 0$ for $x_1^2 + x_2^2 \geq 1$. In particular, all trajectories starting in $M = \{V(x) \leq 1\}$ stay in M for all future time. M contains only one equilibrium point at the origin. Linearization at the origin yields the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, whose eigenvalues are 1 and 1. Since $f(x)$ is an analytic function of x, we conclude that the origin is unstable node. By the Poincaré-Bendixson criterion, there is a periodic orbit in M.

(3) Let $V(x) = 3x_1^2 + 2x_1x_2 + 2x_2^2$.

$$\begin{aligned} f(x) \cdot \nabla V(x) &= (6x_1 + 2x_2)x_2 + (2x_1 + 4x_2)[-x_1 + x_2 - 2(x_1 + 2x_2)x_2^2] \\ &= -2x_1^2 + 4x_1x_2 + 6x_2^2 - 4(x_1 + 2x_2)^2x_2^2 \\ &= -2(x_1^2 + x_2^2) + 4x_2(x_1 + 2x_2) - 4(x_1 + 2x_2)^2x_2^2 \\ &= -2(x_1^2 + x_2^2) + 1 - [1 - 2x_2(x_1 + 2x_2)]^2 \\ &\leq -2(x_1^2 + x_2^2) + 1 \leq 0, \quad \text{for } x_1^2 + x_2^2 \geq \tfrac{1}{2} \end{aligned}$$

Choose a constant $c > 0$ such that the surface $V(x) = c$ contains the circle $\{x_1^2 + x_2^2 = \frac{1}{2}\}$ in its interior. Then, all trajectories starting in $M = \{V(x) \leq c\}$ stay in M for all future time. M contains only one equilibrium point at the origin. Linearization at the origin yields the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, whose eigenvalues are $(1 \pm j\sqrt{3})/2$. Hence, the origin is unstable focus. By the Poincaré-Bendixson criterion, there is a periodic orbit in M.

(4) The equilibrium points are the roots of

$$0 = x_1 + x_2 - x_1 \max\{|x_1|, |x_2|\}, \qquad 0 = -2x_1 + x_2 - x_2 \max\{|x_1|, |x_2|\}$$

From the first equation, we have $x_2 = x_1(\max\{|x_1|, |x_2|\} - 1)$. Substitution in the second equation results in

$$0 = -2x_1 - x_1(\max\{|x_1|, |x_2|\} - 1)^2 = -x_1[2 + (\max\{|x_1|, |x_2|\} - 1)^2] \Rightarrow x_1 = 0 \Rightarrow x_2 = 0$$

Hence there is a unique equilibrium point at the origin. Linearization at the origin yields the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ whose eigenvalues are $1 \pm j1.4142$. Hence, the origin is an unstable focus. Now consider $V(x) = x_1^2 + x_2^2$.

$$\begin{aligned} \nabla V \cdot f &= 2x_1(x_1 + x_2 - x_1 \max\{|x_1|, |x_2|\}) + 2x_2(-2x_1 + x_2 - x_2 \max\{|x_1|, |x_2|\}) \\ &= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2(x_1^2 + x_2^2)\max\{|x_1|, |x_2|\} \\ &= 2\left[x^TPx - \|x\|_2^2 \max\{|x_1|, |x_2|\}\right], \quad \text{where } P = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \end{aligned}$$

The matrix P is positive definite with maximum eigenvalue 1.5. Therefore,

$$\nabla V \cdot f \leq 2[1.5\|x\|_2^2 - \|x\|_2^2 \max\{|x_1|, |x_2|\}] < 0, \quad \text{for } \max\{|x_1|, |x_2|\} > 1.5$$

Thus, by choosing c large enough, $\nabla V \cdot f$ will be negative on the surface $\{V(x) = c\}$. Hence, all trajectories starting in the set $M = \{V(x) \leq c\}$ stay in M for all future time and M contains a single equilibrium point which is unstable focus. It follows from the Poincaré-Bendixson's criterion that there is a periodic orbit in M.

• **2.18**

(a)

$$\dot{V} = x_2\dot{x}_2 + g(x_1)\dot{x}_1 = -x_2g(x_1) + x_2g(x_1) = 0$$

(b) For small $c > 0$, the equation $V(x) = c$ defines a closed curve that encloses the origin. Since $\dot{V} = 0$, a trajectory starting on the curve must remain on the curve for all t. Moreover, from $\dot{x}_1 = x_2$, we see that the trajectory can only move in the clockwise direction. Hence, a trajectory starting at any point on the closed curve $V(x) = c$ must move around the curve until it comes back to the starting point. Thus, the trajectory is a periodic orbit.

(c) Extension of (b) because $V(x) = c$ is a closed curve for all $c > 0$.

(d) $V(x) = \frac{1}{2}x_2^2 + G(x_1) = \text{constant}$. At $x = (A, 0)$, $V = G(A)$. Thus

$$\tfrac{1}{2}x_2^2(t) + G(x_1(t)) \equiv G(A) \Rightarrow x_2(t) \equiv \pm\sqrt{2[G(A) - G(x_1(t))]}$$

(e) Starting from $\dot{x}_1 = x_2$, we have

$$dt = \frac{dx_1}{\sqrt{2[G(A) - G(x_1)]}}$$

for $x_2 \geq 0$. Calculating the line integral of the right-hand side in the upper half of the plane from $(-A, 0)$ to $(A, 0)$, we obtain

$$\frac{T}{2} = \int_{-A}^{A} \frac{dy}{\sqrt{2[G(A) - G(y)]}} \Rightarrow T = 2\sqrt{2}\int_0^A \frac{dy}{\sqrt{G(A) - G(y)}}$$

where we have used the fact that $G(x_1)$ is an even function.

(f) We can generate the trajectories using the equation in part (d). For each value of A, we solve the equation to find x_2 as a function of x_1. The function $G(x_1)$ has a minimum, a maximum, or a point of inflection at each equilibrium point of the system. In particular, It has a minimum at $x_1 = 0$ corresponding to the equilibrium point at the origin. Starting from small values of A, the equation will have a solution defining a closed orbit. As we increase the value of A, the equation will continue to define a closed orbit until A reaches the level of a maximum point of $G(x_1)$. For values of A higher than the maximum, the curves will not be closed. Depending on the shape of $G(x_1)$, the equation may have multiple solutions defining trajectories in different parts of the plane. The conditions of part(c) ensure that $G(x_1)$ will have a global minimum at $x_1 = 0$.

• **2.19** The phase portraits can be generated by solving the equation of the previous exercise either graphically or using a computer. We will only give the function $G(x_1)$ and calculate the period of the trajectory through $(1,0)$.

(1)

$$G(y) = \int_0^y \sin z \; dz = 1 - \cos y$$

$$T = 2\sqrt{2} \int_0^1 \frac{dy}{\sqrt{\cos y - \cos 1}}$$

(2)

$$G(y) = \int_0^y (z + z^3) \; dz = \tfrac{1}{2}y^2 + \tfrac{1}{4}y^4$$

$G(y) \to \infty$ as $|y| \to \infty$ and $zg(z) \geq 0$ for all z. Hence, every solution is periodic.

$$T = 2\sqrt{2} \int_0^1 \frac{dy}{\sqrt{\frac{3}{4} - \frac{1}{2}y^2 - \frac{1}{4}y^4}}$$

(3)

$$G(y) = \int_0^y z^3 \; dz = \tfrac{1}{4}y^4$$

$G(y) \to \infty$ as $|y| \to \infty$ and $zg(z) \geq 0$ for all z. Hence, every solution is periodic.

$$T = 2\sqrt{2} \int_0^1 \frac{dy}{\sqrt{\frac{1}{4} - \frac{1}{4}y^4}}$$

• **2.20**

(1)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 + a \neq 0$$

By Bendixson's criterion, there are no periodic orbits.

(2) The equilibrium points are the roots of

$$0 = x_1(-1 + x_1^2 + x_2^2), \qquad 0 = x_2(-1 + x_1^2 + x_2^2)$$

The system has an isolated equilibrium point at the origin and a continuum of equilibrium points on the unit circle $x_1^2 + x_2^2 = 1$. It can be checked that the origin is a stable node. Transform the system into the polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. It can be verified that

$$\dot{r} = -r(1 - r^2)$$

For $r < 1$, every trajectory starting inside the unit circle approaches the origin as $t \to \infty$. For $r > 1$, every trajectory starting outside the unit circle escapes to ∞ as $t \to \infty$. Thus, there are no limit cycles.

(3) The equilibrium points of the system are the roots of

$$0 = 1 - x_1 x_2^2, \qquad 0 = x_1$$

These equations have no real roots. Thus, there are no equilibrium points. Since, by Corollary 2.1, a closed orbit must enclose an equilibrium point, we conclude that there are no closed orbits.

(4) The x_1-axis is an equilibrium set. Therefore, a periodic orbit cannot cross the x_1-axis; it must lie entirely in the upper or lower halves of the plane. However, there are no equilibrium points other than the x_1-axis. Since, by Corollary 2.1, a periodic orbit must enclose an equilibrium point, we conclude that there are no periodic orbits.

(5) The equilibrium points are the roots of

$$0 = x_2 \cos x_1, \quad 0 = \sin x_1$$

The equilibrium point are $(\pm n\pi, 0)$ for $n = 0, 1, 2, \cdots$. Linearization at the equilibrium points yields the matrix $\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ where $a = \pm 1$. Hence, all equilibrium points are saddles. Since, by Corollary 2.1, a periodic orbit must enclose equilibrium points such that $N - S = 1$, we conclude that there are no periodic orbits.

• **2.21**
(a) Let

$$V(x) = x_2 - \frac{x_1 + b}{x_1 + a}$$

The function $V(x)$ is negative in D and the curve $V(x) = 0$ is the boundary of the set D.

$$f(x) \cdot \nabla V(x) = -cx_1(x_1 + a) + \frac{(b-a)}{(x_1 + a)^2}[-x_1 + x_2(x_1 + a) - b]$$

Evaluating $f(x) \cdot \nabla V(x)$ on the curve $V(x) = 0$ yields

$$f(x) \cdot \nabla V(x)|_{V(x)=0} = -cx_1(x_1 + a) < 0, \quad \forall\, x \in \partial D$$

Hence, trajectories on the boundary of D must move into D, which shows that trajectories starting in D cannot leave it.

(b)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 + x_2 < 0, \quad \forall\, x \in D$$

By Bendixson's criterion, there can be no closed orbits entirely in D. Since trajectories starting in D cannot leave it, a closed orbit through any point in D must lie entirely in D. Thus, we conclude that there are no closed orbits through any point in D.

• **2.22** **(a)** The value of $\dot{x}_2$ on the x_1-axis is $\dot{x}_2 = bx_1^2 \geq 0$. Thus, trajectories starting in D cannot leave it.

(b)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c \leq -(c - a) < 0, \quad \forall\, x \in D$$

By Bendixson's criterion, there can be no closed orbits entirely in D. Since trajectories starting in D cannot leave it, a closed orbit through any point in D must lie entirely in D. Thus, we conclude that there are no closed orbits through any point in D.

• **2.23**
(a)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -a[2b - g(x_1)] = \begin{cases} -2ab & \text{for } |x_1| > 1 \\ -a(2b - k) & \text{for } |x_1| \leq 1 \end{cases}$$

$$k < 2b \Rightarrow \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0, \quad \forall\, x$$

By Bendixson's criterion, there are no closed orbits.

