# Chapter 14: The Laplace Transform

## Exercises

**Ex. 14.3-1** $\cos\omega t = \frac{1}{2}(e^{+j\omega t} + e^{-j\omega t})$ and $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$

$\therefore \mathcal{L}[\cos\omega t] = \frac{1}{2}\left[\frac{1}{s-j\omega}+\frac{1}{s+j\omega}\right]=\frac{s}{s^2+\omega^2} \Rightarrow \underline{F(s)=\frac{s}{s^2+\omega^2}}$

**Ex. 14.3-2** $\mathcal{L}\,[e^{-2t} + \sin t] = \mathcal{L}\left[e^{-2t}\right] + \mathcal{L}\,[\sin t] = \frac{1}{s+2} + \frac{1}{s^2+1}$

$\underline{F(s) = \frac{s^2+s+3}{(s+2)(s^2+1)}}$

**Ex. 14.4-1** $P(s) = \mathcal{L}\,[2u(t) + 3e^{-4t}u(t)] = 2\mathcal{L}\,[u(t)] + 3\mathcal{L}\,[e^{-4t}u(t)] = \underline{\frac{2}{s}+\frac{3}{s+4}}$

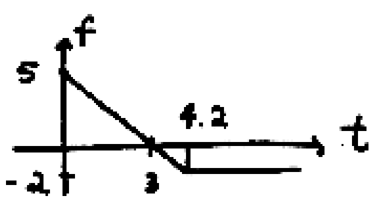
**Ex. 14.4-2** $F(s) = \mathcal{L}\,[\sin(t-2)u(t-2)] = e^{-2s}\mathcal{L}\,[\sin t] = e^{-2s}\left(\frac{1}{s^2+1}\right)$

**Ex. 14.4-3** $F(s) = \mathcal{L}[te^{-t}]$

Now, since $\mathcal{L}[t] = \frac{1}{s^2}$, use a frequency shift $s \to s+1$

$\underline{\therefore F(s) = \frac{1}{(s+1)^2}}$

**Ex. 14.4-4**




$f(t) = f_1(t) + f_2(t)$

$f_1 = \left(-\frac{5}{3}\,t+5\right)u(t)$

$f_2 = \left(\frac{5}{3}(t-4.2)-2\right)u(t-4.2)$

$\therefore F_1(s) = -\frac{5}{3s^2} + \frac{5}{s} \qquad F_2(s) = \frac{5}{3}\frac{e^{-4.2s}}{s^2} - \frac{2e^{-4.2s}}{s}$

$so\ F(s) = F_1(s)+F_2(s) = \underline{\frac{5e^{-4.2s}-6se^{-4.2s}+15s-5}{3s^2}}$

**Ex. 14.4-5** Using eqn. 14-1 $\quad f(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^2 3e^{-st} + 0 = \left.\frac{3e^{-st}}{-s}\right|_0^2 = \frac{3(1-e^{-2s})}{s}$

**Ex. 14.4-6** From Fig. 1 $f(t) = \begin{cases} 5/2\ t & 0<t<2 \\ 0 & \text{otherwise} \end{cases}$

$$So\ f(t) = \frac{5}{2}t\left[u(t)-u(t-2)\right] = \frac{5}{2}t\ u(t) - \frac{5}{2}t\ u(t-2) = \frac{5}{2}\left[t\,u(t)-(t-2)u(t-2)-2u(t-2)\right]$$

$$\therefore F(s) = \mathcal{L}\left[f(t)\right] = \frac{5}{2}\left[\frac{1}{s^2}-\frac{e^{-2s}}{s^2}-\frac{2e^{-2s}}{s}\right] = \underline{\frac{5}{2}\,\frac{1}{s^2}\left[1-e^{-2s}-2se^{-2s}\right]}$$

**Ex. 14.5-1** $F(s) = \dfrac{c+jd}{s+a-j\omega} + \dfrac{c-jd}{s+a+j\omega} = \dfrac{me^{j\theta}}{s+a-j\omega} + \dfrac{me^{j\theta}}{s+a+j\omega}$ $\quad$ *where* $m = \sqrt{c^2+d^2},$ $\quad \theta = \tan^{-1} d/c$

$$\therefore f(t)=e^{-at}\left[c\ \cos\omega t - d\sin\omega t\right] = e^{-at}\left[\sqrt{c^2+d^2}\cos(\omega t+\theta)\right] = \underline{m\ e^{-at}\cos(\omega t+\theta)}$$

**Ex. 14.5-2**

(a) $F(s) = \dfrac{8s-3}{s^2+4s+13} = \dfrac{1}{2}\times\dfrac{2(8s-3)}{(s+2)^2+9}$

$\therefore a=2,\ c=8,\ \omega=3\ \&\ ca-\omega d=-3 \Rightarrow d=\dfrac{-3(8)(2)}{-3}=6.33$

$\therefore \theta=\tan^{-1}\left(\dfrac{6.33}{8}\right)=38.4^\circ,\ m=\sqrt{(8)^2+(6.33)^2}=10.2$

$\Rightarrow \underline{f(t)=10.2e^{-2t}\cos(3t+38.40)}\ t \ge 0$

(b) $F(s) = \dfrac{3e^{-s}}{s^2+2s+17}$

First consider $F_1(s)=\dfrac{3}{s^2+2s+17}=\dfrac{1}{2}\times\dfrac{(2(3))}{(s+1)^2+16}$

$So\ a=1,\ c=0,\ \omega=4\ and\ -\omega d=3\Rightarrow d=-3/4$

$\therefore m=|d|=3/4,\ \theta=\tan^{-1}(-3/(4/0))=-90^\circ$

$So\ f_1(t) = (3/4)e^{-t}\sin 4t$

$Now$ since $F(s) = e^{-s}F_1(s) \Rightarrow f(t) = f_1(t-1)$

$\underline{\therefore f(t)=(3/4)e^{-(t-1)}\ \sin\left[4(t-1)\right]\ t \ge 1}$

**Ex. 14.5-3**

(a) $\mathrm{F(s)} = \dfrac{s^2-5}{s(s+1)^2} = \dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$

$A = sF(s)|_{s=0} = \dfrac{-5}{1} = -5$

$C = (s+1)^2 F(s)|_{s=-1} = \dfrac{1-5}{-1} = 4$

B$\Rightarrow$ multiply both sides by $s(s+1)^2$

$s^2-5=-5(s+1)^2+Bs(s+1)+4s$

equating coefficients $\Rightarrow B = 6$

$\therefore F(s) = -5/s+6/(s+1)+4/(s+1)^2$

$\Rightarrow \underline{f(t)=-5+6e^{-t}+4te^{-t}\ \ t \ge 0}$

(b) $$F(s)=\frac{4s^2}{(s+3)^3}=\frac{A}{(s+3)}+\frac{B}{(s+3)^2}+\frac{C}{(s+3)^3}$$

$$A=(1/2)\ d^2/ds^2\left[(s+3)^2F(s)\right]_{s=-3}=4$$

$$B=d/ds\left[(s+3)^3F(s)\right]_{s=-3}=-24$$

$$C=(s+3)^3F(s)_{s=-3}=36$$

$$\therefore\ F(s)=4/(s+3)-24/(s+3)^2+36/(s+3)^3$$

Using Table 14-3 : $\underline{f(t)=4e^{-3t}-24te^{-3t}+18t^2e^{-3t}}\quad t\ge 0$

**Ex. 14.6-1**

(a) $$F(s)=\frac{6s+5}{s^2+2s+1}$$

$$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\left[\frac{s(6s+5)}{s^2+2s+1}\right]=\underline{6}$$

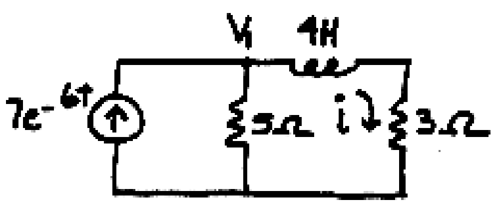
$$f(\infty)=\lim_{s\to 0} sF(s)=\lim_{s\to 0}\left[\frac{s(6s+5)}{s^2+2s+1}\right]=\underline{0}$$

(b) $$F(s)=\frac{6}{s^2-2s+1}$$

$$f(0)=\lim_{s\to\infty}\left[\frac{6s}{s^2-2s+1}\right]=\underline{0}$$

$$f(\infty)=\lim_{s\to 0}\left[\frac{6s}{s^2-2s+1}\right]=\text{undefined}\qquad \underline{\therefore\text{no final value}}$$

**Ex. 14.7-1**




KCL at $v_1$: $v_1/5+i=7e^{-6t}$ (1)

also : $v_1=3i+4\,di/dt$ (2)

(2) into (1) yields : $\frac{di}{dt}+2i=\frac{35}{4}e^{-6t}$

Taking the Laplace Transform of the D.E.

$$s\,I(s)-i(0)+2\,I(s)=\frac{35}{4}\frac{1}{s+6}\qquad \text{where } i(0)=0$$

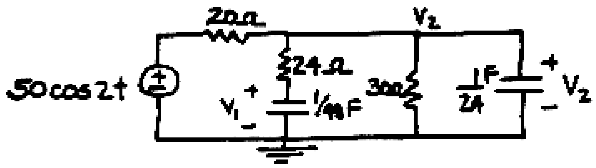
$$\Rightarrow I(s)=\frac{35}{4}\frac{1}{(s+2)(s+6)}$$

Now $\frac{1}{(s+2)\ (s+6)}=\frac{A}{s+2}+\frac{B}{s+6}$ where $A=\frac{1}{s+6}\Big|_{s=-2}=\frac{1}{4}$ and $B=\frac{1}{s+2}\Big|_{s=-6}=-\frac{1}{4}$

$$\therefore I(s)=\frac{35}{16}\frac{1}{s+2}-\frac{35}{16}\frac{1}{s+6}$$

$$\Rightarrow \underline{i(t)=\frac{35}{16}e^{-2t}-\frac{35}{16}e^{-6t}}$$

**Ex. 14.7-2**




$v_1(0) = 10$

$v_2(0) = 25$

KCL at $v_1$ : $(1/48)\ dv_1/dt + (v_1 - v_2)/24 = 0$

$$\Rightarrow 2\,v_1 + dv_1/dt - 2v_2 = 0 \qquad (1)$$

KCL at $v_2$ : $(v_2 - 50\cos 2t)/20 + (v_2 - v_1)/24 + v_2/30 + (1/24)\,dv_2/dt = 0$

$$\Rightarrow -v_1 + 3v_2 + dv_2/dt = 60\cos 2t \qquad (2)$$

Taking Laplace Transform of (1) & (2)

$$(2+s)\ V_1(s) - 2V_2(s) = 10 \qquad (3)$$

$$-V_1(s) + (3+s)\ \ V_2(s) = \frac{25s^2 + 60s + 100}{s^2 + 4} \qquad (4)$$

Using Cramers rule with (3) & (4)

$$V_2(s) = \frac{(2+s)\left(\frac{25s^2+60s+100}{s^2+4}\right)+10}{(2+s)\ (3+s)-2} = \frac{(2+s)\left(25s^2+60s+100\right)+10\left(s^2+4\right)}{\left(s^2+4\right)(s+1)(s+4)} = \frac{25s^3+120s^2+220s+240}{\left(s^2+4\right)(s+1)(s+4)}$$

$$Let\ V_2(s) = \frac{A}{s+j2} + \frac{A^*}{s-j2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = \left.\frac{25s^3+120s^2+220s+240}{(s+1)(s+4)(s-j2)}\right|_{s=-j2} = \frac{-240-j240}{-40} = 6+j6$$

$$\therefore A^* = 6 - j6$$

$$B = \left.\frac{25s^3+120s^2+220s+240}{\left(s^2+4\right)(s+4)}\right|_{s=-1} = \frac{115}{15} = \frac{23}{3}$$

$$C = \left.\frac{25s^3+120s^2+220s+240}{\left(s^2+4\right)(s+1)}\right|_{s=-4} = \frac{-320}{-60} = \frac{16}{3}$$

$$\therefore V_2(s) = \frac{6+j6}{s+j2} + \frac{6-j6}{s-j2} + \frac{23/3}{s+1} + \frac{16/3}{s+4}$$

$\Rightarrow \underline{v_2(t) = 12\cos 2t + 12\sin 2t + (23/3)e^{-t} + (16/3)e^{-4t}\ \text{V},}$ $\qquad t \geq 0$

**Ex. 14.7-3** Taking Laplace Transform of the D.E.

$$s^2F(s)=s\,f(0)-f^{'}(0)+5\left[s\,F(s)-f(0)\right]+6F(s)=\frac{10}{s+3}$$

Plugging in initial conditions

$$(s^2+5s+6)\ F(s) = \frac{10}{s+3}+2s+10 = \frac{2s^2+16s+40}{s+3}$$

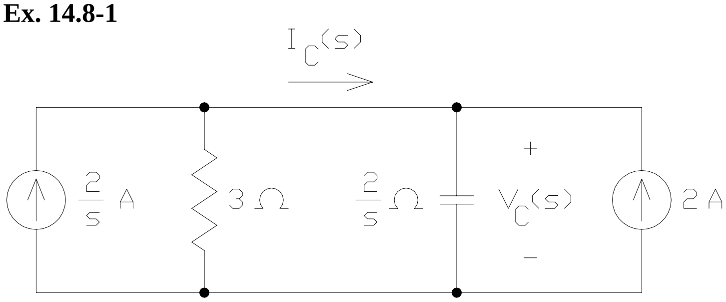
$$\Rightarrow F(s)=\frac{2s^2+16s+40}{(s+3)(s+2)(s+3)}=\frac{A}{(s+3)^2}+\frac{B}{(s+3)}+\frac{C}{(s+2)}$$

$$\Rightarrow A = -10,\ B = -14,\ C = 16$$

$$F(s) = \frac{-10}{(s+3)^2} + \frac{-14}{s+3} + \frac{16}{s+2}$$

$$\therefore\ \underline{f(t) = -10te^{-3t} - 14e^{-3t} + 16e^{-2t}} \quad \text{for } t \ge 0$$

**Ex. 14.8-1**




KCL at top node:

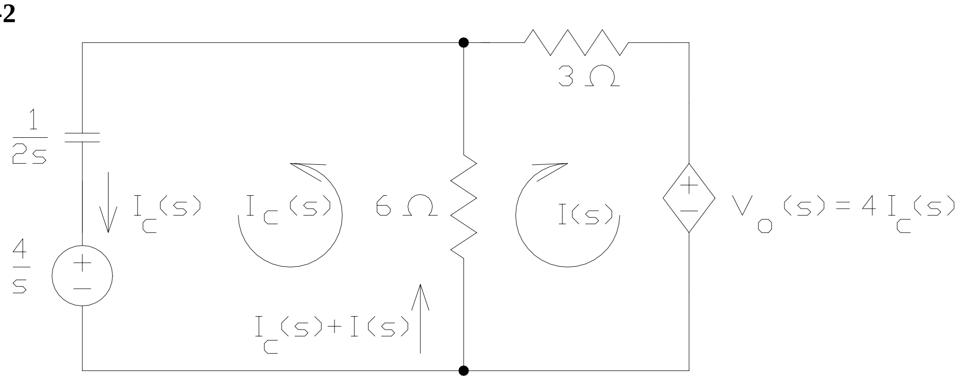
$$\frac{V_C(s)}{3}+\frac{s}{2}V_C(s)=\frac{2}{s}+2$$

$$V_C(s)=\frac{6}{s}-\frac{2}{s+\frac{2}{3}}$$

$$v_C(t)=\left(6-2e^{-0.67t}\right)u(t)\ \text{V}$$

$$I_C(s)=\frac{V_C(s)}{\frac{2}{s}}-2=\frac{\frac{2}{3}}{s+\frac{2}{3}} \ \text{so}\ i_C(t)=\frac{2}{3}e^{-0.67t}\,u(t)\ \text{A}$$

**Ex. 14.8-2**



Mesh Equations:

$$-\frac{4}{s}-\frac{1}{2s}I_C(s)-6\big(I(s)-I_C(s)\big)=0 \quad\Rightarrow\quad -\frac{4}{s}=\left(6+\frac{1}{2s}\right)I_C(s)+6I(s)$$

$$6\big(I(s)-I_C(s)\big)+3I(s)+4I_C(s)=0 \quad\Rightarrow\quad I(s)=-\frac{10}{9}I_C(s)$$

Solving for $I_c(s)$:

$$-\frac{4}{s}=\left(-\frac{2}{3}+\frac{1}{2s}\right)I_C(s) \quad\Rightarrow\quad I_C(s)=\frac{6}{s-\frac{3}{4}}$$

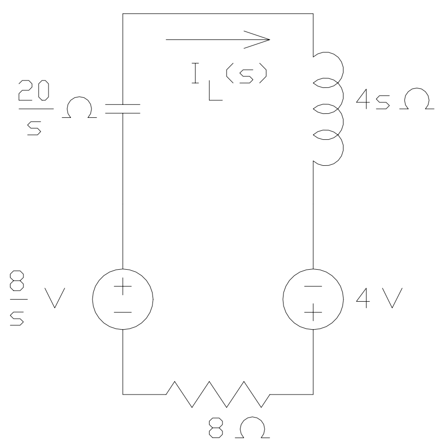
So $V_o(s)$ is

$$V_o(s)=4I_C(s)=\frac{24}{s-\frac{3}{4}}$$

Back in the time domain:

$$v_o(t)=24e^{0.75t}\,u(t)\text{ V}$$

**Ex. 14.8-3**




KVL:

$$\frac{8}{s}+4=\left(\frac{20}{s}+8+4s\right)I_L(s)$$

so

$$I_L(s)=\frac{2+s}{s^2+2s+5}=\frac{(s+1)+1}{(s+1)^2+4}$$

Taking the inverse Laplace transform:

$$i(t)=\left(e^{-t}\cos 2t+\frac{1}{2}e^{-t}\sin 2t\right)u(t)\text{ A}$$

**Ex 14.9-1**

(a) $\textit{impulse response}=\mathfrak{L}^{-1}\left[\frac{5}{s+5}-\frac{10}{s+10}\right]=\left(5e^{-5t}-10e^{-10t}\right)u(t)$

(b) $\textit{step response}=\mathfrak{L}^{-1}\left[\frac{1}{s+10}-\frac{1}{s+5}\right]=\left(e^{-10t}-e^{-5t}\right)u(t)$

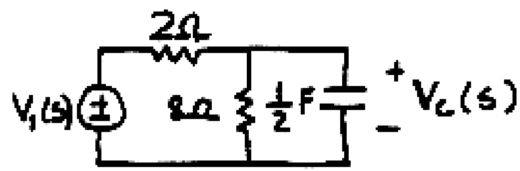
**Ex 14.9-2**

$$H(s)=\mathfrak{L}\left[5e^{-2t}\sin(4t)u(t)\right]=\frac{5(4)}{(s+2)^2+4^2}=\frac{20}{s^2+4s+20}$$

$$\textit{step response}=\mathfrak{L}^{-1}\left[\frac{H(s)}{s}\right]=\mathfrak{L}^{-1}\left[\frac{1}{s}-\frac{s+4}{s^2+4s+20}\right]=(1-e^{-2t}(\cos 4t-\frac{1}{2}\sin 4t))u(t)$$

**Ex. 14.10-1**

(a) Voltage divider yields




$$H(s) = \frac{V_c(s)}{V_1(s)} = \frac{\dfrac{8/s/2}{8+s/2}}{2+\dfrac{8/s/2}{8+s/2}} = \frac{8/(1+4s)}{2+8/(1+4s)}$$

$$= \frac{8}{10+8s} = \underline{\frac{1}{1.25+s}}$$

(b) $h(t) = \mathfrak{L}^{-1}[H(s)] = \underline{e^{-1.25t}}$

**Ex. 14.10-2**

$$h(t) = e^{-2t} \quad \Rightarrow \quad H(s) = 1/(s+2)$$

$$f(t) = u(t) \quad \Rightarrow \quad F(s) = 1/s$$

$$\text{so } h(t) * f(t) = \mathfrak{L}^{-1}[H(s)F(s)] = \mathfrak{L}^{-1}\left[\frac{1}{s(s+2)}\right] = \mathfrak{L}^{-1}\left[\frac{1/2}{s} + \frac{-1/2}{s+2}\right] = \underline{1/2\,u(t) - 1/2\,e^{-2t}}$$

**Ex. 14.11-1** For the poles to be in the left half of the s-plane, the s-term needs to be positive.

$$\therefore 8 - k > 0 \text{ or } k < 8 \quad \text{so } \underline{0 \le k < 8}$$

**Ex. 14.11-2** To get oscillation, we want the s-term to go to zero

$$\therefore \underline{k = 8} \text{ will give oscillation}$$

**Ex. 14.12-1**

$$V_0(s) = 0.1\left[\frac{2}{s} - \left(2\frac{s+5}{(s+5)^2+10^2} + \frac{10}{(s+5)^2+10^2}\right)\right]$$

$$= 0.1\left[\frac{2}{s} - \frac{2s+20}{s^2+10s+125}\right]$$

$$= 0.1\,\frac{2(s^2+10s+125)-(2s+20)s}{s^2+10s+125}$$

$$= 0.1\,\frac{250}{s^2+10s+125} = \frac{25}{s^2+10s+125}$$

These specifications are consistent.