(b) Consider the case $k > 2b$. We shall show the existence of a periodic orbit by imitating the steps of Example 2.9. We track the trajectory starting at point $A = (0, p)$; see Figure 2.28. At the starting point,

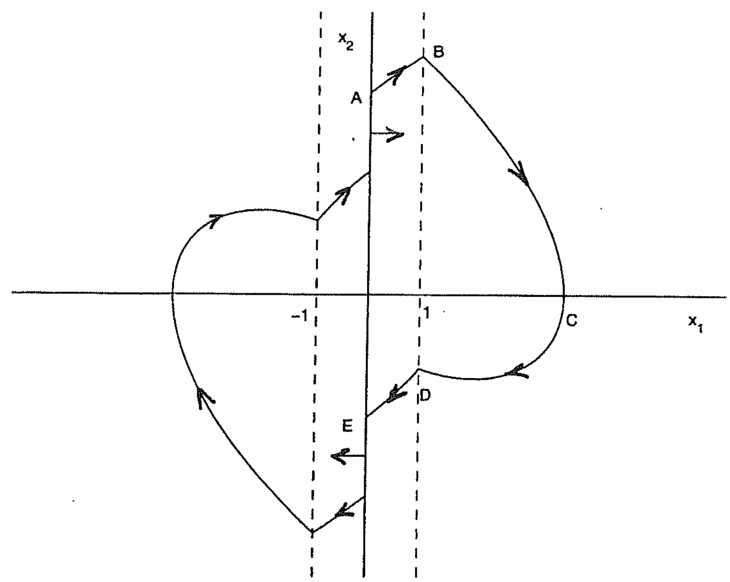

Figure 2.28: Exercise 2.23.

$f_1 = x_2 > 0$ and $f_2 = (k - 2b)ax_2 > 0$. So, the trajectory starts with a positive slope. Within the segment $0 \le x_1 \le 1$, the trajectory will continue to have a positive slope, provided p is large enough, until it arrives at $B = (1, \beta(p))$. As the trajectory leaves B, we have $f_2 = -2abx_2 - a_2x_1 < 0$. Thus, the trajectory will turn around forming the curve BCD. Let $D = (1, -\gamma(p))$ and consider the motion on the curve BCD. Let $V(x) = a^2x_1^2 + x_2^2$. Then

$$\dot{V} = 2a^2x_1x_2 + 2x_2(-a^2x_1 - 2abx_2) = -4abx_2^2 \le 0$$

$$V(D) - V(B) = \int_{BD} \dot{V}\ dt$$

Noting that $V(D) - V(B) = a^2 + \gamma^2(p) - a^2 - \beta^2(p)$, we obtain

$$\gamma^2(p) - \beta^2(p) = -4ab \int_{BD} x_2^2 \frac{dt}{dx_2}\ dx_2 = 4ab \int_{BD} \frac{x_2^2}{a^2x_1 + 2abx_2}\ dx_2$$

As p increases, the arc BCD moves to the right and the domain of integration increases. It follows that $\gamma^2(p) - \beta^2(p)$ decreases as p increases and

$$\lim_{p\to\infty} \{\gamma^2(p) - \beta^2(p)\} \to -\infty \text{ as } p \to \infty$$

Thus, for sufficiently large p, by the time the trajectory reaches the point $E = (0, -\delta(p))$ on the x_2-axis, we have $\delta(p) < p$. Similar to Example 2.9, we form a closed curve of the arc $ABCDE$, its reflection through the origin, and segments on the x_2-axis connecting these arcs. Let M be the region enclosed by this closed curve. Every trajectory starting in M stays in M for all future time. Linearization at the origin yields the matrix $\begin{bmatrix} 0 & 1 \\ -a^2 & k-2b \end{bmatrix}$, which has both eigenvalues in the right-half plane. Thus, the origin is unstable node or unstable focus. By the Poincaré-Bendixson criterion, there is a periodic orbit in M.

• **2.24** Suppose M does not contain an equilibrium point. Then, by the Poincaré-Bendixson criterion, there is a periodic orbit in M. But, by Corollary 2.1, the periodic orbit must contain an equilibrium point: A contradiction. Thus, M contains an equilibrium point.

• **2.25** Verifying Lemma 2.3 by examining the vector fields is simple, but requires drawing several sketches. Hence, it is skipped.

• **2.26**

(1) Linearization at the origin yields $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Hence, the origin is not hyperbolic. The index of the origin is zero. This can be easily seen by noting that $f_1 = x_1^2$ is always nonnegative. Clearly, the vector field cannot make a full rotation as we encircle the origin because this will require f_1 to be negative.

(2) Linearization at the origin yields $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, the origin is not hyperbolic. The index of the origin is two. This can be seen by sketching the vector field along a closed curve around the origin.

• **2.27** **(1)** The equilibrium points are the real roots of

$$0 = x_2, \quad 0 = \mu(x_1 + x_2) - x_2 - x_1^3 - 3x_1^2 x_2$$

$$x_2 = 0 \;\Rightarrow\; 0 = x_1(\mu - x_1^2) \;\Rightarrow\; x_1 = 0 \text{ or } x_1^2 = \mu$$

For $\mu > 0$, there are three equilibrium points at $(0,0)$, $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \mu - 3x_1^2 - 6x_1x_2 & \mu - 1 - 3x_1^2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1,\ \mu \;\Rightarrow\; (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(\sqrt{\mu},0)} = \begin{bmatrix} 0 & 1 \\ -2\mu & -1 - 2\mu \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -2\mu,\ -1 \;\Rightarrow\; (\sqrt{\mu}, 0) \text{ is a stable node}$$

Similarly, $(-\sqrt{\mu}, 0)$ is a stable node. For $\mu < 0$, there is a unique equilibrium point at $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1,\ \mu \;\Rightarrow\; (0,0) \text{ is a stable node}$$

Thus, there is supercritical pitchfork bifurcation at $\mu = 0$.

(2) The equilibrium points are the real roots of

$$0 = -x_1^3 + x_2, \quad 0 = -(1 + \mu^2)x_1 + 2\mu x_2 - \mu x_1^3 + 2(x_2 - \mu x_1)^3$$

$$x_2 = x_1^3 \;\Rightarrow\; 0 = x_1\left\{-1 + (x_1^2 - \mu)[\mu + 2x_1^2(x_1^2 - \mu)^2]\right\}$$

For all values of μ, there is an equilibrium point at $(0,0)$. At $\mu = 0$, there are two other equilibrium points at (a, a^3) and $(-a, -a^3)$, where $a^8 = 0.5$. It can be checked that these two equilibrium points are saddles. By continuous dependence of the roots of a polynomial equation on its parameters, we see that there is a range of values of μ around zero for which these two saddle points will persist. We will limit our attention to such values of μ and study local bifurcation at $\mu = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -(1 + \mu^2) & 2\mu \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = \mu \pm j$$

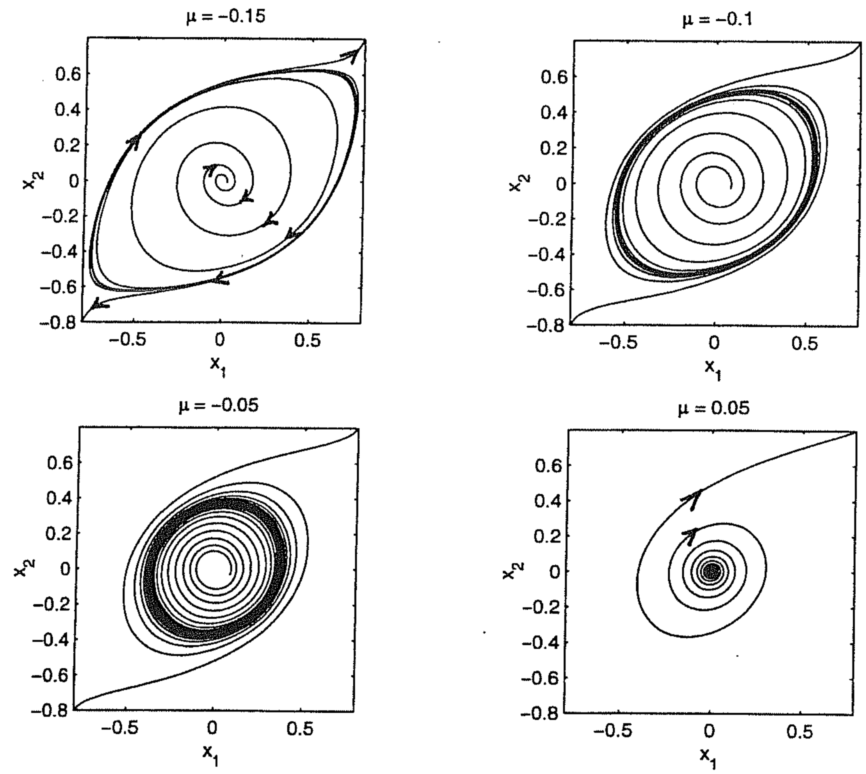

Figure 2.29: Exercise 2.27(2).

Hence, the origin is a stable focus for $\mu < 0$ and unstable focus for $\mu > 0$. The phase portrait for different values of μ is shown in Figure 2.29. For $\mu < 0$, there is a stable focus at the origin and unstable limit cycle around the origin. The size of the limit cycle shrinks as μ tends to zero. For $\mu > 0$, the origin is an unstable focus and the limit cycle disappears. Hence, there is a subcritical Hopf bifurcation at $\mu = 0$.

(3) The equilibrium points are the real roots of

$$0 = x_2, \quad 0 = \mu - x_2 - x_1^2 - 2x_1x_2$$

$$x_2 = 0 \;\Rightarrow\; x_1^2 = \mu$$

When $\mu < 0$, there are no equilibrium points. When $\mu > 0$, there are two equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(\sqrt{\mu},0)} = \begin{bmatrix} 0 & 1 \\ -2\sqrt{\mu} & -(1+2\sqrt{\mu}) \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1,\; -2\sqrt{\mu} \;\Rightarrow\; (\sqrt{\mu}, 0) \text{ is a stable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-\sqrt{\mu},0)} = \begin{bmatrix} 0 & 1 \\ 2\sqrt{\mu} & -(1-2\sqrt{\mu}) \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1,\; 2\sqrt{\mu} \;\Rightarrow\; (\sqrt{\mu}, 0) \text{ is a saddle}$$

There is a saddle–node bifurcation at $\mu = 0$.

(4) The equilibrium points are the real roots of

$$0 = x_2, \quad 0 = -(1+\mu^2)x_1 + 2\mu x_2 + \mu x_1^3 - x_1^2 x_2$$

$$x_2 = 0 \;\Rightarrow\; 0 = -(1+\mu^2)x_1 + \mu x_1^3$$

For all values of μ, there is an equilibrium point at $(0,0)$. For $\mu > 0$ there are two other equilibrium points at $(a,0)$ and $(-a,0)$, where $a = \sqrt{(1+\mu^2)/\mu}$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -(1+\mu^2) & 2\mu \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = \mu \pm j$$

Hence, the origin is a stable focus for $\mu < 0$ and unstable focus for $\mu > 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{(\pm a,0)} = \begin{bmatrix} 0 & 1 \\ 2(1+\mu^2) & (-1+\mu^2)/\mu \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = \frac{1}{2}\left[-\left(\frac{1-\mu^2}{\mu}\right) \pm \sqrt{\left(\frac{1-\mu^2}{\mu}\right)^2 + 8(1+\mu^2)}\right]$$

Hence, $(a,0)$ and $(-a,0)$ are saddle points. The phase portrait for different values of μ is shown in Figure 2.30. For $\mu < 0$, there is a stable focus at the origin. For $\mu > 0$, the origin is an unstable focus and there is a stable limit cycle around the origin. The size of the limit cycle shrinks as μ tends to zero. Hence, there is a supercritical Hopf bifurcation at $\mu = 0$. Also, as μ becomes positive, the saddle points appear on the x_1-axis at $x_1 = \pm\sqrt{(1+\mu^2)/\mu}$. The saddle points start at infinity and they move toward the origin as μ increases, until they reach ± 2 at $\mu = 1$. Then they move again toward infinity. For $\mu = 0.2$, the phase portrait is shown in a larger area that includes the saddle points.