## Problems

## Section 14-3: Laplace Transform

**P14.3-1** $\mathcal{L}\left[A\, f_1(t)\right] = A\, F_1(s)$

$f_1(t) = \cos(\omega t) \;\Rightarrow\; F_1(s) = s \big/ \left(s^2+\omega^2\right)$

$\therefore\; F(s) = \dfrac{As}{s^2+\omega^2}$

**P14.3-2** From Table 14.4-1: $f(t) = t^n,\; F(s) = \dfrac{n!}{s^{n+1}}$, but we have n = 1

$\therefore F(s) = \dfrac{1}{s^2}$

**P14.3-3** Linearity: $\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$

Here $a_1 = a_2 = 1$

$$\mathcal{L}\left[f_1(t)\right] = \mathcal{L}\left[e^{-3t}\right] = \frac{1}{s+3} = F_1(s)$$

$$\mathcal{L}\left[f_2(t)\right] = \mathcal{L}[t] = \frac{1}{s^2} = F_2(s)$$

so $F(s) = \dfrac{1}{s+3} + \dfrac{1}{s^2}$

**P14.3-4** $f(t) = A\left(1-e^{-bt}\right)u(t) = \mathfrak{L}\left[Af_1(t)\right] = AF_1(s)$

$f_i(t) = A\left(1-e^{-bt}\right)u(t) = 1u(t) - e^{-bt}u(t) = f_2(t) + f_3(t)$

$F_2(s) = \dfrac{1}{s},\; F_3(s) = \dfrac{-1}{s+b}$

$\therefore F(s) = A\left[\dfrac{1}{s} - \dfrac{1}{s+b}\right] = \dfrac{Ab}{s(s+b)}$

## Section 14-4: Impulse Function and Time Shift Property

**P14.4-1** $f(t) = A\left[u(t) - u(t-T)\right]$

$$F(s) = A\mathcal{L}\left[u(t)\right] - A\mathcal{L}\left[u(t-T)\right] = A/s - \frac{Ae^{-sT}}{s} = A\frac{\left(1-e^{-sT}\right)}{s}$$

**P14.4-2** Have $f(t) = 1\left[u(t) - u(t-T)\right]e^{at}$

$F(s) = \mathcal{L}\left[e^{at}\left[u(t) - u(t-T)\right]\right]$

Now $\mathcal{L}\left[u(t) - u(t-T)\right] = \dfrac{1-e^{-sT}}{s}$ and $\mathcal{L}\left[e^{at}g(t)\right] = G(s-a)$

$\Rightarrow F(s) = \dfrac{1-e^{(s-a)T}}{(s-a)}$

**P14.4-3**

(a) $\underline{F(s) = 2/(s+3)^3}$ using Table 14-3

(b) $f(t) = \delta(t-T)\,u(t-T)$ & using time shift, *get* : $\underline{F(s) = e^{-sT}\,[\delta(t)] = e^{-sT}}$

(c) $F(s) = \dfrac{5}{(s+4)^2 + (5)^2} = \underline{\dfrac{5}{(s+4)^2 + 25}}$ using Table 14.4-1

**P14.4-4**

$$g(t) = e^{-t}u(t-0.5),\ \tau = 0.5$$

$$\mathcal{L}[f(t-\tau)\,u(t-\tau)] = e^{-s\tau}F(s)$$

$$f(t-\tau) = e^{-t-\tau}(e^{\tau});\ f(t) = e^{\tau}\,e^{-t}$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a},\ \text{here } a=1$$

$$F(s) = e^{\tau}\left(\frac{1}{s+1}\right)e^{-0.5s} = \left(e^{0.5}\,e^{-0.5s}\right)\left(\frac{1}{s+1}\right) = \underline{\frac{e^{0.5-0.5s}}{s+1}}$$

**P14.4-5**

$$\mathcal{L}[f_1(t-\tau)u(t-\tau)] = e^{-s\tau}\,F(s)$$

$$f_1(t-\tau) = \frac{-t-T}{T};\ \tau = T;\ f_1(t) = -t/\tau$$

$$\therefore F_1(s) = \frac{1}{\tau}\left(\frac{1}{s^2}\right) \quad and \quad \underline{F(s) = \left[\frac{-1}{Ts^2}\right]e^{-sT}}$$

## Section 14-5: Inverse Laplace Transform

**P14.5-1**

$$F(s) = \frac{s+3}{s^3+3s^2+6s+4} = \frac{s+3}{(s+1)\left[(s+1)^2+3\right]} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+4}$$

$$A = \left.\frac{s+3}{(s+1)^2+3}\right|_{s=-1} = 2/3$$

$$\therefore \frac{(s+3)}{(s+1)(s^2+2s+4)} = \frac{2/3}{s+1} + \frac{Bs+C}{s^2+2s+4}$$

$$\Rightarrow (s+3) = (2/3 + B)\ s^2 + (4/3 + B + C)\,s + 8/3 + C$$

Equating coefficients $\quad s^2 : 0 = 2/3 + B \ \Rightarrow\ B = 2/3$

$\quad s : 1 = 4/3 - 2/3 + C \ \Rightarrow\ C = 1/3$

$$\text{So } F(s) = \frac{2/3}{s+1} + \frac{-(2/3)s+1/3}{(s+1)^2+3} = \frac{2/3}{s+1} + \frac{-(2/3)(s+1)}{(s+1)^2+3} + \frac{(1/3)\sqrt{3}}{(s+1)^2+3}$$

$$\therefore \underline{f(t) = (2/3)\,e^{-t} - (2/3)\,e^{-t}\ \cos\sqrt{3}t\ + 1/\sqrt{3}e^{-t}\ \sin\sqrt{3}}$$

**P14.5-2**

$$F(s)=\frac{s^2-2s+1}{s^3+3s^2+4s+2}=\frac{s^2-2s+1}{(s+1)(s+1-j)(s+1+j)}=\frac{a}{s+1-j}+\frac{a^*}{s+1+j}+\frac{b}{s+1}$$

$$b=\left.\frac{s^2-2s+1}{(s+1)^2+1}\right|_{s=-1}=4$$

$$a=\left.\frac{s^2-2s+1}{(s+1)(s+1+j)}\right|_{s=-1+j}=\frac{3-j4}{-2}=-3/2+j2$$

$$a^*=-3/2-j2$$

$$\Rightarrow\ F(s)=\frac{-3/2+j2}{s+1-j}+\frac{-3/2-j2}{s+1+j}+\frac{4}{s+1}$$

for first two terms : $m=\sqrt{(-3/2)^2+(2)^2}=5/2$

$$\theta=\tan^{-1}(2/-3/2)=126.9^\circ$$

$$\therefore\ \underline{f(t)=5e^{-t}\cos\left(t+233^\circ\right)+4e^{-t}}$$

**P14.5-3**

$$F(s)=\frac{5s-1}{(s+1)^2(s-2)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s-2}$$

$$B=\left.\frac{5s-1}{s-2}\right|_{s=-1}=2$$

$$C=\left.\frac{5s-1}{(s+1)^2}\right|_{s=2}=1$$

$$A=\frac{d}{ds}\left[(s+1)^2F(s)\right]\Big|_{s=-1}=-9\Big/(s-2)^2\Big|_{s=-1}=-1$$

$$\therefore F(s)=-1\Big/(s+1)+2\Big/(s+1)^2+1\Big/(s-2)$$

$$\Rightarrow\ \underline{f(t)=-e^{-t}+2te^{-t}+e^{2t}}$$

**P14.5-4**

$$Y(s)=\frac{1}{(s+1)\left(s^2+2s+2\right)}=\frac{1}{(s+1)\left[(s+1)^2+1\right]}=\frac{A}{s+1}+\frac{Bs+C}{(s+1)^2+1}$$

$$A=\left.\frac{1}{s^2+2s+2}\right|_{s=-1}=1$$

$$B\Rightarrow\frac{1}{(s+1)\left(s^2+2s+2\right)}=\frac{1}{s+1}+\frac{Bs+C}{s^2+2s+2}$$

$$\Rightarrow 1=s^2+2s+2+(Bs+C)(s+1)$$

$$1=(B+1)s^2+(B+C+2)s+C+2$$

Equating coefficients : $s^2$ : $0=B+1\quad\Rightarrow B=-1$

$s$ : $0=B+C+2\Rightarrow C=-1$

$$\therefore\ Y(s)=\frac{1}{s+1}-\frac{s+1}{(s+1)^2+1}\Rightarrow\underline{y(t)=e^{-t}-e^{-t}\cos t}$$

**P14.5-5** $F(s)=\dfrac{2(s+3)}{(s+1)(s+1+2j)(s+1-2j)}$ using partial fractions yields

$$F(s)=\frac{1}{s+1}+\frac{-(s+1)}{(s+1)^2+4}+\frac{2}{(s+1)^2+4}$$

So $\underline{f(t)=e^{-t}-e^{-t}\cos(2t)+e^{-t}\sin(2t)}$

**P14.5-6** Use partial fractions : $\dfrac{2(s+3)}{s(s+1)(s+2)}=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{s+2}$

$$sF(s)=\left.\frac{2(s+3)}{(s+1)(s+2)}\right|_{s=0}=A=3;\quad (s+1)\,F(s)=\left.\frac{2(s+3)}{s(s+2)}\right|_{s=-1}=B=-4$$

and $(s+2)\,F(s)=\left.\dfrac{2(s+3)}{s(s+1)}\right|_{s=-2}=C=1$

So $F(s)=\dfrac{3}{s}+\dfrac{-4}{s+1}+\dfrac{1}{s+2}$ yields $\underline{f(t)=\left(3-4e^{-t}+e^{-2t}\right)}\,u(t)$

## Section 14-6: Initial and Final Value Theorems

**P14.6-1**

(a) $f(0)=\lim\limits_{s\to\infty}sF(s)=\lim\limits_{s\to\infty}\dfrac{2s^2-3s+4}{s^2+3s+2}=\dfrac{2s^2}{s^2}=\underline{2}$

(b) $f(\infty)=\lim\limits_{s\to 0}sF(s)=\dfrac{4}{2}=\underline{2}$

**P14.6-2** initial value

$$\lim_{s\to\infty}sV(s)=\lim_{s\to\infty}\frac{s(s+16)}{s^2+4s+12}=\lim_{s\to\infty}\frac{s^2+16s}{s^2+4s+12}=1\ \therefore\ \underline{v(0)=1}$$

final value

$$\lim_{s\to 0}s\left(\frac{s+16}{s^2+4s+12}\right)=\lim_{s\to 0}\frac{s^2+16s}{s^2+4s+12}=0\ \therefore\ \underline{v(\infty)=0}$$

$\mathrm{Re}\{p_i\}<0$ since $p_i=-2\pm 2.828j$

**P14.6-3** initial value : $\lim\limits_{s\to\infty}sV(s)=\lim\limits_{s\to\infty}\dfrac{s^2+10s}{3s^3+2s^2+1}=0$ $\therefore\ \underline{v(0)=0}$

final value

poles : $(3s^2+2s+1)$ yields $p_i=-0.333\pm 0.471i$ $\therefore\ \mathrm{Re}\{p_i\}<0$

So $\lim\limits_{s\to 0}sV(s)=\lim\limits_{s\to 0}\dfrac{s(s+10)}{s\left(3s^2+2s+1\right)}=10$ $\therefore\ \underline{v(\infty)=10}$

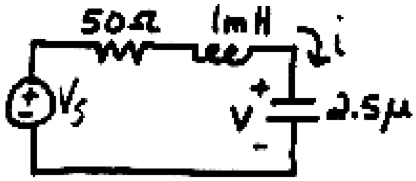
**P14.6-4**

initial value: $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \dfrac{-2s^2 - 14s}{s^2 - 2s + 10} = -2 \therefore \underline{f(0) = -2}$

final value: $s^2 - 2s + 10$ yields $p_i = 1 \pm 3i$ $\quad \text{Re}\{p_1\} > 0 \quad \therefore$ <u>no final value exists</u>

## <u>Section 14-7:</u> Solution of Differential Equations Describing a Circuit

**P14.7-1**




$v_s = 2e^{-2\times 10^4 t}\ \text{V}$

$i(0) = 1\text{A},\ v(0) = 8\text{V}$

$$\text{KVL}: 50I + 0.001\frac{di}{dt} + v = 2e^{-2\times 10^4 t} \qquad (1)$$

$$\text{also}: \ = 2.5 \times 10^{-6}\frac{dv}{dt} \qquad (2)$$

Take the transform of (1) & (2)

$$50\, I(s) + 0.001\left[s\, I(s) - i(0)\right] + V(s) = \frac{2}{s + 2\times 10^4} \qquad (3)$$

$$I(s) = 2.5\times 10^{-6}\left[s\, V(s) - v(0)\right] \qquad (4)$$

Solving for V(s) in (4) and plugging into (3) yields

$$I(s) = \frac{s^2 + 1.4\times 10^4 s - 1.6\times 10^8}{\left(s+10^4\right)\left(s+2\times 10^4\right)\left(s+4\times 10^4\right)} = \frac{A}{s+10^4} + \frac{B}{s+2\times 10^4} + \frac{C}{s+4\times 10^4}$$

$$A = \frac{s^2 + 1.4\times 10^4 s - 1.6\times 10^8}{\left(s+2\times 10^4\right)\left(s+4\times 10^4\right)}\Bigg|_{s=-10^4} = \frac{-2\times 10^8}{3\times 10^8} = \frac{-2}{3}$$

$$B = \frac{s^2 + 1.4\times 10^4 s - 1.6\times 10^8}{\left(s+10^4\right)\left(s+4\times 10^4\right)}\Bigg|_{s=-2\times 10^4} = \frac{.4\times 10^8}{2\times 10^8} = \frac{1}{5}$$

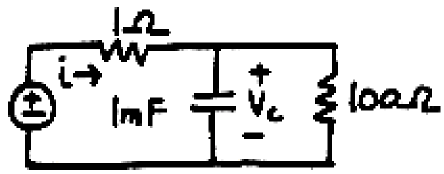
$$C = \frac{s^2 + 1.4\times 10^4 s - 1.6\times 10^8}{\left(s+10^4\right)\left(s+2\times 10^4\right)}\Bigg|_{s=-4\times 10^4} = \frac{8.8\times 10^8}{6\times 10^8} = \frac{22}{15}$$

$$\therefore I(s) = -\frac{2/3}{s+10^4} + \frac{1/5}{s+2\times 10} + \frac{22/15}{s+4\times 10^4}$$

$$\Rightarrow \underline{i(t) = \frac{1}{15}\left[-10e^{-10^4 t} + 3e^{-2\times 10^4 t} + 22e^{-4\times 10^4 t}\right]\text{A}}$$

**P14.7-2**

v = 160 cos400t




At $t = 0^+$: $V_c = 0$

$\Rightarrow i(0^+) = 160\text{A}$

$$\text{KCL at top}: \; 10^{-3}\, dv_c/dt + v_c/100 = i \qquad (1)$$

$$\text{also}: \; i = (v - v_c)/1 \Rightarrow v_c = v - i \qquad (2)$$

$$\text{(2) into (1) yields}: \; \frac{di}{dt} + 1010i = 1600\cos 400t - 6.4 \text{ x } 10^4 \sin 400t$$

Taking the Laplace Transform yields:

$$sI(s) - i(0) + (1010)I(s) = \frac{1600s}{s^2 + (400)^2} - \frac{(6.4\,x\,10^2)(400)}{s^2 + (400)^2}$$

$$\Rightarrow I(s) = \frac{160}{s + 1010} + \frac{1600s - 2.5 \text{ x } 10^7}{(s + 1010)\left[s^2 + (400)^2\right]}$$

$$I(s) = I_1 + I_2(s)$$

$$\text{Now let } I_2(s) = \frac{A}{s + 1010} + \frac{B}{s + j400} + \frac{B^*}{s - j400}$$

$$A = \left.\frac{1600s - 2.5 \text{ x } 10^7}{s^2 + (400)^2}\right|_{s = -1010} = -23.1$$

$$B = \left.\frac{1600s - 2.5 \text{ x } 10^7}{(s + 1010)(s - j400)}\right|_{s = -j400} = \frac{2.56 \text{ x } 10^7 \angle 1.4^\circ}{8.69 \text{ x } 10^5 \angle 68.4^\circ} = 11.5 - j27.2$$

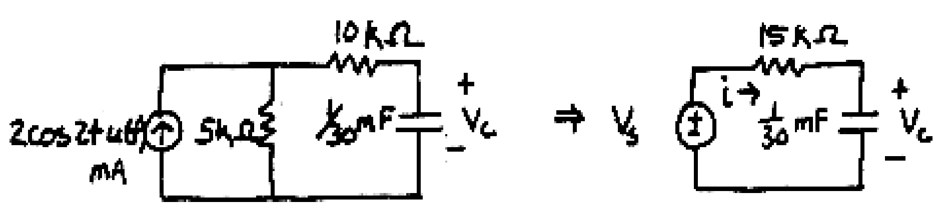
$$B^* = 11.5 + j27.2$$

$$\therefore I(s) = I_1(s) + I_2(s) = \frac{136.9}{s + 1010} + \frac{11.5 - j27.2}{s + j400} + \frac{11.5 + j27.2}{s - j400}$$

$$\Rightarrow i(t) = 136.9e^{-1010t} + 2(11.5)\cos 400t - 2(27.2)\sin 400t$$

$$\underline{i(t) = 136.9e^{-1010t} + 23.0\cos 400t - 54.4\sin 400t}$$

**P14.7-3**




where $v_s = 10\cos 2t\, u(t)$ & $v_c(0) = 0$

$$\text{KVL}: v_c + 15i = 10\cos 2t \quad \text{but} \quad i = \frac{1}{30}\frac{dv_c}{dt} \quad \Rightarrow \quad \frac{dv_c}{dt} + 2v_c = 20\cos 2t$$

Taking the Laplace Transform yields:

$$sV_c(s) - v_c(0) + 2V_c(s) = 20s\big/\left(s^2+4\right)$$

$$\Rightarrow V_c(s) = \frac{20s}{(s+2)\left(s^2+4\right)} = \frac{A}{s+2} + \frac{B}{s+j2} + \frac{B^*}{s-j2}$$

$$A = \frac{20s}{s^2+4}\Big|_{s=-2} = \frac{-40}{8} = -5$$

$$B = \frac{20s}{(s+2)(s-j2)}\Big|_{s=-j2} = j5/(1+j) = 5/2 + j5/2$$

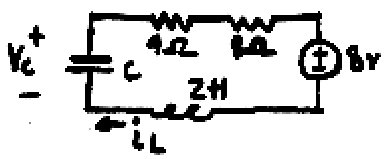
$$B^* = 5/2 - j5/2$$

$$\therefore V_c(s) = -5/(s+2) + (5/2 - j5/2)/(s+j2) + (5/2 - j5/2)/(s-j2)$$

$$\Rightarrow \underline{v_c(t) = -5e^{-2t} + 5(\cos 2t + \sin 2t)\ \text{V}}$$

**P14.7-4** From Prob. 9.9-11 : $v_c(0) = 0,\ i_L(0) = 0$

after source transformation




KVL : $v_c + 12i_L + 2\,di_L/dt = -8$ (1)

also : $i_L C\, dv_c/dt$ (2)

Laplace Transform of (1) & (2) yields:

$$V_c(s) + 12\,I_L(s) + 2\left[sI_L(s) - i_L(0)\right] = -8/s \qquad (3)$$

$$I_L(s) = C\left[sV_c(s) - v_c(0)\right] \qquad (4)$$

(4) into (3) yields : $V_c(s) = \dfrac{-(8/2)\,C}{s\left(s^2 + 6s + 1/2C\right)}$

(a) $C = 1/18 : V_c(s) = -72\big/s(s+3)^2 = a/s + b/(s+3) + c\big/(s+3)^2$

$$\Rightarrow a = -8,\ b = 8,\ c = 24, \text{ so } V_C(s) = -8/s + 8/(s+3) + 24\big/(s+3)^2$$

$$\therefore \underline{v_c(t) = -8 + 8e^{-3t} + 24te^{-3t}}$$

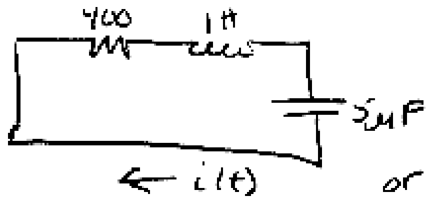
(b) $C = 1/10 : V_c(s) = -40/s(s+1)(s+5) = a/s + b/(s+1) + c/(s+5)$

$$\Rightarrow a = -8,\ b = 10,\ c = -2, \text{ so} V_c(s) = -8/5 + b/(s+1) - 2/(s+5)$$

$$\therefore \underline{v_c(t) = -8 + 10e^{-t} - 2e^{-5t}}$$

**P14.7-5** By inspection $v_c(0^-) = 10V, i_L(0^-) = 0_A$

@ t > 0




$$L\frac{d^2i}{dt^2} + 400\frac{di}{dt} + \frac{i_c}{C} = 0$$

$$\frac{di}{dt} = 10$$

$$L\left(s^2 I(s) - si(0) - \frac{di(0)}{dt}\right) + R\left(sI(s) - i(0)\right) + \frac{I(s)}{C} = 0$$

Solving for I(s) and plugging in values yields

$$I(s) = \frac{-\left[1s(0) + 1(10) + 400(0)\right]}{1s^2 + 400s + 2\times 10^5} = \frac{-10}{s^2 + 400s + 2\times 10^5}$$

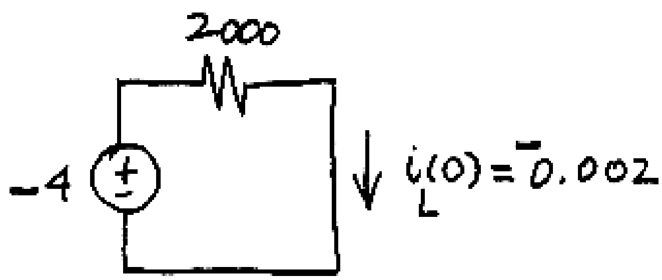
The poles are $p_i = -200 \pm 400i$

$$\therefore i(t) = -\frac{1}{40}e^{-200t}\sin(400t)A$$

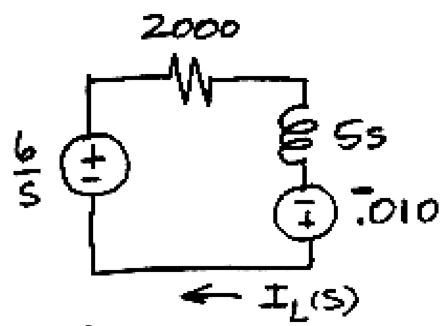
## Section 14-8: Circuit Analysis Using Impedance and Initial Conditions

**P14.8-1**

t < 0




t > 0

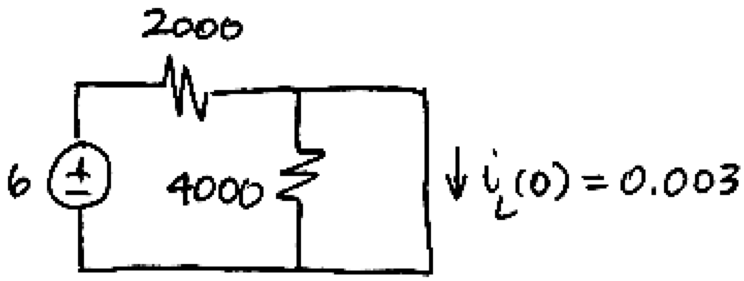



$$I_L(s) = \frac{\frac{6}{s} - 0.010}{5s + 2000} = \frac{\frac{6}{s} - 0.002s}{s(s+400)} = \frac{.003}{s} = \frac{.005}{s+400}$$