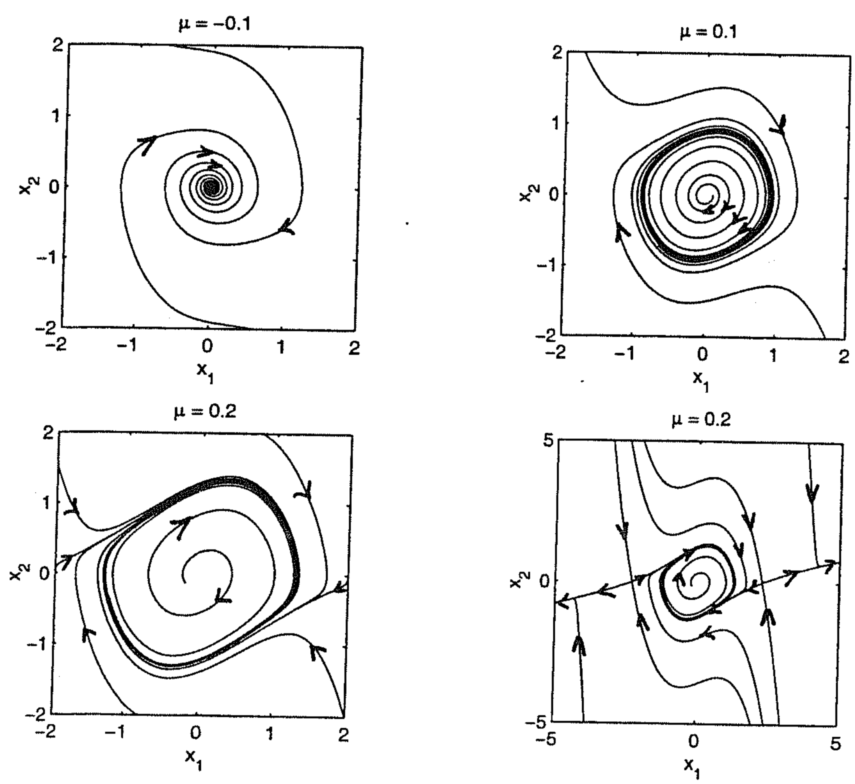

Figure 2.30: Exercise 2.27(4).

(5) The equilibrium points are the real roots of

$$0 = x_2, \qquad 0 = \mu(x_1 + x_2) - x_2 - x_1^3 + 3x_1^2 x_2$$

$$x_2 = 0 \;\Rightarrow\; 0 = x_1(\mu - x_1^2)$$

For all values of μ, there is an equilibrium point at $(0,0)$. For $\mu > 0$ there are two other equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1,\ \mu$$

Hence, the origin is a stable node for $\mu < 0$ and a saddle for $\mu > 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{(\pm\sqrt{\mu},0)} = \begin{bmatrix} 0 & 1 \\ -2\mu & (4\mu - 1) \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = \frac{-(1-4\mu) \pm \sqrt{(1-4\mu)^2 - 8\mu}}{2}$$

The following table gives the type of the equilibrium points $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$ for various positive values of μ.

Range	Type
$0 < \mu < 0.067$	Stable node
$0.067 < \mu < 0.25$	Stable focus
$0.25 < \mu < 0.933$	Unstable focus
$0.933 < \mu$	Unstable node

Thus, there is a supercritical pitchfork bifurcation at $\mu = 0$. We also examine $\mu = 0.25$, where the equilibrium points $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$ change from stable focus to unstable focus. The phase portraits for $\mu = 0.24$ and $\mu = 0.26$ are shown in Figure 2.31. As μ crosses 0.25 new stable limit cycles are created around the points $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. Thus, there is a supercritical Hopf bifurcation at $\mu = 0.25$.

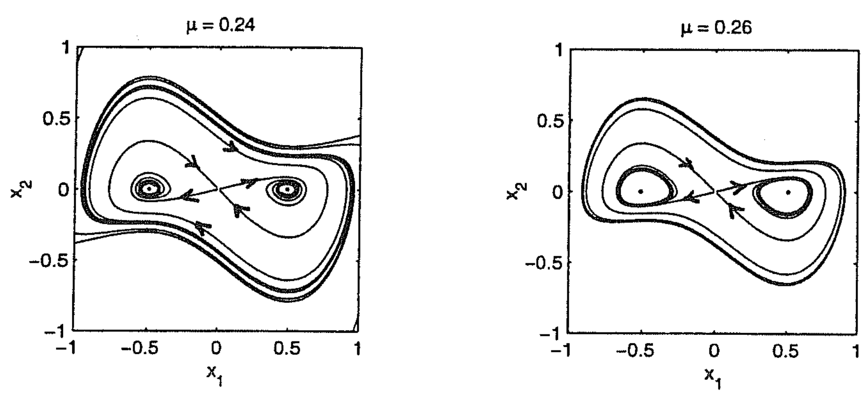

Figure 2.31: Exercise 2.27(5).

(6) The equilibrium points are the real roots of

$$0 = x_2, \qquad 0 = \mu(x_1 + x_2) - x_2 - x_1^2 - 2x_1x_2$$

$$x_2 = 0 \;\Rightarrow\; 0 = x_1(\mu - x_1)$$

There are two equilibrium points at $(0,0)$ and $(\mu, 0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -1,\ \mu$$

Hence, the origin is a stable node for $\mu < 0$ and a saddle for $\mu > 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{(\mu,0)} = \begin{bmatrix} 0 & 1 \\ -\mu & -(\mu+1) \end{bmatrix} \Rightarrow \lambda_{1,2} = -1,\ -\mu$$

Hence, $(\mu, 0)$ is a saddle for $\mu < 0$ and a stable node for $\mu > 0$. There is a transcritical bifurcation at $\mu = 0$.

• **2.28** **(a)** The equilibrium points are the real roots of

$$0 = -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2), \qquad 0 = -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)$$

By adding and subtracting the two equations, we see that the equilibrium points are the intersections of the two curves

$$x_2 = -x_1 + 2\tau \tanh(\lambda x_1), \qquad x_1 = x_2 - 2\tau \tanh(\lambda x_2)$$

Clearly there is an equilibrium point at the origin $(0, 0)$. By plotting the two curves for different values of $\lambda\tau$ (see Figure 2.32), it can be seen that the origin is the only intersection point. In fact, the two curves touch each other asymptotically as $\lambda\tau \to \infty$. Thus, we conclude that the origin is the only equilibrium point. Next we use linearization to determine the type of the equilibrium point.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\frac{1}{\tau} + \lambda\frac{1}{\cosh^2(\lambda x_1)} & -\lambda\frac{1}{\cosh^2(\lambda x_2)} \\ \lambda\frac{1}{\cosh^2(\lambda x_1)} & -\frac{1}{\tau} + \lambda\frac{1}{\cosh^2(\lambda x_2)} \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} -\frac{1}{\tau} + \lambda & -\lambda \\ \lambda & -\frac{1}{\tau} + \lambda \end{bmatrix}, \quad \text{Eigenvalues}: \frac{(\lambda\tau - 1)}{\tau} \pm j\lambda$$

Hence, the origin is a stable focus for $\lambda\tau < 1$ and unstable focus for $\lambda\tau > 1$. To apply the Poincare-Bendixson criterion when $\lambda\tau > 1$, we need to find a set M that satisfies the conditions of the criterion. We do it by transforming the equation into the polar coordinates

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\dot{r} = -\frac{1}{\tau}r + \cos(\theta)[\tanh(\lambda r\cos(\theta)) - \tanh(\lambda r\sin(\theta))] + \sin(\theta)[\tanh(\lambda r\cos(\theta)) + \tanh(\lambda r\sin(\theta))]$$

Using $|\tanh(\cdot)| \le 1$, $|\cos(\cdot)| \le 1$, and $|\sin(\cdot)| \le 1$, we see that

$$\dot{r} \le -\frac{1}{\tau}r + 4$$

Choosing $r = c > 4\tau$, we conclude that on the circle $r = c$, $\dot{r} < 0$. Hence, vector fields on $r = c$ point to the inside of the circle. Thus, the set $M = \{r \le c\}$ has the property that every trajectory starting in M stays in M for all future time. Moreover, M is closed, bounded, and contains only one equilibrium point which is unstable focus. By the Poincare-Bendixson criterion, we conclude that there is a periodic orbit in M.

(b) The phase portrait is shown in Figure 2.33. The origin is an unstable focus and there is a stable limit cycle around it. All trajectories, except the trivial solution $x = 0$, approach the limit cycle asymptotically.

(c) The phase portrait is shown in Figure 2.33. The origin is a stable focus. All trajectories approach the origin asymptotically.

(d) For $\lambda\tau < 1$, there a stable focus at the origin. For $\lambda\tau > 1$, there is an unstable focus at the origin and a stable limit cycle around the origin. Hence, there is a supercritical Hopf bifurcation at $\lambda\tau = 1$.

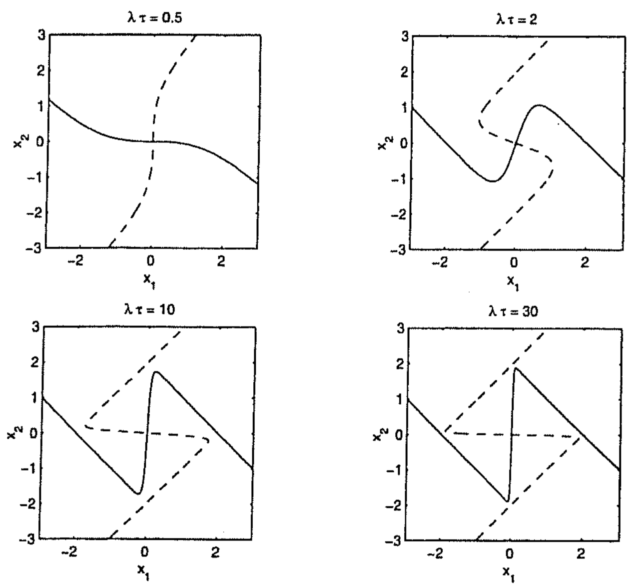

Figure 2.32: Exercise 2.28.

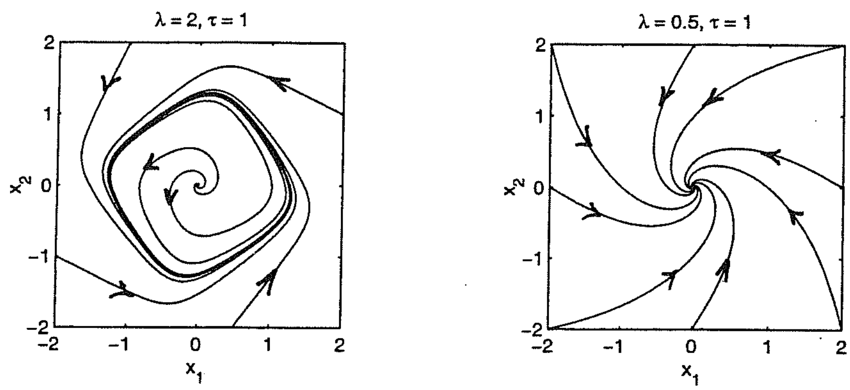

Figure 2.33: Exercise 2.28.