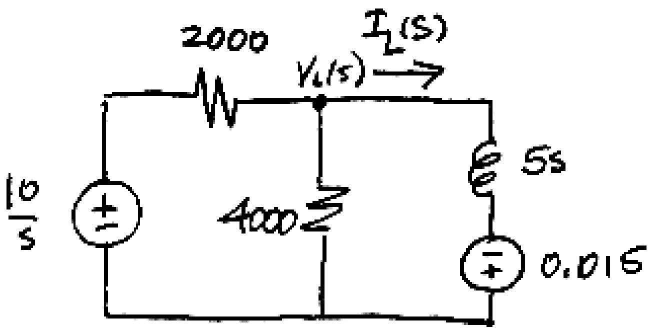
$$i_L(t) = \begin{cases} 2mA & t < 0 \\ 3 - 5e^{-400t}mA & t > 0 \end{cases}$$

**P14.8-2**

t < 0




t > 0




$$0 = \frac{V_L(s) - \frac{10}{s}}{2000} + \frac{V_L(s)}{4000} + \frac{V_L(s) - (-.015)}{5s}$$

$$\Rightarrow V_L(s) = \frac{\frac{8}{3}}{s + \frac{4000}{15}}$$

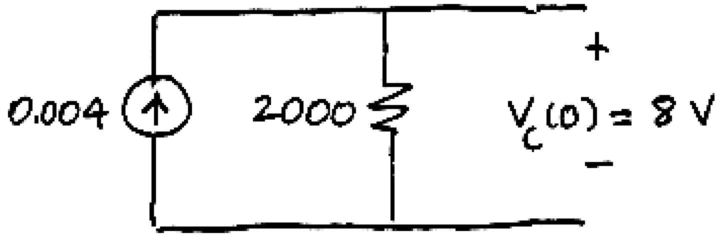
$$I_L(s) = \frac{V_L(s) + 0.15}{5s} = \frac{\frac{8}{15}}{s\left(s + \frac{4000}{15}\right)} + 0.015$$

$$I_L(s) = \frac{0.005}{5} - \frac{0.002}{s + \frac{4000}{15}}$$

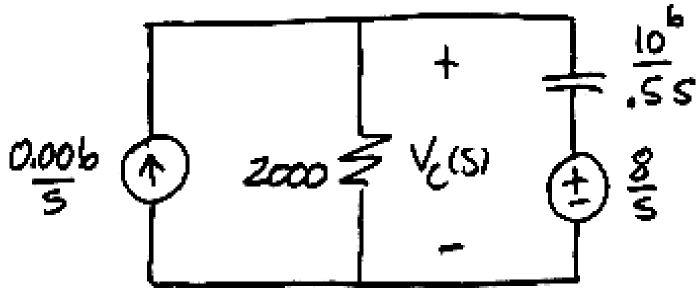
$$i_L(t) = 5 - 3e^{-\frac{4000}{15}t} \text{ mA, } t > 0$$

**P14.9-3**

t < 0




t > 0




$$-\frac{0.006}{s}+\frac{V_c(s)}{2000}+\frac{V_c(s)-\frac{8}{s}}{\frac{10^6}{.5s}}=0$$

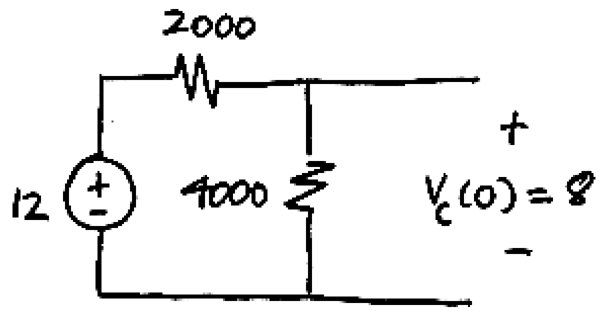
$$-\frac{6000}{s}+500\,V_c(s)+0.5s\left(V_c(s)-\frac{8}{s}\right)=0$$

$$V_c(s)=\frac{8s+12000}{s(s+1000)}=\frac{12}{s}-\frac{4}{s+1000}$$

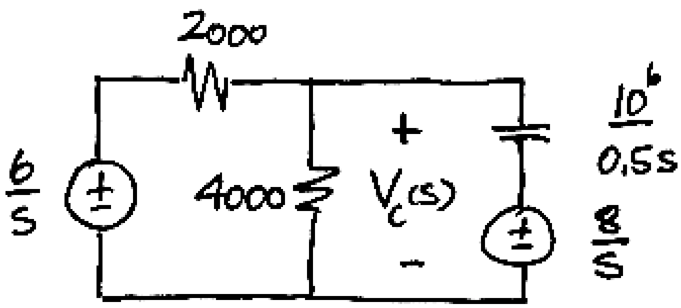
$$V_c(t)=12-4e^{-1000t}\text{ V},\ t>0$$

**P14.8-4**

t < 0




t > 0




$$\frac{V_c(s) - \frac{6}{s}}{2000} + \frac{V_c(s)}{4000} + \left(\frac{0.5s}{10^6}\right)\left(V_c(s) - \frac{8}{s}\right) = 0$$

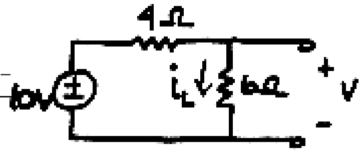
$$500\left(V_c(s) - \frac{6}{s}\right) + 250\,V_c(s) + 0.5s\left(V_c(s) - \frac{8}{s}\right) = 0$$

$$V_c(s) = \frac{6000 + 8s}{s(s+1500)} = \frac{4}{s} + \frac{4}{s+1500}$$

$$v_c(t) = 4 + 4e^{-1500t}\ \text{V},\ t > 0$$

**P14.8-5**

t = $0^-$




Steady - state

$$i_L(0) = \frac{10\text{V}}{10\Omega} = 1\text{A},\ V(0) = 10\left(\frac{6}{6+4}\right) = 6\text{V}$$

t > 0


KVL rt. mesh: $v - 6i_L - di_{L/dt} = 0$ (1)

KCL top node: $v + dV/dt + 4i_L = 6e^{-3t}$ (2)

Taking Laplace Transforms of (1) & (2) :

$V(s) - (6+s)\, I_L(s) = -1$ (3)

$(1+s)\, V(s) + 4I_L(s) = 6 + 6/(s+3) = (6s+24)/(s+3)$ (4)

Using Cramer's rule on (3) & (4) yields

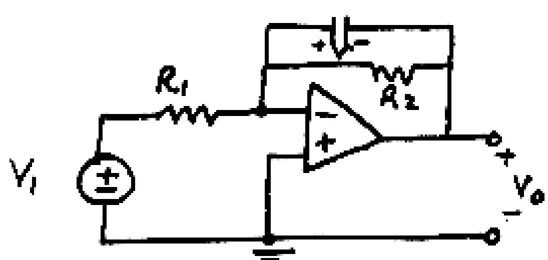
$$V(s) = \frac{-4 + (6+s)\left(\dfrac{6s+24}{s+3}\right)}{-4+(1+s)(6+s)} = \frac{6s^2+56s+132}{(s+3)(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+5}$$

$A = (s+2)\, V(s)\big|_{s=-2} = 44/3,\ B = (s+3)\, V(s)\big|_{s=-3} = -9,\ C = (s+5)\, V(s)\big|_{s=-5} = 1/3$

$\therefore V(s) = 44/3/(s+2) - 9/(s+3) + 1/3/(s+5)$

$\Rightarrow \underline{v(t) = 44/3e^{-2t} - 9e^{-3t} + (1/3)e^{-5t}\ \text{V}}$

**P14.8–6**




$v_2 = 0$

KCL node $v_2$

$$\frac{v_2 - v_1}{R_1} + C\frac{d}{dt}\left[v_2 - v_0\right] + \frac{v_2 - v_0}{R_2} = 0$$

$$\Rightarrow \frac{v_1}{R_1} + C\frac{dv_0}{dt} + \frac{v_0}{R_2} = 0$$

Taking Laplace Transform

$$\frac{V_1(s)}{1000} + 10^{-6}\left[sV_0(s) - 5\right] + \frac{V_0(s)}{1000} = 0$$

$$V_1(s) = \frac{10}{s} \Rightarrow V_0(s) = -\frac{10}{s} + \frac{15}{s+1000}$$

$\therefore\ \underline{v_0(t) = 15e^{-1000t} - 10}\ , t \ge 0$

**P14.8-7**

$$v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \quad \Rightarrow \; V_1(s) = 3sI_1 + sI_2 - 9 \qquad (1)$$

$$v_2(t) = L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \quad \Rightarrow V_2(s) = \; sI_1 + 2sI_2 - 8 \qquad (2)$$

from KCL : $I_3 = I_1 + I_2$ $\qquad (3)$

from KVL : $\frac{5}{s} = 2I_3 + V_1$ $\qquad (4)$

& $V_1 = V_2 + 1I_2$ $\qquad (5)$

Plugging (1) and (3) into (4) $\Rightarrow (3s+2)I_1 + (s+2)I_2 = 9 + \frac{5}{s}$

Plugging (1) and (2) into (5) $\Rightarrow 2sI_1 - (s+1)I_2 = 1$

$$\therefore \; \Delta = \begin{vmatrix} 3s+2 & s+2 \\ 2s & -(s+1) \end{vmatrix} = -\left[5s^2 + 9s + 2\right]$$

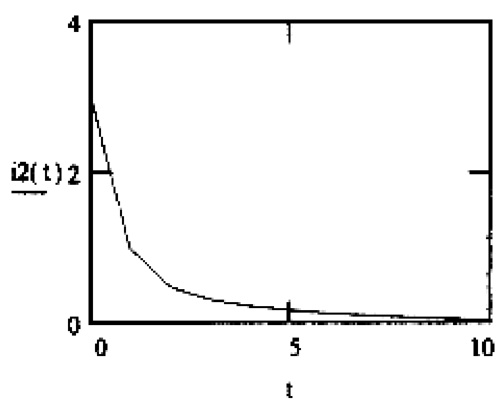
$$\text{So } \; I_2 = \frac{1}{\Delta}\begin{vmatrix} 3s+2 & \frac{5}{s}+9 \\ 2s & 1 \end{vmatrix} = \frac{15s+8}{5s^2+9s+2} = \frac{3s+1.6}{(s+0.26)(s+1.54)} = \frac{A}{s+0.26} + \frac{B}{s+1.54}$$

$$A = \left.\frac{3s+1.6}{s+1.54}\right|_{s=-0.26} = 0.64$$

$$B = \left.\frac{3s+1.6}{s+0.26}\right|_{s=-1.54} = 2.36$$

$$I_2 = \frac{0.64}{s+0.26} + \frac{2.36}{s+1.54}$$

$$i_2(t) = 0.64e^{-0.26t} + 2.36e^{-1.54t} \;\; u(t) \text{ A}$$



## Section 14-9: Transfer Function and Impedance

**P14.9-1**

We want: $H(s) = a$

We have: $H(s) = \dfrac{Z_2}{Z_1 + Z_2}$ where $Z_1 = \dfrac{R_1 1/C_1 s}{R_1 + 1/C_1 s} = \dfrac{R_1}{R_1 C_1 s + 1}$ and $Z_2 = \dfrac{R_2}{R_2 C_2 s + 1}$

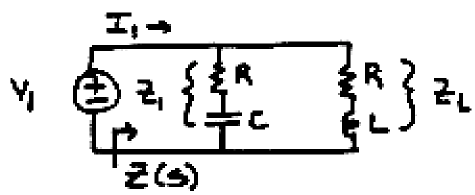
let $\tau_1 = R_1 C_1$ and $\tau_2 = R_2 C_2$

$$\therefore\ H(s) = \frac{R_2(\tau_1 s + 1)}{R_1(\tau_2 s + 1) + (\tau_1 s + 1) R_2}$$