• **2.29** **(a)** The equilibrium points are the real roots of

$$0 = a - x_1 - \frac{4x_1x_2}{1+x_1^2}, \qquad 0 = bx_1\left(1 - \frac{x_2}{1+x_1^2}\right)$$

From the second equation we have $x_1 = 0$ or $x_2 = 1 + x_1^2$. The first equation cannot be satisfied with $x_1 = 0$. Substitution of $x_2 = 1 + x_1^2$ in the first equation results in $x_1 = a/5$. Thus, there is a unique equilibrium point at $((a/5), 1 + (a/5)^2)$. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 - \frac{4x_2}{1+x_1^2} + \frac{8x_1^2x_2}{(1+x_1^2)^2} & -\frac{4x_1}{1+x_1^2} \\ b\left(1 - \frac{x_2}{1+x_1^2}\right) + \frac{2bx_1^2x_2}{(1+x_1^2)^2} & -\frac{bx_1}{1+x_1^2} \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{((a/5),1+(a/5)^2)} = \frac{1}{1+(a/5)^2}\begin{bmatrix} -5+3(a/5)^2 & -4(a/5) \\ 2b(a/5)^2 & -b(a/5) \end{bmatrix} = \frac{1}{1+(a/5)^2}B$$

The characteristic equation of B is

$$s^2 + \beta s + \gamma = 0$$

where

$$\beta = 5 - 3(a/5)^2 + b(a/5), \quad \gamma = 5[1 + (a/5)^2]b(a/5)$$

We have $\gamma > 0$. If $\beta < 0$, the eigenvalues will be real and positive or complex with positive real parts; hence the equilibrium point will be unstable node or unstable focus. β is negative if $b < 3(a/5) - 25/a$. To apply the Poincare-Bendixson criterion, we need to choose the set M. Figure 2.34 sketches the two curves whose intersection determines the equilibrium point. On the sketch we identify a rectangle with vertices at A, B, C and D. On the line AB, $\dot{x}_2 > 0$; hence the vector fields point upward. On the line BC, $\dot{x}_1 < 0$; hence the vector fields point to the left. On the line CD, $\dot{x}_2 < 0$; hence the vector fields point downward. On the line DA, $\dot{x}_1 > 0$; hence the vector fields point to the right. Thus, taking the set M to be the rectangle $ABCD$, we see that every trajectory starting in M stays in M for all future time. Moreover, M is closed, bounded, and contains only one equilibrium point which is unstable node or unstable focus. By the Poincare-Bendixson criterion, we conclude that there is a periodic orbit in M.

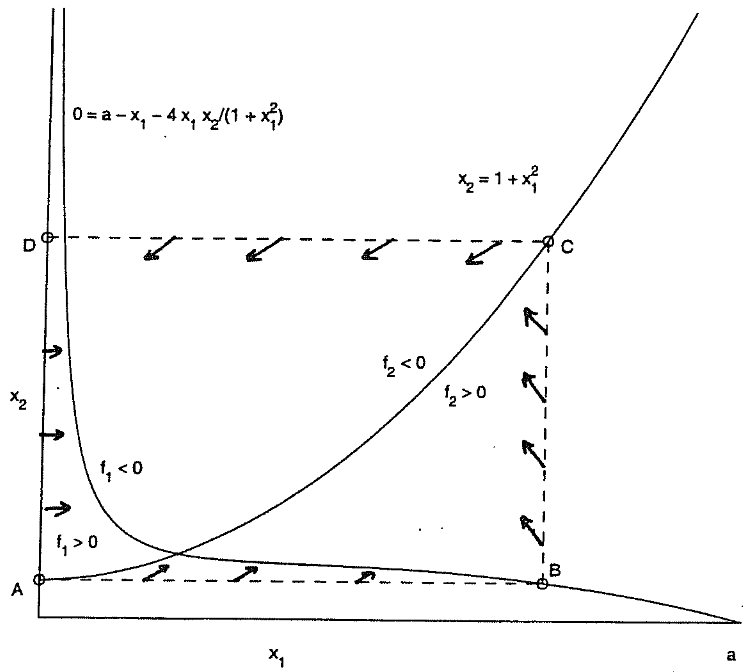

Figure 2.34: Exercise 2.29.

(b) For $a = 10$ and $b = 2$, we have $b < 3a/5 - 25/a$. The equilibrium point is $(2, 5)$ and it is unstable focus. The phase portrait is shown in Figure 2.35. The system has a stable limit cycle. All trajectories, except the equilibrium solution $x = (2, 5)$, approach the limit cycle asymptotically

(c) For $a = 10$ and $b = 4$, we have $b > 3a/5 - 25/a$. The equilibrium point is $(2, 5)$ and it is a stable focus. The phase portrait is shown in Figure 2.35. All trajectories approach the equilibrium point asymptotically.

(d) For $b < 3a/5 - 25/a$, β is negative. Moreover, when b is close to $3a/5 - 25/a$, β will be close to zero. Hence, $4\gamma > \beta^2$ and the equilibrium point is unstable focus. As we saw from the phase portrait, there

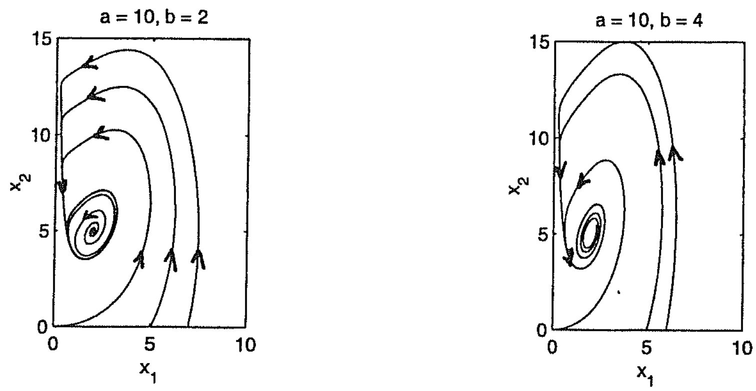

Figure 2.35: Exercise 2.29.

is a stable limit cycle around the equilibrium point. For $b > 3a/5 - 25/a$, β is positive. Once again, when b is close to $3a/5 - 25/a$, β will be close to zero. Hence, $4\gamma > \beta^2$ and the equilibrium point is a stable focus. There is a supercritical Hopf bifurcation at $b = 3a/5 - 25/a$.

• **2.30** **(a)** The equilibrium points are the roots of

$$0 = \left(\frac{\mu_m x_2}{(k_m + x_2)} - d\right) x_1, \quad 0 = d(x_{2f} - x_2) - \frac{\mu_m x_1 x_2}{Y(k_m + x_2)}$$

The first equation has two solutions: $x_1 = 0$ or the solution of $d = \mu_m x_2/(k_m + x_2)$, which we denote by α. When $d < \mu_m$, there is a unique solution α. Substitution of $x_1 = 0$ in the second equation yields $x_2 = x_{2f}$. Substitution of $x_2 = \alpha$ in the second equation yields $x_1 = Y(x_{2f} - \alpha)$, which will be a feasible solution if $\alpha \le x_{2f}$; that is, $d \le \mu_m x_{2f}/(k_m + x_{2f})$. Thus, when $d \le \mu_m x_{2f}/(k_m + x_{2f})$, there are two equilibrium points at $(0, x_{2f})$ and $(Y(x_{2f} - \alpha), \alpha)$. When $d > \mu_m x_{2f}/(k_m + x_{2f})$, there is a unique equilibrium point at $(0, x_{2f})$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\mu_m x_2}{k_m + x_2} - d & \frac{k_m \mu_m x_1}{(k_m + x_2)^2} \\ \frac{-\mu_m x_2}{Y(k_m + x_2)} & -d - \frac{k_m \mu_m x_1}{Y(k_m + x_2)^2} \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0, x_{2f})} = \begin{bmatrix} \frac{\mu_m x_{2f}}{k_m + x_{2f}} - d & 0 \\ \frac{-\mu_m x_{2f}}{Y(k_m + x_{2f})} & -d \end{bmatrix}$$

Hence, $(0, x_{2f})$ is a saddle if $d < \mu_m x_{2f}/(k_m + x_{2f})$ and a stable node if $d > \mu_m x_{2f}/(k_m + x_{2f})$.

$$\left.\frac{\partial f}{\partial x}\right|_{(Y(x_{2f} - \alpha), \alpha)} = \begin{bmatrix} 0 & \frac{k_m \mu_m Y(x_{2f} - \alpha)}{(k_m + \alpha)^2} \\ -\frac{d}{Y} & -d - \frac{k_m \mu_m (x_{2f} - \alpha)}{(k_m + \alpha)^2} \end{bmatrix}$$

The eigenvalues are $-d$ and $-k_m \mu_m (x_{2f} - \alpha)/(k_m + \alpha)^2$. For $d < \mu_m x_{2f}/(k_m + x_{2f})$, $(Y(x_{2f} - \alpha), \alpha)$ is a stable node. For the given numerical data, $\mu_m x_{2f}/(k_m + x_{2f}) = 0.4878$. When $d > \mu_m$, there is a unique equilibrium point at $(0, x_{2f})$, which is a stable node.

(b) The bifurcation diagram is shown in Figure 2.36. As d increases toward 0.4878, the saddle at $(0, x_{2f})$ and the stable node at $(Y(x_{2f} - \alpha), \alpha)$ collide and bifurcate into a stable node at $(0, x_{2f})$.

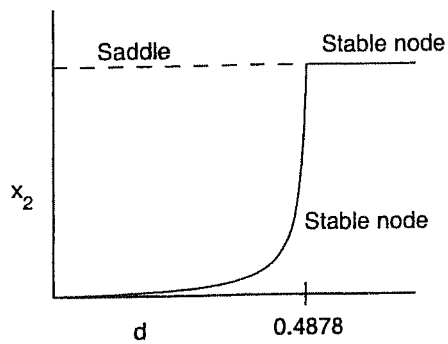

Figure 2.36: Exercise 2.30: Bifurcation diagram.

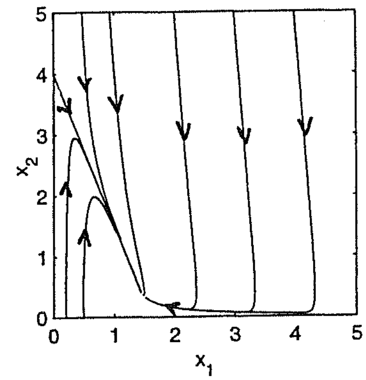

Figure 2.37: Exercise 2.30.

(c) For $d = 0.4$, $\alpha = k_m d/(\mu_m - d) = 0.4$. Since $d = 0.4 < 0.4878$, there is a saddle at $(0,4)$ and a stable node at $(1.44, 0.4)$. The phase portrait is shown in Figure 2.37.

• **2.31** (a) Let

$$\mu(x_2) = \frac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2}$$

The equilibrium points are the roots of

$$0 = x_1[\mu(x_2) - d], \quad 0 = d(x_{2f} - x_2) - \frac{x_1 \mu(x_2)}{Y}$$

From the first equation, $x_1 = 0$ or $\mu(x_2) = d$.

$$x_1 = 0 \;\Rightarrow\; x_2 = x_{2f}$$

$$\mu(x_2) = d \;\Rightarrow\; x_1 = Y(x_{2f} - x_2)$$

By sketching the function $\mu(x_2)$, it can be seen that if $d < \max_{x_2 \geq 0}\{\mu(x_2)\}$, the equation $d = \mu(x_2)$ will have two solutions. Let us denote them by α_1 and α_2. In this case, there are three equilibrium points at $(0, x_{2f})$, $(Y(x_{2f} - \alpha_1), \alpha_1)$, and $(Y(x_{2f} - \alpha_2), \alpha_2)$, provided $x_{2f} - \alpha_1$ and $x_{2f} - \alpha_2$ are nonnegative numbers. If one of these numbers is negative, the corresponding equilibrium point is not feasible. If $d > \max_{x_2 \geq 0}\{\mu(x_2)\}$, the equation $d = \mu(x_2)$ has no solutions. In this case, there is only one equilibrium point at $(0, x_{2f})$. The plot of μ as a function of x_2 is shown in Figure 2.38. By differentiation, it can be seen that μ has a maximum value $(\mu_m \sqrt{k_m/k_1})/(2k_m + \sqrt{k_m/k_1}) = 0.3455$ at $x_2 = \sqrt{k_m/k_1} = 0.4472$. When $d > 0.3455$, there is a unique equilibrium point at $(0,4)$, and when $d < 0.3455$ there are three equilibrium points at $(0,4)$, $(0.4(4-\alpha_1), \alpha_1)$, and $(0.4(4-\alpha_2), \alpha_2)$, where $\alpha_1 < 0.4472$ and $\alpha_2 > 0.4472$ are the solutions of $d = \mu(x_2)$. In the case of α_2, the equilibrium point $(0.4(4-\alpha_2), \alpha_2)$ is not feasible if $\alpha_2 > 4$. It can be checked that $\mu = 0.1653$ at $x_2 = 4$. Hence, for $d < 0.1653$, there are only two equilibrium points at $(0,4)$ and $(0.4(4-\alpha_1), \alpha_1)$. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -d + \frac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2} & \frac{\mu_m x_1 (k_m - k x_2^2)}{(k_m + x_2 + k_1 x_2^2)^2} \\ \frac{-\mu_m x_2}{Y(k_m + x_2 + k_1 x_2^2)} & -d - \frac{\mu_m x_1 (k_m - k x_2^2)}{Y(k_m + x_2 + k_1 x_2^2)^2} \end{bmatrix} = \begin{bmatrix} -d + \frac{0.5 x_2}{0.1 + x_2 + 0.5 x_2^2} & \frac{0.5 x_1 (0.1 - k x_2^2)}{(0.1 + x_2 + 0.5 x_2^2)^2} \\ \frac{-0.5 x_2}{0.4(0.1 + x_2 + 0.5 x_2^2)} & -d - \frac{0.5 x_1 (0.1 - k x_2^2)}{0.4(0.1 + x_2 + 0.5 x_2^2)^2} \end{bmatrix}$$