When $\tau_2 = \tau_2 = \tau \ \Rightarrow H(s) = \dfrac{R_2(\tau_1 s + 1)}{(R_1 + R_2)(\tau_s + 1)} = \dfrac{R_2}{R_1 + R_2} = \text{constant}$

$\therefore$ require $\underline{R_1 C_1 = R_2 C_2}$

**P14.9-2**




$Z_1 = R + 1/Cs \qquad Z_L = R + Ls$

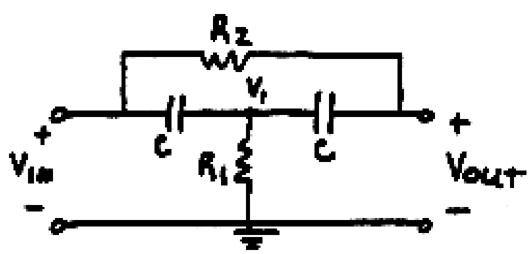
$$Z(s) = \frac{Z_1 Z_L}{Z_1 + Z_L} = \frac{(R + 1/sC)(R + Ls)}{R + 1/sC + R + Ls}$$

$$= R\,\frac{LCs^2 + (RC + L/R)\,s + 1}{LCs^2 + 2RCs + 1}$$

Now require : $RC + L/R = 2RC$

$\Rightarrow \underline{L = R^2 C}$ then $\underline{Z = R}$

**P14.9-3**




KCL at $V_{in}$ : $(V_{in} - V_1)\,sC + (V_{in} - V_{out})/R_2 = 0$

$$\Rightarrow V_1 = \frac{(1 + sCR_2)\,V_{in} - V_{out}}{s\,CR_2} \qquad (1)$$

KCL at$V_1$ : $V_1/R_1 + (V_1 - V_{in})\,sC + (V_1 - V_{out})\,sC = 0 \qquad (2)$

Plugging (1) into (2) yields :

$$\frac{V_{out}}{V_{in}} = \frac{1 + (2R_1 + R_2)\,Cs + R_1 R_2 C^2 s^2}{1 + 2R_1 Cs + R_1 R_2 C^2 s^2}$$

**P14.9-4**

(a) From voltage division

$$V_0(s) = V_1(s)\left[\frac{150 + 2\times 10^{-3}s}{150 + 2\times 10^{-3}s + 100 + 3\times 10^{-3}s}\right]$$

$$\Rightarrow H(s) = \frac{V_o(s)}{V_1(s)} = \frac{1.5\times 10^5 + 2s}{2.5\times 10^5 + 2s}$$

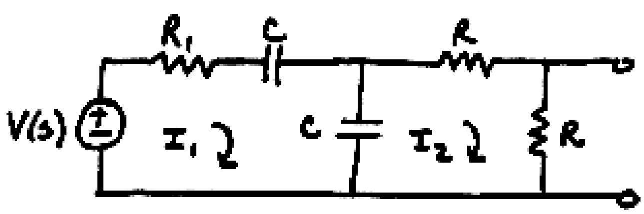
(b) Now $V_1(s) = 100/s$

$$\text{so } V_0(s) = (100/s)\frac{\left(1.5\times 10^5 + 2s\right)}{25\times 10^5 + 5s} = \frac{100\left(1.5\times 10^5 + 2s\right)}{s\left(2.5\times 10^5 + 5s\right)} = \frac{A}{s} + \frac{B}{S + 5\times 10^4}$$

$$A = sV_0(s)\Big|_{s=0} = 3 \ , \qquad B = \left(s + 5\times 10^4\right) V_0(s)\Big|_{s=-5\times 10^4} = -20$$

$$\Rightarrow V_0(s) = 3/s + -20/\left(s + 5\times 10^4\right) \qquad \therefore\ v_0(t) = 3 - 20e^{-5\times 10^4 t}\ \text{V} \quad t > 0$$

**P14.9-5**



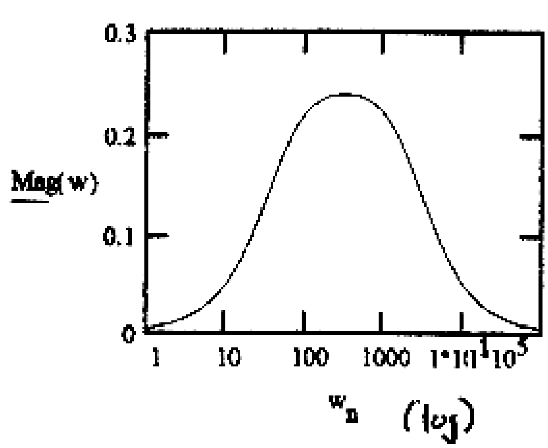

(a)
$$V(s) = \left(R_1 + \frac{1}{Cs} + \frac{1}{Cs}\right) I_1(s) - \frac{1}{Cs} I_2 \qquad (1)$$

$$0 = \left(R + R + \frac{1}{Cs}\right) I_2 - \frac{1}{Cs} I_1 \qquad (2)$$

$$\text{Solving for } I_2 \Rightarrow I_2 = \frac{V(s)\left(\frac{1}{Cs}\right)}{\left(R_1 + \frac{2}{Cs}\right)\left(2R + \frac{1}{Cs}\right) - \frac{1}{(Cs)^2}}$$

$$\therefore \frac{V_0(s)}{V(s)} = \frac{RCs}{\left[R_1Cs + 2\right]\left[2RCs + 1\right] - 1} = \frac{s}{2R_1C\left[s^2 + \frac{4RC + R_1C}{2RR_1C^2}s + \frac{1}{\left(2RR_1C^2\right)^2}\right]}$$

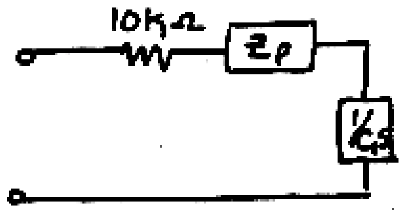
(b)



**P14.9-6**

$V_s(s) = 1/s$

Use voltage divider




$$Z_p = \frac{(1/Cs)(Ls)}{\frac{1}{Cs} + Ls} = \frac{Ls}{1 + LCs^2}$$

$$LC = 1.67 \times 10^{-8}$$

$$H(s) = \frac{1/C_1 s}{1/C_1 s + \frac{Ls}{1 + LCs^2} + 10^4} = \frac{\left(1 + LCs^2\right)}{\left(1 + LCs^2\right) + LC_1 s^2 + 10^4 C_1 s \left(1 + LCs^2\right)}$$

$$= \frac{1 + 1.67 \times 10^{-8}\ s^2}{1 + 5\times10^{-4} s + 10^{-7} s^2 + 8.33\times10^{-12} s^3}$$

$$\text{So } V(s) = \frac{1}{s} H(s) = \frac{1}{s} + \frac{2.65}{s + 4644} + \frac{0.826 + j\,.604}{s + s_1} + \frac{0.826 + j\,.604}{s + s_1^*}$$

where $s_1 = -3678 + j\,3509$

$$\therefore\ v(t) = 1 - 2.65 e^{-4644t} + e^{-3678t}\left[1.65 \cos 3509 - 1.209 \sin 3509t\right]$$

$$\underline{v(t)_0 = 1 - 2.65 e^{-4644t} + 2.05 e^{-3678t} \cos(3509t + 0.632)}$$

**P14.9-7** $\dfrac{V_0(s)}{V(s)} = \dfrac{1/Cs}{R + Ls + 1/Cs} = \dfrac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = H(s)$

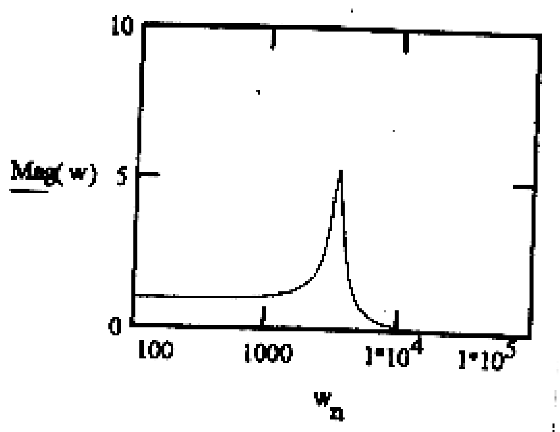
Mathcad spreadsheet:

N : 100   n := 0..N

$\omega min := 100$  $\omega max := 100000$  $m := \ln\left(\dfrac{\omega max}{\omega min}\right)$  $\omega_n := \omega\, min \cdot e^{\frac{n}{N} \cdot m}$

$s_n := \omega_n \cdot j$

$$H_n := \frac{1.136 \cdot 10^7}{(s_n)^2 + 625 \cdot s_n + 1.136 \cdot 10^7} \quad Mag(\omega) := |H_n|$$

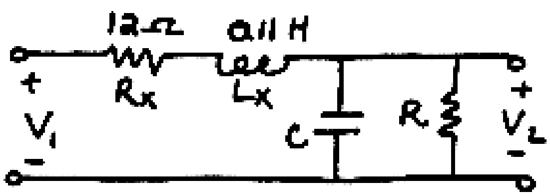



Now if $\omega = 2\pi f$ , then $\omega := 9.42.10^4$ rad/sec

and plugging this into the transfer function: $H := \dfrac{1.136\cdot10^7}{(\omega\cdot j)^2 + 625\cdot\omega\cdot j + 1.136\cdot10^7}$

yields a magnitude of $|H| = 0.001$

**P14.9-8**

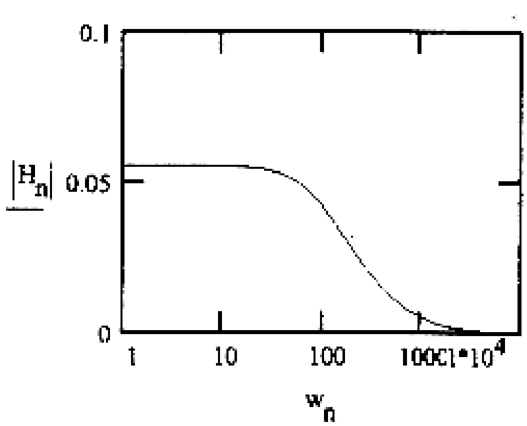



a) $$Z_2 = \frac{R\left(\frac{1}{Cs}\right)}{R+\frac{1}{Cs}} = \frac{R}{RCs+1}$$

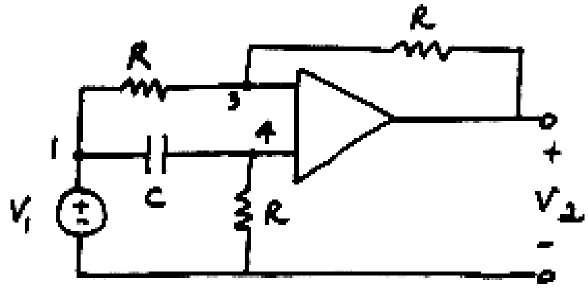
$$Z_1 = R_x + L_x s$$

$$\frac{V_2}{V_1} = \frac{Z_2}{Z_1+Z_2} = \frac{\frac{R}{RCs+1}}{R_x + L_x s + \frac{R}{RCs+1}} = \frac{R}{LRCs^2 + (L+R_xRC)\,s + R_x + R}$$

$$\frac{V_2}{V_1} = \frac{R/LRC}{s^2 + \frac{(L+R_xRC)}{LRC}\,s + \frac{R_x+R}{LRC}}$$