At $x = (0,4)$, the Jacobian matrix is

$$\begin{bmatrix} -d + 0.1653 & 0 \\ \star & -d \end{bmatrix}$$

When $d > 0.1653$, the equilibrium point is a stable node. When $d < 0.1653$, it is a saddle. At $x = (0.4(4-\alpha_1), \alpha_1)$, the Jacobian matrix is

$$\begin{bmatrix} 0 & d^2\beta Y \\ -d/Y & -d - d^2\beta \end{bmatrix}$$

where $\beta = (x_{2f} - \alpha_1)(k_m - k_1\alpha_1^2)/\mu_m\alpha_1^2 > 0$. The eigenvalues of this matrix are $-d$ and $-d^2\beta$. Hence, the equilibrium point is a stable node. At $x = (0.4(4-\alpha_2), \alpha_2)$ with $\alpha_2 < 4$, the Jacobian matrix is

$$\begin{bmatrix} 0 & d^2\gamma Y \\ -d/Y & -d - d^2\gamma \end{bmatrix}$$

where $\gamma = (x_{2f} - \alpha_2)(k_m - k_1\alpha_2^2)/\mu_m\alpha_2^2 < 0$. The eigenvalues of this matrix are $-d$ and $-d^2\gamma$. Hence, the equilibrium point is a saddle. In summary, we have the following three cases:

- When $d > 0.3455$, there is one equilibrium point at $(0,4)$ which is a stable node.
- When $0.1653 < d < 0.3455$, there are three equilibrium points: a stable node at $(0,4)$, a stable node at $(0.4(4-\alpha_1), \alpha_1)$, and a saddle at $(0.4(4-\alpha_2), \alpha_2)$.
- When $d < 0.1653$, there are two equilibrium points: a saddle at $(0,4)$ and a stable node at $(0.4(4-\alpha_1), \alpha_1)$.

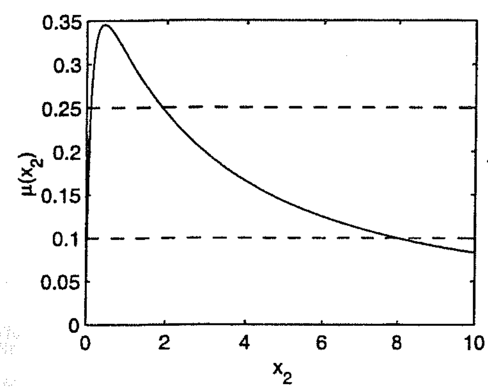

Figure 2.38: Exercise 2.31.

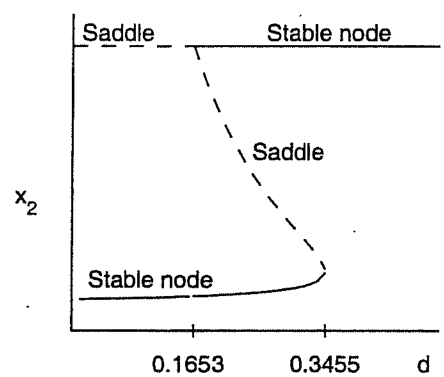

Figure 2.39: Exercise 2.31: Bifurcation diagram.

(b) The bifurcation diagram is shown in Figure 2.39. There is a saddle–node bifurcation at $d = 0.3455$. At $d = 0.1653$ there is a type of bifurcation that is not shown in Figure 2.28. A saddle point bifurcates into a stable node and a new saddle is created.

(c) When $d = 0.1$, there are two equilibrium points: $(0,4)$ is a saddle and $(1.59, 0.0251)$ is a stable node. The phase portrait is shown in Figure 2.40. The stable trajectories of the saddle are on the x_2-axis. All trajectories in the first quadrant approach the stable node.

(d) When $d = 0.25$, there are three equilibrium points: $(0,4)$ is a stable node, $(1.5578, 0.1056)$ is a stable node, and $(0.8426, 1.8936)$ is a saddle. The phase portrait is shown in Figure 2.41. The stable trajectories of the saddle form a separatrix which divides the first quadrant into two halves. All trajectories in the right half approach the stable node $(1.5578, 0.1056)$, while all trajectories in the left half approach the stable node $(0,4)$.

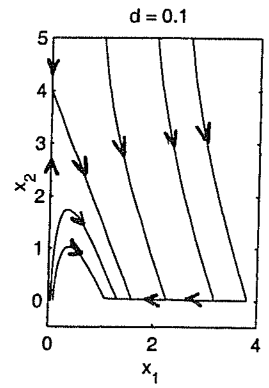

Figure 2.40: Exercise 2.31(c).

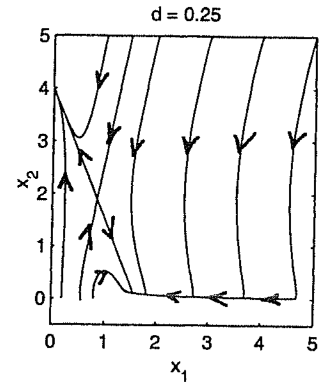

Figure 2.41: Exercise 2.31(d).

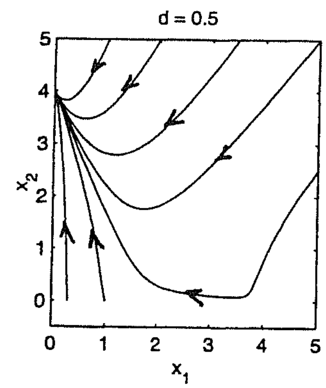

Figure 2.42: Exercise 2.31(e).

(e) When $d = 0.5$, there is one equilibrium point at $(0, 4)$ which is a stable node. The phase portrait is shown in Figure 2.42. All trajectories in the first quadrant approach the stable node.

Chapter 3

• **3.1** **(1)** The term $|x|$ is not continuously differentiable at $x = 0$, but it is globally Lipschitz. The term x^2 is continuously differentiable, but its partial derivative is not globally bounded. Thus $f = x^2 + |x|$ is not continuously differentiable at $x = 0$. It is continuously differentiable on a domain that does not include $x = 0$. It is locally Lipschitz, hence continuous, but not globally Lipschitz.
(2) The term $\text{sgn}(x)$ is discontinuous at $x = 0$. Thus, $f(x) = x + \text{sgn}(x)$ does not have any of the four properties in a domain that contains $x = 0$.
(3) $f(x) = \sin(x)\,\text{sgn}(x)$ is globally Lipschitz. This can be seen as follows. If both x and y are nonnegative, we have

$$|f(x) - f(y)| = |\sin(x) - \sin(y)| \leq |x - y|$$

If $x \geq 0$ and $y \leq 0$, we have

$$|f(x) - f(y)| = |\sin(x) + \sin(y)| = |2\sin(\tfrac{1}{2}(x+y))\cos(\tfrac{1}{2}(x-y))| \leq |x - y|$$

Other cases can be dealt with similarly to conclude that $|f(x) - f(y)| \leq |x - y|$ for all x, y. It follows that f is both locally Lipschitz and continuous. It is not continuously differentiable at $x = 0$ because $\lim_{x\to 0+} f'(x) = +1$ while $\lim_{x\to 0-} f'(x) = -1$.
(4) $f(x) = -x + a\sin x$ is continuously differentiable. Hence, it is locally Lipschitz and continuous. $\frac{df}{dx} = -1 + a\cos x$ is globally bounded. Hence, it is globally Lipschitz.
(5) $f(x) = -x + 2|x|$ is not continuously differentiable. It is globally Lipschitz because both x and $|x|$ are so. Hence, it is locally Lipschitz.
(6) $f(x) = \tan(x)$ is continuously differentiable in the open interval $-\pi/2 < x < \pi/2$. Hence, it is locally Lipschitz and continuous in the same interval. Its derivative $\sec^2(x)$ is not globally bounded; hence, it is not globally Lipschitz.
(7) The function $\tanh(y)$ is continuously differentiable and its derivative $1/cosh^2(y)$ is globally bounded; hence it is globally Lipschitz. Clearly, the linear function y is both continuously differentiable and globally Lipschitz. Hence, f has all four properties.
(8) f is not continuously differentiable due to the term $|x_2|$ in f_1. Check the Lipschitz property component by component. f_1 is globally Lipschitz as can be easily checked. f_2 is continuously differentiable, but its partial derivatives are not globally bounded. Hence f_2 is locally Lipschitz but not globally so. Since both f_1 and f_2 are locally Lipschitz, so is f. Since f is locally Lipschitz, it is continuous. f is not globally Lipschitz since f_2 is not so.

• **3.2** **(1)**

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$[\partial f/\partial x]$ is globally bounded. Hence f is globally Lipschitz, which implies that it is locally Lipschitz on D_r for any $r > 0$.
(2)

$$f(x) = \begin{bmatrix} -\frac{1}{C}h(x_1) + \frac{1}{C}x_2 \\ -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} -\frac{1}{C}h'(x_1) & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$$

$[\partial f/\partial x]$ is continuous everywhere; hence it is bounded on the bounded set D_r. Thus f is locally Lipschitz on D_r for any finite $r > 0$. It is not globally Lipschitz since $[\partial f/\partial x]$ is not globally bounded.

(3)

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

$\eta(x_1, x_2)$ is discontinuous at $x_2 = 0$. Hence it is not locally Lipschitz at the origin. This means it is not locally Lipschitz on D_r for any $r > 0$.

(4)

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon(1 - x_1^2)x_2 \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & -\varepsilon(1 - x_1^2) \end{bmatrix}$$

$[\partial f/\partial x]$ is continuous on D_r; hence f is locally Lipschitz on D_r for any $r > 0$. $[\partial f/\partial x]$ is not globally bounded; hence f is not globally Lipschitz.

(5) Let $x = [e_o, \phi_1, \phi_2]^T$.

$$f(t,x) = \begin{bmatrix} a_m x_1 + k_p x_2 r(t) + k_p x_3(x_1 + y_m(t)) \\ -\gamma x_1 r(t) \\ -\gamma x_1(x_1 + y_m(t)) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} a_m + k_p x_3 & k_p r(t) & k_p(x_1 + y_m(t)) \\ -\gamma r(t) & 0 & 0 \\ -\gamma(2x_1 + y_m(t)) & 0 & 0 \end{bmatrix}$$

$[\partial f/\partial x]$ is continuous and bounded on D_r for bounded $r(t)$ and $y_m(t)$. Hence f is locally Lipschitz. It is not globally Lipschitz since $[\partial f/\partial x]$ is not globally bounded.

(6)

$$f(x) = Ax - B\psi(Cx)$$

where $\psi(\cdot)$ is a dead-zone nonlinearity. The dead-zone nonlinearity is globally Lipschitz. Hence f is globally Lipschitz, which implies that it is locally Lipschitz on D_r for any $r > 0$.

• **3.3** For each $x_0 \in R$, there exist positive constants r, L_1, L_2, k_1, and k_2 such that

$$|f_1(x) - f_1(y)| \le L_1|x - y|, \; |f_2(x) - f_2(y)| \le L_2|x - y|, \; |f_1(x)| \le k_1, \; |f_2(x)| \le k_2$$

for all $x, y \in \{x \in \dot{R} \mid |x - x_0| < r\}$. For $f = f_1 + f_2$, we have

$$|f(x) - f(y)| = |f_1(x) - f_1(y) + f_2(x) - f_2(y)| \le L_1|x - y| + L_2|x - y| \le (L_1 + L_2)|x - y|$$

For $f = f_1 f_2$, we have

$$\begin{aligned} |f(x) - f(y)| &= |f_1(x)f_2(x) - f_1(y)f_2(y)| = |f_1(x)f_2(x) - f_1(x)f_2(y) + f_1(x)f_2(y) - f_1(y)f_2(y)| \\ &\le |f_1(x)|\,|f_2(x) - f_2(y)| + |f_2(y)|\,|f_1(x) - f_1(y)| \le k_1 L_2|x - y| + k_2 L_1|x - y| \\ &\le (k_1 L_2 + k_2 L_1)|x - y| \end{aligned}$$

For $f = f_2 \circ f_1$, we have

$$|f(x) - f(y)| = |f_2(f_1(x)) - f_2(f_1(y))| \le L_2|f_1(x) - f_1(y)| \le L_2 L_1|x - y|$$

• **3.4** The function f can be written as $f(x) = g(x)Kxh(\psi(x))$ where

$$h(\psi) = \begin{cases} \frac{1}{\psi}, & \text{if } \psi \ge \mu > 0 \\ \frac{1}{\mu}, & \text{if } \psi < \mu \end{cases} \quad \text{and} \quad \psi(x) = g(x)\|Kx\|$$

The norm function $\|Kx\|$ is Lipschitz since

$$|\,\|Kx\| - \|Ky\|\,| \le \|Kx - Ky\| \le \|K\|\,\|x - y\|$$

Using the previous exercise, we see that $\psi(x)$ is Lipschitz on any compact set. Furthermore, $g(x)Kx$ is also Lipschitz. Thus, $f(x)$ will be Lipschitz on any compact set if we can show that $h(\psi)$ is Lipschitz in ψ over any compact interval $[0, b]$. Now if $\psi_1 \geq \mu$ and $\psi_2 \geq \mu$, we have

$$|h(\psi_2) - h(\psi_1)| = \left|\frac{1}{\psi_2} - \frac{1}{\psi_1}\right| = \left|\frac{\psi_1 - \psi_2}{\psi_1\psi_2}\right| \leq \frac{1}{\mu^2}|\psi_2 - \psi_1|$$

If $\psi_2 \geq \mu$ and $\psi_1 < \mu$, we have

$$|h(\psi_2) - h(\psi_1)| = \left|\frac{1}{\psi_2} - \frac{1}{\mu}\right| = \frac{1}{\mu} - \frac{1}{\psi_2} = \frac{\psi_2 - \mu}{\mu\psi_2} \leq \frac{\psi_2 - \psi_1}{\mu\psi_2} \leq \frac{1}{\mu^2}|\psi_2 - \psi_1|$$

If $\psi_1 < \mu$ and $\psi_2 < \mu$, we have

$$|h(\psi_2) - h(\psi_1)| = \left|\frac{1}{\mu} - \frac{1}{\mu}\right| = 0 \leq \frac{1}{\mu^2}|\psi_2 - \psi_1|$$

Thus $h(\psi)$ is Lipschitz with a Lipschitz constant $1/\mu^2$.

- **3.5** There are positive constants c_1 and c_2 such that

$$c_1\|x\|_\alpha \leq \|x\|_\beta \leq c_2\|x\|_\alpha, \quad \forall\, x \in R^n$$

Suppose

$$\|f(y) - f(x)\|_\alpha \leq L_\alpha\|y - x\|_\alpha$$

Then

$$\|f(y) - f(x)\|_\beta \leq c_2\|f(y) - f(x)\|_\alpha \leq c_2L_\alpha\|y - x\|_\alpha \leq \frac{c_2L_\alpha}{c_1}\|y - x\|_\beta$$

Similarly, it can be shown that if f is Lipschitz in the β-norm, it will be Lipschitz in the α-norm.

- **3.6 (a)**

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\; d\tau \\ \|x(t)\| &\leq \|x_0\| + \int_{t_0}^{t} \|f(\tau, x(\tau))\|\; d\tau \\ &\leq \|x_0\| + \int_{t_0}^{t} [k_1 + k_2\|x(\tau)\|]\; d\tau \\ &= \|x_0\| + k_1(t - t_0) + k_2\int_{t_0}^{t} \|x(\tau)\|\; d\tau \end{aligned}$$

By Gronwall-Bellman inequality

$$\|x(t)\| \leq \|x_0\| + k_1(t - t_0) + \int_{t_0}^{t} [\|x_0\| + k_1(s - t_0)]k_2e^{k_2(t-s)}\; ds$$

Integrating by parts, we obtain

$$\|x(t)\| \leq \|x_0\|\exp[k_2(t - t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t - t_0)] - 1\}, \quad \forall\, t \geq t_0$$

(b) The upper bound on $\|x(t)\|$ is finite for every finite t. It tends to ∞ as $t \to \infty$. Hence the solution of the system cannot have a finite escape time.

• **3.7** It can be easily verified that $f(x)$ is continuously differentiable. Hence, local existence and uniqueness follows from Theorem 3.1. Furthermore,

$$\|f(x)\|_2 = \frac{\|g(x)\|_2}{1+\|g(x)\|_2^2} \le \frac{1}{2}$$

Hence

$$\|x(t)\|_2 \le \|x_0\|_2 + \frac{1}{2}(t-t_0)$$

which shows that the solution is defined for all $t \ge t_0$.

• **3.8** It can be easily seen that $f(x)$ is continuously differentiable and

$$\|f(x)\| \le k_1 + k_2\|x\|, \quad \forall\, x \in R^2$$

for some positive constants k_1 and k_2. Apply Exercise 3.6.

• **3.9** Due to uniqueness of solution, trajectories in the plane cannot intersect. Therefore, all trajectories starting in the region enclosed by the limit cycle must remain in that region. The closure of this region is a compact set. Therefore, the solution must stay in a compact set. Apply Theorem 3.3.

• **3.10**

$$\dot{x}_1 = \frac{1}{C}[-h(x_1)+x_2], \qquad \dot{x}_2 = \frac{1}{L}(-x_1 - Rx_2 + u)$$

where $R = 1.5$, $u = 1.2$, and the nominal values of C and L are 2 and 5, respectively. Let $\lambda = [C, L]^T$. The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \lambda]$, are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\frac{1}{C}h'(x_1) & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad \frac{\partial f}{\partial \lambda} = \begin{bmatrix} -\frac{1}{C^2}[-h(x_1)+x_2] & 0 \\ 0 & -\frac{1}{L^2}(-x_1 - Rx_2 + u) \end{bmatrix}$$

Evaluate these Jacobian matrices at the nominal values $C = 2$ and $L = 5$. Let

$$S = \left.\frac{\partial x}{\partial \lambda}\right|_{\text{nominal}} = \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix}$$

Then

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}}, \quad S(0) = 0$$

The augmented equation (3.7) is given by

$$\begin{aligned}
\dot{x}_1 &= 0.5[-h(x_1)+x_2] \\
\dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2) \\
\dot{x}_3 &= 0.5[-h'(x_1)x_3 + x_4] - 0.25[-h(x_1)+x_2] \\
\dot{x}_4 &= 0.2(-x_3 - 1.5x_4) \\
\dot{x}_5 &= 0.5[-h'(x_1)x_5 + x_6] \\
\dot{x}_6 &= 0.2(-x_5 - 1.5x_6) - 0.04(-x_1 - 1.5x_2 + 1.2)
\end{aligned}$$

with the initial conditions

$$x_1(0) = x_{10},\ x_2(0) = x_{20},\ x_3(0) = x_4(0) = x_5(0) = x_6(0) = 0$$

- **3.11**

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2$$

Denote the nominal values of ε by ε_0. The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \varepsilon]$, are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon(1 - x_1^2) \end{bmatrix}, \quad \frac{\partial f}{\partial \varepsilon} = \begin{bmatrix} 0 \\ (1 - x_1^2)x_2 \end{bmatrix}$$

Let

$$S = \left.\frac{\partial x}{\partial \varepsilon}\right|_{\text{nominal}} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Then

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \varepsilon}\right|_{\text{nominal}}, \quad S(0) = 0$$

The augmented equation (3.7) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon_0(1 - x_1^2)x_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -[1 + 2\varepsilon_0 x_1 x_2]\, x_3 + \varepsilon_0(1 - x_1^2)x_4 + (1 - x_1^2)x_2 \end{aligned}$$

with the initial conditions

$$x_1(0) = x_{10}, \; x_2(0) = x_{20}, \; x_3(0) = x_4(0) = 0$$

- **3.12**

$$\dot{x}_1 = \frac{1}{\varepsilon}x_2, \quad \dot{x}_2 = -\varepsilon\left(x_1 - x_2 + \frac{1}{3}x_2^3\right)$$

Denote the nominal values of ε by ε_0. The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \varepsilon]$, are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & \frac{1}{\varepsilon} \\ -\varepsilon & \varepsilon(1 - x_2^2) \end{bmatrix}, \quad \frac{\partial f}{\partial \varepsilon} = \begin{bmatrix} -\frac{1}{\varepsilon^2}x_2 \\ -\left(x_1 - x_2 + \frac{1}{3}x_2^3\right) \end{bmatrix}$$

Let

$$S = \left.\frac{\partial x}{\partial \varepsilon}\right|_{\text{nominal}} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Then

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \varepsilon}\right|_{\text{nominal}}, \quad S(0) = 0$$

The augmented equation (3.7) is given by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\varepsilon_0}x_2 \\ \dot{x}_2 &= -\varepsilon_0\left(x_1 - x_2 + \frac{1}{3}x_2^3\right) \\ \dot{x}_3 &= \frac{1}{\varepsilon_0}x_4 - \frac{1}{\varepsilon_0^2}x_2 \\ \dot{x}_4 &= -\varepsilon_0 x_3 + \varepsilon_0(1 - x_2^2)x_4 - \left(x_1 - x_2 + \frac{1}{3}x_2^3\right) \end{aligned}$$

with the initial conditions

$$x_1(0) = x_{10}, \; x_2(0) = x_{20}, \; x_3(0) = x_4(0) = 0$$

• **3.13**

$$\dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2, \quad \dot{x}_2 = bx_1^2 - cx_2$$

Let $\lambda = [a, b, c]^T$. The nominal values are $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$. The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \lambda]$, are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{a}{1+a^2x_1^2} - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix}, \quad \frac{\partial f}{\partial \lambda} = \begin{bmatrix} \frac{x_1}{1+a^2x_1^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}$$

Let

$$S = \left.\frac{\partial x}{\partial \lambda}\right|_{\text{nominal}} = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}$$

Then

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}}, \quad S(0) = 0$$

The augmented equation (3.7) is given by

$$\begin{aligned} \dot{x}_1 &= \tan^{-1}(x_1) - x_1x_2 \\ \dot{x}_2 &= -x_2 \\ \dot{x}_3 &= \left(\frac{1}{1+x_1^2} - x_2\right) x_3 - x_1x_4 + \frac{x_1}{1+x_1^2} \\ \dot{x}_4 &= -x_4 \\ \dot{x}_5 &= \left(\frac{1}{1+x_1^2} - x_2\right) x_5 - x_1x_6 \\ \dot{x}_6 &= -x_6 + x_1^2 \\ \dot{x}_7 &= \left(\frac{1}{1+x_1^2} - x_2\right) x_7 - x_1x_8 \\ \dot{x}_8 &= -x_8 - x_2 \end{aligned}$$

with the initial conditions

$$x_1(0) = x_{10}, \; x_2(0) = x_{20}, \; x_3(0) = x_4(0) = x_5(0) = x_6(0) = x_7(0) = x_8(0) = 0$$

• **3.14** **(a)** Let $p = \begin{bmatrix} \tau \\ \lambda \end{bmatrix}$ be the vector of parameters.