**P14.9-9**




node 3: $-GV_2 + 2GV_3 - GV_1 = 0$ (1)

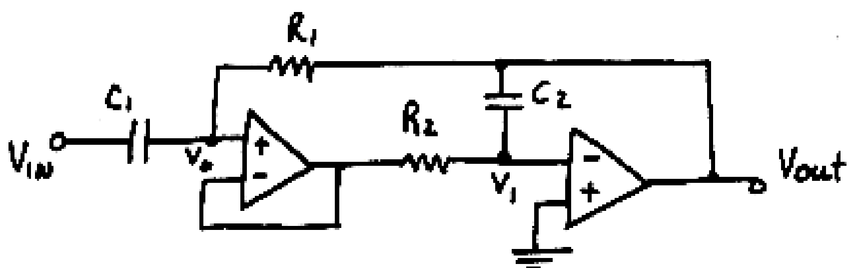
node 4: $(G+sC)\,V_4 - sCV_1 = 0$ (2)

also : $V_3 = V_4$ (3)

Solving (1) $\rightarrow$ (3) for $V_2/V_1$ yields

$$H(\omega) = \frac{V_2}{V_1} = \frac{(G-Cs)}{(G+Cs)} \qquad |H| = 1 \text{ for } \omega \geq 0$$

**P14.9-10**




KCL at $V_0$ : $(V_0 - V_{in})\,sC_1 + \frac{V_0 - V_{out}}{R_1} = 0$ or $(R_1C_1s+1)\,V_0 = R_1C_1s\,V_{in} + V_{out}$

KCL at $V_1 = 0$ : $\frac{V_0}{R_2} + V_{out} + sC_2 = 0$ or $V_0 = -R_2C_2s\,V_{out}$

Solving for $V_{out}/V_{in}$ from (1) & (2) get

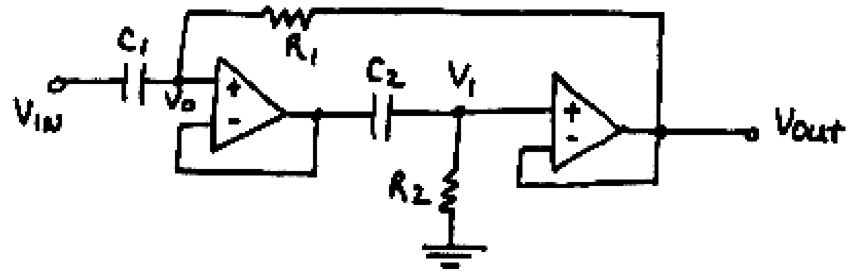
$$\frac{V_{out}}{V_{in}} = \frac{-R_1C_1s}{R_1R_2C_1C_2s^2 + R_2C_2s+1}$$

$$\text{or } T(s) = \frac{-\frac{1}{R_2C_2}s}{s^2 + \frac{1}{R_1C_1}s + \frac{1}{R_1R_2C_1C_2}} = \frac{-Q\,\omega_0\,s}{s^2 + \frac{\omega_0}{Q}s + \omega_0{}^2}$$

where $R_1C_1 = Q/\omega_o$

$R_2C_2 = 1/Q\omega_o$

**P14.9-11**




$V_{out} = V_1$

KCL at $V_0$ : $\left(V_0 - V_{in}\right) sC_1 + \dfrac{V_0 - V_{out}}{R_1} = 0$ or $\left(R_1C_1s+1\right) V_0 = R_1C_1s\,V_{in} + V_{out}$

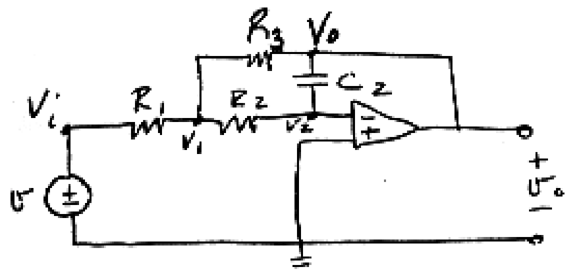
KCL at $V_1$ : $\left(V_1 - V_0\right) sC_2 + \dfrac{V_1}{R_2}=0$ or $(R_2C_2s+1)V_{out} = R_2C_2sV_0$

Solving for $V_{out}/V_{in}$ from (1) and (2) yields

$$\frac{V_{out}}{V_{in}} = \frac{R_1R_2C_1C_2s^2}{R_1R_2C_1C_2s^2 + R_1C_1s+1}$$

$$T(s) = \frac{s^2}{s^2 + \dfrac{1}{R_2C_2}s + \dfrac{1}{R_1R_2C_1C_2}} = \frac{s^2}{s^2 + \dfrac{\omega_0}{Q}s + {\omega_0}^2}$$ where $R_1C_1 = 1/Q\omega_o$ and $R_2C_2 = Q/\omega_o$

**P14.9-12**




at node $V_1$ :

$$\frac{V_1 - V_i}{R_1} + \frac{V_1 - V_2}{R_2} + \frac{V_1 - V_0}{R_3} = 0$$

Rearranging @ node $V_1$ yields :

$$V_1\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - \frac{V_0}{R_3} = \frac{V_i}{R_1} \qquad (1)$$

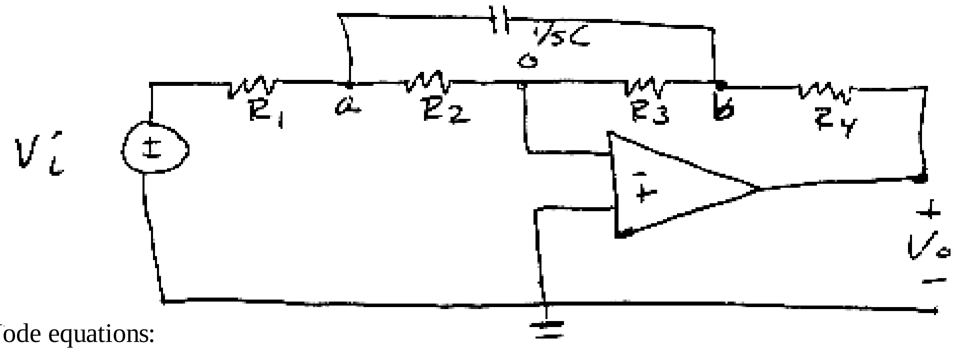
@ node $V_2$: $\dfrac{V_2 - V_1}{R_2} + \left(V_2 - V_0\right)sC_2 = 0$, but $V_2 = 0$

$$\therefore V_1 = -sC_2\, R_2\, V_0 \qquad (2)$$

Plugging (2) into (1) and rearranging yields

$$\frac{V_0}{V_i} = \frac{-R_3}{sC_2R_2R_3 + sC_2R_1R_3 + sC_2R_1R_2 + R_1}$$

**P14.9-13**



Node equations:

$$\frac{V_a - V_i}{R_1} + \frac{V_a}{R_2} + sC(V_a - V_b) = 0 \qquad (1)$$

$$\frac{V_6}{R_3} + \frac{V_b - V_o}{R_4} - sC(V_a - V_b) = 0 \qquad (2)$$

$$-\frac{V_b}{R_3} - \frac{V_a}{R_2} = 0 \Rightarrow V_b = -\frac{R_3}{R_2} V_a \text{ or } V_a = -\frac{R_2}{R_3} V_b \quad (3)$$

$$\Rightarrow (V_a - V_b) = \frac{R_2 + R_3}{R_2} V_a = -\frac{R_2 + R_3}{R_3} V_b$$

Solving yields

$$\frac{V_o}{V_i} = \frac{-R_3 + R_4 + sCR_4 (R_2 + R_3)}{R_1 + R_2 + sC R_1 (R_2 + R_3)}$$

**P14.9-14** $H(s) = \dfrac{2}{s(s+2)(s+1)^2} = \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{s+1} + \dfrac{D}{(s+1)^2}$

$$\Rightarrow A = sH(s)\big|_{s=0} = 1$$

$$B = (s+2)\,H(s)\big|_{s=-2} = -2$$

$$D = (s+1)^2\ H(s)\big|_{s=1} = -1$$

$$C = \frac{d}{ds}\left[(s+1)^2\ H(s)\right]\Big|_{s=-1} = 0$$

$$\Rightarrow h(t) = \left(1 - 2te^{-t} - e^{-2t}\right)u(t)$$

$$h(0) = 0 \Rightarrow h(0) = \lim_{s\to\infty} sH(s)$$

$$h(\infty) = 1 \Rightarrow h(\infty) = \lim_{s\to 0} sH(s)$$

**P14.9-15** $H(s) = 400/(s^2 + 400s + 2\times10^5)$

$$F(s) = \frac{\omega}{(s+a)^2+\omega^2} \Rightarrow f(t) = e^{-at}\sin\omega t \qquad \text{Table 14.4-3}$$

gives $\omega = 400,\ \omega^2 = 160000$

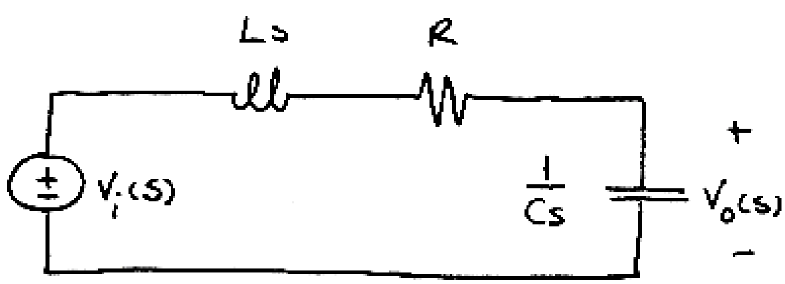
$$\therefore H(s) = \frac{400}{(s+200)^2+160000} \qquad \therefore \underline{h(t) = e^{-200t}\sin(400t)\ u(t)}$$

**P14.9-16** $H(s) = \dfrac{4(s+3)}{s(s+2)^2} = \dfrac{A}{s} + \dfrac{B}{(s+2)} + \dfrac{C}{(s+2)^2}$

Solving partial fractions yields: $A=3,\ B=-3,\ C=-2$

$$\therefore H(s) = \frac{3}{s} - \frac{3}{s+2} - \frac{2}{(s+2)^2} \qquad \therefore \underline{h(t) = 3-3e^{-1t}-2te^{-2t}}$$

**P14.9-17**




$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{Ls+R+\frac{1}{Cs}} = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

| L | C | R | T(s) |
|---|---|---|---|
| 2 | .025 | 18 | $\dfrac{20}{s^2+9s+20} = \dfrac{20}{(s+4)(s+5)}$ |
| 2 | .025 | 8 | $\dfrac{20}{s^2+4s+20}$ |
| 1 | .391 | 4 | $\dfrac{2.56}{s^2+4s+2.56} = \dfrac{2.56}{(s+.8)(s+3.2)}$ |
| 2 | .125 | 8 | $\dfrac{20}{s^2+4s+4} = \dfrac{4}{(s+2)}$ |

a) $T(s) = \frac{20}{(s+4)(s+5)}$

$\mathcal{L}\{\text{impulse response}\} = T(s) = \frac{20}{s+4} - \frac{20}{s+5}$

impulse response = $\left(20e^{-4t} - 20e^{-5t}\right)u(t)$

$\mathcal{L}\{\text{step response}\} = \frac{T(s)}{s} = \frac{20}{s(s+4)(s+5)} = \frac{1}{s} + \frac{-5}{s+4} + \frac{4}{s+5}$

step response = $\left(1+4e^{-5t} - 5e^{-4t}\right)u(t)$

b) $T(s) = \frac{20}{s^2 + 4s + 20}$

$\mathcal{L}\{\text{impulse response}\} = T(s) = \frac{5(4)}{(s+2)^2 + 4^2}$

impulse response $= 5e^{-2t}\sin 4t\ u(t)$

$\mathcal{L}\{\text{step response}\} = \frac{T(s)}{s} = \frac{20}{s(s^2 + 4s + 20)} = \frac{1}{s} + \frac{K_1 s + K_2}{s^2 + 4s + 20}$