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} -(1/\tau) + \lambda/\cosh^2(\lambda x_1) & -\lambda/\cosh^2(\lambda x_2) \\ \lambda/\cosh^2(\lambda x_1) & -(1/\tau) + \lambda/\cosh^2(\lambda x_2) \end{bmatrix}$$

$$B = \frac{\partial f}{\partial p} = \begin{bmatrix} -(1/\tau^2)x_1 & x_1/\cosh^2(\lambda x_1) - x_2/\cosh^2(\lambda x_2) \\ -(1/\tau^2)x_2 & x_1/\cosh^2(\lambda x_1) + x_2/\cosh^2(\lambda x_2) \end{bmatrix}$$

The sensitivity equation is given by

$$\dot{S} = A_0S + B_0, \quad S(0) = 0$$

where A_0 and B_0 are evaluated at the nominal parameters. This equation should be solved simultaneously with the nominal state equation.

ıd

(b)

$$\begin{aligned} r\dot{r} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= -(1/\tau)r^2 + x_1[\tanh(\lambda x_1) - \tanh(\lambda x_2)] + x_2[\tanh(\lambda x_1) + \tanh(\lambda x_1)] \\ &= -(1/\tau)r^2 + r\cos(\theta)[\tanh(\lambda x_1) - \tanh(\lambda x_2)] \\ &\quad + r\sin(\theta)[\tanh(\lambda x_1) + \tanh(\lambda x_1)] \\ &\le -(1/\tau)r^2 + 2r(|\cos(\theta)| + |\sin(\theta)|) \\ &\le -(1/\tau)r^2 + 2\sqrt{2}r \end{aligned}$$

(c) By the comparison lemma, $r(t) \le u(t)$ where u satisfies the scalar differential equation

$$\dot{u} = -(1/\tau)u + 2\sqrt{2}, \quad u(0) = r(0) = \|x(0)\|_2$$

The solution of this differential equation is

$$\begin{aligned} u(t) &= \exp(-t/\tau)\|x(0)\|_2 + \int_0^t exp[-(t-\sigma)/\tau]2\sqrt{2}\, d\sigma \\ &= \exp(-t/\tau)\|x(0)\|_2 + 2\sqrt{2}\tau[1 - \exp(-t/\tau)] \end{aligned}$$

• **3.15** Let $V = \|x\|_2^2 = x_1^2 + x_2^2$. Then

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \;=\; -2x_1^2 - 2x_2^2 + \frac{4x_1x_2}{1+x_2^2} + \frac{4x_1x_2}{1+x_1^2} \\ &\le -2V + 4|x_1|\frac{|x_2|}{1+x_2^2} + 4|x_2|\frac{|x_1|}{1+x_1^2} \\ &\le -2V + 2|x_1| + 2|x_2| \quad \left(\text{since } \frac{|y|}{1+y^2} \le \frac{1}{2}\right) \\ &\le -2V + 2\sqrt{2}\sqrt{V} \quad (\text{since } \|x\|_1 \le \sqrt{n}\|x\|_2) \end{aligned}$$

Taking $W = \sqrt{V} = \|x\|_2$, we see that for $V \ne 0$,

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} \le -W + \sqrt{2}$$

At $V = 0$, we have

$$\frac{|W(t+h) - W(t)|}{h} = \frac{|W(t+h)|}{h} = \frac{1}{h}\|x(t+h)\|_2$$

Similar to Example 3.9 of the textbook, it can be shown that

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \|f(x(\tau))\|_2 \, d\tau = 0$$

Thus $D^+W(t) \le -W(t) + \sqrt{2}$ for all $t \ge 0$. Let $u(t)$ be the solution of the differential equation

$$\dot{u} = -u + \sqrt{2}, \quad u(0) = \|x(0)\|_2$$

By the comparison lemma,

$$\|x(t)\|_2 \le u(t) = e^{-t}\|x(0)\|_2 + \sqrt{2}\,(1 - e^{-t})$$

• **3.16** Let $v = x^2$.

$$\dot{v} = 2x\dot{x} = -2x^2 + \frac{2x \sin t}{1+x^2} \le -2v + 1$$

let $u(t)$ be the solution of the differential equation

$$\dot{u} = -2u + 1, \quad u(0) = 2$$

Then

$$v(t) \le u(t) = 2e^{-2t} + \int_0^t e^{-2(t-\tau)} \, d\tau = \frac{1+3e^{-2t}}{2}$$

Thus

$$|x(t)| = \sqrt{v(t)} \le \sqrt{\frac{1+3e^{-2t}}{2}}$$

• **3.17 (a)**

$$\frac{d}{dt} x^T x = 2x^T \dot{x} = 2x^T f(t,x)$$

$$\left| \frac{d}{dt} x^T x \right| \le 2\|x\|_2 \|f(t,x)\|_2 \le 2L\|x\|_2^2$$

(b) Let $V(t) = x^T(t)x(t)$ and $V_0 = x_0^T x_0$, then from part (a) we have

$$-2LV(t) \le \dot{V}(t) \le 2LV(t)$$

Divide through by V(t), multiply by dt, and integrate to obtain

$$-\int_{t_0}^t 2L \, dt \le \int_{V_0}^V \frac{dV}{V} \le \int_{t_0}^t 2L \, dt$$

$$-2L(t-t_0) \le \ln\left(\frac{V(t)}{V_0}\right) \le 2L(t-t_0)$$

$$V_0 \exp[-2L(t-t_0)] \le V(t) \le V_0 \exp[2L(t-t_0)]$$

Taking the square root yields

$$\|x_0\|_2 \exp[-L(t-t_0)] \le \|x(t)\|_2 \le \|x_0\|_2 \exp[L(t-t_0)]$$

• **3.18** Let $z(t) = y(t)e^{\alpha(t-t_0)}$. Then

$$\begin{aligned} z(t) \;&\le\; k_1 + \int_{t_0}^t e^{\alpha(\tau-t_0)}[k_2 y(\tau) + k_3] \, d\tau \;=\; k_1 + \int_{t_0}^t \left[k_2 z(\tau) + k_3 e^{\alpha(\tau-t_0)}\right] \, d\tau \\ &=\; k_1 + k_2 \int_{t_0}^t z(\tau) \, d\tau + (k_3/\alpha)\left[\exp^{\alpha(t-t_0)} - 1\right] \end{aligned}$$

From Gronwall-Bellman inequality,

$$z(t) \le k_1 + (k_3/\alpha)\left[\exp^{\alpha(t-t_0)} - 1\right] + \int_{t_0}^t \left\{k_1 + (k_3/\alpha)\left[\exp^{\alpha(s-t_0)} - 1\right]\right\} k_2 e^{k_2(t-s)} \, ds$$

By evaluating the integral, it can be shown that

$$z(t) \le \frac{k_3}{\alpha} e^{\alpha(t-t_0)} + \left(k_1 - \frac{k_3}{\alpha}\right) e^{k_2(t-t_0)} + \frac{k_2 k_3}{\alpha(\alpha-k_2)} \left[e^{\alpha(t-t_0)} - e^{k_2(t-t_0)}\right]$$

Hence

$$\begin{aligned} y(t) &= z(t)e^{-\alpha(t-t_0)} \leq \frac{k_3}{\alpha}\left[1+\frac{k_2}{(\alpha-k_2)}\right] + \left[k_1 - \frac{k_3}{\alpha} - \frac{k_2k_3}{\alpha(\alpha-k_2)}\right]e^{(k_2-\alpha)(t-t_0)} \\ &= \frac{k_3}{(\alpha-k_2)} + \left[k_1 - \frac{k_3}{(\alpha-k_2)}\right]e^{(k_2-\alpha)(t-t_0)} \\ &= k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{(\alpha-k_2)}\left[1-e^{-(\alpha-k_2)(t-t_0)}\right] \end{aligned}$$

• **3.19** Choose the covering of S as described in the hint. Within each neighborhood we have

$$\|f(x)-f(y)\| \leq L_i\|x-y\|, \quad \forall\; x,y \in N(a_i,r_i)$$

If $x,y \in S \cap N(a_i,r_i)$ for some i, then the Lipschitz condition holds with $L = L_i$. Otherwise, $\|x-y\| \geq \min_i r_i$. Since $f(x)$ is uniformly bounded on S, we have

$$\|f(x)-f(y)\| \leq C, \quad \forall\; x,y \in S, \; C > 0$$

Therefore, whenever $\|x-y\| \geq \min_i r_i$, we have

$$\|f(x)-f(y)\| \leq \frac{C}{\min_i r_i}\|x-y\|$$

Take

$$L = \max\left\{L_1, L_2, \ldots, L_k, \frac{C}{\min_i r_i}\right\}$$

• **3.20** We have

$$\|f(x)-f(y)\| \leq L\|x-y\|, \quad \forall\; x,y \in W$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/L$. For all $\|x-y\| < \delta$, we have $\|f(x)-f(y)\| < L\delta = \varepsilon$, which implies uniform continuity.

• **3.21** The vector y is defined only for $x \neq 0$. For $x = 0$, we can take y as any vector with $\|y\|_q = 1$. Now, for $x \neq 0$ we have

$$y^Tx = \sum_{i=1}^n y_ix_i = \sum_{i=1}^n \frac{x_i^p \operatorname{sign}(x_i^p)}{\|x\|_p^{p-1}} = \frac{1}{\|x\|_p^{p-1}}\sum_{i=1}^n |x_i|^p = \frac{\|x\|_p^p}{\|x\|_p^{p-1}} = \|x\|_p$$

$$\|y\|_q^q = \sum_i^n |y_i|^q = \frac{1}{\|x\|_p^{pq-q}}\sum_{i=1}^n |x_i|^{pq-p} = \frac{1}{\|x\|_p^p}\sum_{i=1}^n |x_i|^p = 1$$

For $p = \infty$, take

$$y_i = \begin{cases} 1, & \text{if } i = \arg\max|x_i| \\ 0 & \text{otherwise} \end{cases}$$

Then, $y^Tx = \|x\|_\infty$ and $\|y\|_1 = 1$.

• **3.22** If

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\| \leq L, \quad \forall\; (t,x) \in [a,b] \times R^n$$

then, from Lemma 3.1,

$$\|f(t,y)-f(t,x)\| \leq L\|x-y\|, \quad \forall\; (t,x) \in [a,b] \times R^n$$

Alternatively, suppose $f(t,x)$ is globally Lipschitz. By the mean value theorem

$$f_i(t,y) - f_i(t,x) = \frac{\partial f_i}{\partial x}(t,z)\ (y-x)$$

where $z = \alpha x + (1-\alpha)y$ and $0 < \alpha < 1$. Then

$$\left\| \frac{\partial f_i}{\partial x}(t,z)\ (y-x) \right\| = \|f_i(t,y) - f_i(t,x)\| \le L_i \|y-x\|, \quad \forall\ x,y \in R^n, \forall\ t \in [a,b]$$

Hence

$$\frac{\left\| \frac{\partial f_i}{\partial x}(t,z)\ (y-x) \right\|}{\|y-x\|} \le L_i, \quad \forall\ x,y \in R^n, \forall\ t \in [a,b]$$

Taking $y = \beta x$ with $\beta > 1$, we have $z = [\alpha + (1-\alpha)\beta]x \stackrel{\text{def}}{=} \gamma x$, and

$$\frac{\left\| \frac{\partial f_i}{\partial x}(t,\gamma x)\ (\beta-1)x \right\|}{(\beta-1)\|x\|} \le L_i, \quad \forall\ x \in R^n, \forall\ t \in [a,b]$$

Thus

$$\frac{\left\| \frac{\partial f_i}{\partial x}(t,\gamma x)\ x \right\|}{\|x\|} \le L_i, \quad \forall\ x \in R^n, \forall\ t \in [a,b]$$

By letting β approach 1, we conclude that

$$\frac{\left\| \frac{\partial f_i}{\partial x}(t,x)\ x \right\|}{\|x\|} \le L_i, \quad \forall\ x \in R^n, \forall\ t \in [a,b]$$

which shows that

$$\left\| \frac{\partial f_i}{\partial x}(t,x) \right\| \le L_i, \quad \forall\ x \in R^n, \forall\ t \in [a,b]$$

Since this inequality holds for every $1 \le i \le n$, we conclude that the Jacobian matrix $[\partial f/\partial x]$ is globally bounded.

- **3.23** Set $g(\sigma) = f(\sigma x)$ for $0 \le \sigma \le 1$. Since D is convex, $\sigma x \in D$ for $0 \le \sigma \le 1$.

$$g'(\sigma) = \frac{\partial f}{\partial x}(\sigma x)\frac{\partial \sigma x}{\partial \sigma} = \frac{\partial f}{\partial x}(\sigma x)\ x$$

$$f(x) = f(x) - f(0) = g(1) - g(0) = \int_0^1 g'(\sigma)\ d\sigma = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\ d\sigma\ x$$

- **3.24** (a)

$$V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t,\sigma x)\ d\sigma\ x \le \int_0^1 \left\| \frac{\partial V}{\partial x}(t,\sigma x) \right\|\ \|x\|\ d\sigma \le \int_0^1 c_4 \sigma\ d\sigma\ \|x\|^2 \le \frac{1}{2}c_4\|x\|^2$$

(b) Since

$$c_1\|x\|^2 \le V(t,x) \le \frac{1}{2}c_4\|x\|^2, \quad \forall\ x \in D$$

we must have $c_1 \le \frac{1}{2}c_4$.
(c) Consider two points x_1 and x_2 such that $\alpha x_1 + (1-\alpha)x_2 \ne 0$ for all $0 \le \alpha \le 1$; that is, the origin does

not lie on the line connecting x_1 and x_2. The Jacobian $[\partial W/\partial x]$ is defined for every $x = \alpha x_1 + (1-\alpha)x_2$ and given by

$$\frac{\partial W}{\partial x}(t,x) = \frac{1}{2\sqrt{V(t,x)}}\frac{\partial V}{\partial x}(t,x)$$

By the mean value theorem, there is $\alpha^* \in (0,1)$ such that, with $z = \alpha^* x_1 + (1-\alpha^*)x_2$,

$$W(t,x_2) - W(t,x_1) = \frac{\partial W}{\partial x}(t,z)\ (x_2 - x_1) = \frac{1}{2\sqrt{V(t,z)}}\frac{\partial V}{\partial x}(t,z)\ (x_2 - x_1)$$

Hence

$$|W(t,x_2) - W(t,x_1)| \le \frac{1}{2\sqrt{c_1}\ \|z\|}\ c_4\|z\|\ \|x_2 - x_1\| \le \frac{c_4}{2\sqrt{c_1}}\ \|x_2 - x_1\|$$

Consider now the case when the origin lies on the line connecting x_1 and x_2; that is, $0 = \alpha_0 x_1 + (1-\alpha_0)x_2$ for some $\alpha_0 \in [0,1]$. We have

$$|W(t,x_2) - W(t,0)| = |W(t,x_2)| = \sqrt{V(t,x_2)} \le \sqrt{\frac{c_4}{2}}\|x_2\|$$

$$|W(t,x_1) - W(t,0)| = |W(t,x_1)| = \sqrt{V(t,x_1)} \le \sqrt{\frac{c_4}{2}}\|x_1\|$$

$$|W(t,x_2) - W(t,x_1)| = |W(t,x_2) - W(t,0) + W(t,0) - W(t,x_1)| \le \sqrt{\frac{c_4}{2}}(\|x_2\| + \|x_1\|)$$

Since the origin lies on the line connecting x_1 and x_2, we have $\|x_2\| + \|x_1\| = \|x_2 - x_1\|$. We also have $1 \le \sqrt{c_4/2c_1}$. Therefore,

$$|W(t,x_2) - W(t,x_1)| \le \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\|$$

Thus, the preceding inequality is satisfied for all $x_1, x_2 \in D$.

• **3.25**
(a)

$$x(t) = x(\alpha) + \int_\alpha^t f(\tau, x(\tau))\ d\tau, \quad \forall\ [\alpha, t) \subset [t_0, T)$$

Since $f(t,x)$ is piecewise continuous in t and continuous in x, there exists a constant $M > 0$ such that $\|f(t,x(t))\| \le M$ for all $t \in [t_0, T)$. Therefore

$$\|x(t) - x(\alpha)\| = \left\| \int_\alpha^t f(\tau, x(\tau))\ d\tau \right\| \le \int_\alpha^t M\ d\tau = M(t-\alpha)$$

which shows that $x(t)$ is uniformly continuous on $[t_0, T)$.
(b)

$$x(T) = x(t_0) + \lim_{t\to T}\int_{t_0}^t f(\tau, x(\tau))\ d\tau = x(t_0) + \int_{t_0}^T f(\tau, x(\tau))\ d\tau$$

since $x(t)$ is uniformly continuous. Thus

$$x(t) = x(t_0) + \int_{t_0}^t f(\tau, x(\tau))\ d\tau, \quad \forall\ t \in [t_0, T]$$

is a solution on $[t_0, T]$. Since W is closed, $x(T) \in W$.
(c) Apply the local existence and uniqueness theorem at $(T, x(T))$.

• **3.26** Suppose there is no such t. Then, $y(t) \in W$ for all $t \in [t_0, T)$. From the previous exercise we can extend the solution beyond T, which contradicts the claim that $[t_0, T)$ is the maximal interval of existence.

• **3.27** Set $y(t) = x_1(t) - x_2(t)$ and $\mu = \mu_1 + \mu_2$.

$$\begin{aligned} \|\dot{y}(t)\| &= \|\dot{x}_1(t) - \dot{x}_2(t)\| \\ &= \|\dot{x}_1(t) - f_1(t, x_1(t)) - \dot{x}_2(t) + f_2(t, x_2(t)) + f_1(t, x_1(t)) - f_2(t, x_2(t))\| \\ &\le \mu_1 + \mu_2 + L\|x_1(t) - x_2(t)\| \ = \ \mu + \|y(t)\| \end{aligned}$$

$$\begin{aligned} \|y(t)\| &= \|y(t_0) + \int_{t_0}^{t} \dot{y}(s)\ ds\| \ \le \ \gamma + \int_{t_0}^{t} \|\dot{y}(s)\|\ ds \\ &\le \gamma + \mu(t-a) + \int_a^t L\|y(s)\|\ ds \end{aligned}$$

Application of Gronwall-Bellman inequality yields

$$\|y(t)\| \le \gamma + \mu(t-a) + \int_a^t [\gamma + \mu(s-a)]Le^{\int_s^t L\, d\tau}\ ds$$

After integrating the right-hand side by parts, we obtain

$$\|y(t)\| \le \gamma e^{L(t-a)} + \frac{\mu}{L}\left[e^{L(t-a)} - 1\right]$$

• **3.28** Let

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\ ds, \quad y(t) = x_0 + \int_{t_0'}^{t} f(s, y(s))\ ds$$

where $t_0' \ge t_0$. Then

$$x(t) - y(t) = \int_{t_0}^{t_0'} f(s, x(s))\ ds + \int_{t_0'}^{t} [f(s, y(s)) - f(s, x(s))]\ ds$$

We have $\|f(s, x(s))\| \le M$ for all $t \in [t_0, t_1]$, and $\|f(s, y) - f(s, x)\| \le L\|x - y\|$. Therefore

$$\|x(t) - y(t)\| \le M(t_0' - t_0) + \int_{t_0'}^{t} L\|x(s) - y(s)\|\ ds$$

By Gronwall-Bellman inequality

$$\|x(t) - y(t)\| \le M(t_0' - t_0)e^{L(t - t_0')}$$

Hence, over any compact interval of time, we have

$$\|x(t) - y(t)\| \le K(t_0' - t_0)$$

• **3.29**

$$\dot{x} = f(t, x), \quad x(t_0) = \eta$$

Setting $y = x - \eta$, we obtain

$$\dot{y} = f(t, y + \eta), \quad y(t_0) = 0$$

Since f is continuously differentiable in x, the solution will be continuously differentiable in η. Let

$$y_\eta(t, \eta) = \frac{\partial y(t, \eta)}{\partial \eta} = \frac{\partial x(t, \eta)}{\partial \eta} - I = x_\eta(t, \eta) - I$$

From (3.4)

$$\frac{\partial}{\partial t}y_\eta(t,\eta) = A(t,\eta)y_\eta(t,\eta) + B(t,\eta), \quad y_\eta(t_0,\eta) = 0$$

where

$$\begin{aligned} A(t,\eta) &= \frac{\partial f}{\partial y}(t, y(t,\eta)+\eta) = \frac{\partial f}{\partial x}(t, x(t,\eta)) \\ B(t,\eta) &= \frac{\partial f}{\partial \eta}(t, y(t,\eta)+\eta) = \frac{\partial f}{\partial x}(t, x(t,\eta)) = A(t,\eta) \end{aligned}$$

Thus, $x_\eta(t,\eta)$ satisfies the variational equation

$$\frac{\partial}{\partial t}x_\eta(t,\eta) = A(t,\eta)x_\eta(t,\eta), \quad x_\eta(t_0,\eta) = I$$

- **3.30**

$$x(t,a,\eta) = \eta + \int_a^t f(s, x(s,a,\eta))\, ds$$

$$\begin{aligned} x_a(t) &= \frac{\partial}{\partial a}x(t,a,\eta) = -f(a,\eta) + \int_a^t \frac{\partial f}{\partial x}(s, x(s,a,\eta))\frac{\partial}{\partial a}x(s,a,\eta)\, ds \\ x_\eta(t) &= \frac{\partial}{\partial \eta}x(t,a,\eta) = I + \int_a^t \frac{\partial f}{\partial x}(s, x(s,a,\eta))\frac{\partial}{\partial \eta}x(s,a,\eta)\, ds \end{aligned}$$

Therefore

$$x_a(t) + x_\eta(t)f(a,\eta) = \int_a^t \left\{\frac{\partial f}{\partial x}(s, x(s,a,\eta))[x_a(s) + x_\eta(s)f(a,\eta)]\right\} ds$$

Differentiating with respect to t, we see that $x_a(t) + x_\eta(t)f(a,\eta)$ satisfies the differential equation

$$\frac{\partial}{\partial t}[x_a(t) + x_\eta(t)f(a,\eta)] = \frac{\partial f}{\partial x}(t, x(t,a,\eta))[x_a(t) + x_\eta(t)f(a,\eta)]$$

with initial condition

$$x_a(a) + x_\eta(a)f(a,\eta) = -f(a,\eta) + f(a,\eta) = 0$$

Thus

$$x_a(t) + x_\eta(t)f(a,\eta) \equiv 0, \quad \forall\, t \in [a, t_1]$$

- **3.31** Put

$$z(t) = x(a) + \int_a^t f(s, y(s))\, ds$$

so that $z(a) = x(a)$ and $z(t) \le y(t)$ for $a \le t \le b$.

$$\dot{z} = f(t, y(t)) \le f(t, z(t))$$

From the comparison lemma, we conclude that

$$z(t) \le x(t) \;\Rightarrow\; y(t) \le x(t), \quad \forall\, a \le t \le b$$