$20 = s^2 + 4s + 20 + s\left(K_1 s + K_2\right) = s^2\left(1+K_1\right) + s\left(4+K_2\right) + 20 \Rightarrow K_1 = -1,\ K_2 = -4$

$\mathcal{L}\{\text{step response}\} = \frac{1}{s} + \frac{-(s+2)}{(s+2)^2 + 4^2} + \frac{-\frac{1}{2}(4)}{(s+2) + 4^2}$

step response $= \left(1 - e^{-2t}\left(\cos 4t + \frac{1}{2}\sin 4t\right)\right)u(t)$

c) $T(s) = \frac{2.56}{(s + .8)(s + 3.2)}$

$\mathcal{L}\{\text{impulse response}\} = T(s) = \frac{1.07}{s + .8} - \frac{1.07}{s+3.2}$

impulse response $= 1.07\left(e^{-.8t} - e^{-3.2t}\right)u(t)$

$\mathcal{L}\{\text{step response}\} = \frac{T(s)}{s} = \frac{2.56}{s\,(s + .8)(s+3.2)} = \frac{1}{s} + \frac{\frac{-4}{3}}{s + .8} + \frac{\frac{1}{3}}{s+3.2}$

step response $= \left(1 + \frac{1}{3}e^{-3.2t} - \frac{4}{3}e^{-.8t}\right)u(t)$

d) impulse response $= 4te^{-2t}u(t)$

step response $= \left(1 - (1+2t)e^{-2t}\right)u(t)$

**P14.9-18**

$$h(t)=\sqrt{2}e^{-t/\sqrt{2}}\sin(t/\sqrt{2})$$

(a) Using Table 14.5-1 with $\omega_{nb}=1/\sqrt{2}$ and $\frac{\omega_n}{b}=\sqrt{2}$

$$\Rightarrow\left(\frac{\omega_n}{b}\right)(\omega_n b)=\omega_n^{\,2}=\left(\sqrt{2}\right)\left(1/\sqrt{2}\right)=1$$

$$\therefore b=\frac{\omega_n}{\sqrt{2}}=1/\sqrt{2}$$

$$\therefore b=\sqrt{1-\xi^2}\Rightarrow\xi=1/\sqrt{2}$$

$$\Rightarrow H(s)=\frac{\omega_n^{\,2}}{s^2+2\xi\omega_n s+\omega_n^{\,2}}=\frac{1}{s^2+\sqrt{2}s+1}$$

(b) $$H(j\omega)=\frac{1}{(j\omega)^2+\sqrt{2}j\omega+1}=\frac{1}{1-\omega^2+j\sqrt{2}\omega}$$

$$\therefore\left|H(j\omega)\right|=\frac{1}{\sqrt{\left(1-\omega^2\right)^2+2\omega^2}}=\frac{1}{\sqrt{1+\omega^4}}$$

**P14.8-19**

$$V_1(s)=1\ ,\ V_0(s)=\frac{3(s+2)}{s(s+3-j2)\,(s+3+j2)}=\frac{A}{s}+\frac{B}{s+3-j2}+\frac{B^*}{s+3+j2}$$

$$\Rightarrow A=sV_0(s)\Big|_{s=0}=.462$$

$$B=(s+3-j2)V_0(s)\Big|_{s=-3+j2}=0.47\angle-119.7^o$$

$$B^*=0.47\ \angle 119.7^o$$

$$\therefore V_0(s)=\frac{0.462}{s}+\frac{0.47\ \angle 119.7^o}{s+3-j2}+\frac{0.47\ \angle 119.7^o}{s+3+j2}$$

So $v_0(t)=.462+2(.47)e^{-3t}\cos\left(2t-119.7^o\right)\qquad t\geq 0$

**P14.9-20**

(a) $$H(s)=\frac{V_o}{V_1}=\frac{R/(1+RCs)}{LS+R/(1+RCs)}=\frac{R}{s^2LCR+Ls+R}$$

(b) have underdamped response $\Rightarrow s=-\frac{1}{2RC}\pm j\sqrt{1/LC-1/(2RC)^2}$ (1)

$T=8\text{ ms}\Rightarrow f=\frac{1}{8\times10^{-3}}=125\text{ Hz}$ *or* $\omega=250\ \pi\text{rad/s}$

Using (1): $s=\alpha+j\omega$

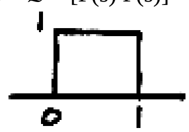
$250\pi=\sqrt{1/10^{-6}-10^{12}/4R^2}\quad\Rightarrow\quad \underline{R=807.8\ \Omega}$

(c) Critically damped when $\sqrt{1/LC-1/(2RC)^2}=0\quad\Rightarrow \underline{R=1/2\ \sqrt{L/C}=1/2\ \sqrt{10^6}=500\Omega}$

## Section 14-10: Convolution Theorem

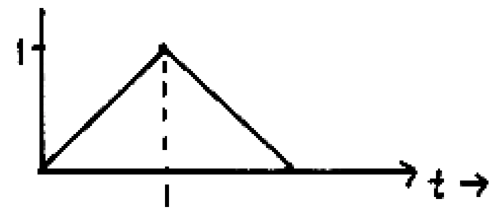
**P14.10-1**

f(t) * f(t) = $\mathcal{L}^{-1}$ [F(s) F(s)]

pulse: 


$\therefore\ F(s)=\dfrac{1-e^{-s}}{s}$

f * f




$$\therefore\ f*f=\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s}\right]^2=\mathcal{L}^{-1}\left[\frac{1-2e^{-s}+e^{-2s}}{s^2}\right]$$

Now $\mathcal{L}^{-1}\left[1/s^2\right]=t\,u(t)$

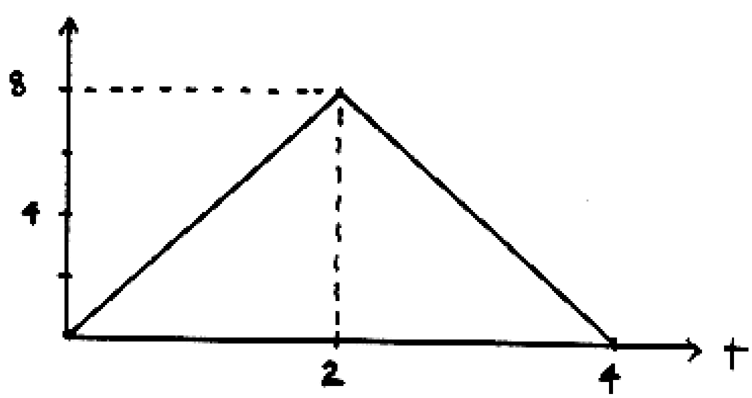
$$\therefore\ \underline{f*f=t\,u(t)-2(t-1)\,u(t-1)+(t-2)u(t-2)}$$

**P14.10-2** $f(t)=2\left[u(t)-u(t-2)\right]$

$$F(s)=\frac{2}{s}-\frac{2e^{-2s}}{s}$$

$$f*f=\mathcal{L}^{-1}\left[F(s)F(s)\right]=\mathcal{L}^{-1}\left[\frac{4}{s^2}-\frac{8e^{-2s}}{s^2}+\frac{4e^{-4s}}{s^2}\right]=4t\,u(t)-8(t-2)u(t-2)+4(t-4)u(t-4)$$

f * f




**P14.10-3** $H(s)=\dfrac{V_2(s)}{V_1(s)}=\dfrac{1/Cs}{R+1/Cs}=\dfrac{1/RC}{s+1/RC}$

$$V_2(t)=h(t)*v_1(t)=\mathcal{L}^{-1}\left[V_1(s)\,H(s)\right]$$

Now if $v_1(t)=tu(t),\ V_1(s)=\dfrac{1}{s^2}$

$$\text{or } V_1(s)\,H(s)=\left(\frac{1}{s^2}\right)\left(\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right)=V_2(s)$$

and $\underline{v_2(t)=t-RC\left(1-e^{-t/RC}\right),\ t\ge 0}$

**P14.10-4** $h(t) * f(t) = \mathcal{L}^{-1}\left[H(s)\,F(s)\right]$ where $H(s) = \frac{1}{s^2}$ and $F(s) = \frac{1}{s+a}$

So $H(s)\,F(s) = \left(\frac{1}{s^2}\right)\left(\frac{1}{s+a}\right) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+a}$

Solving the partial fractions yields: $A = -1/a^2$, $B = 1/a$, $C = 1/a^2$

So $\underline{h(t) * f(t) = \frac{-1}{a^2} + \frac{t}{a} + \frac{e^{-(at)}}{a^2}, \quad t \geq 0}$

## Section 14-11: Stability

**P14.11-1** To have stable operation, the roots of the denominator must be in the left-hand s-plane

$$\therefore 3 - k > 0 \text{ or } k < 3 \text{ so } \underline{0 \leq k < 3 \text{ for stability}}$$

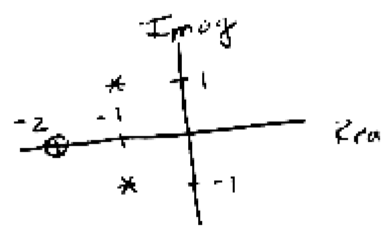
**P14.11-2** We want the roots of the denominator to be in the left-hand side of the s-plane for stability,

so $6 - k < 0$ or $0 \leq k < 6$ will give this result

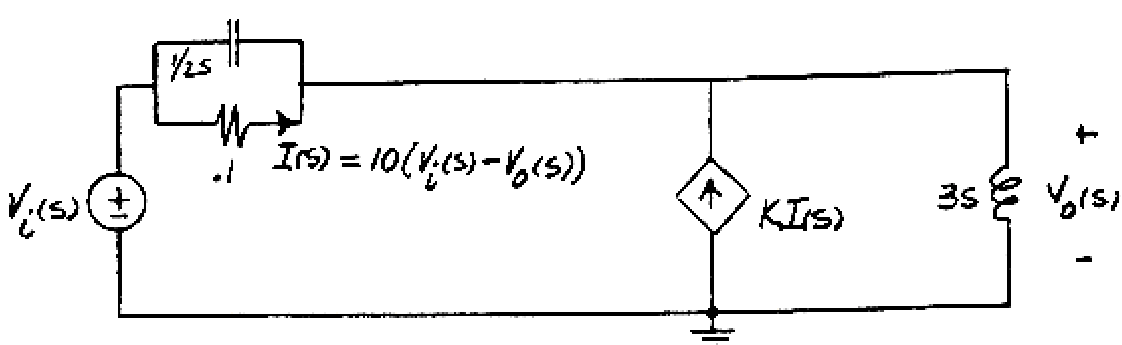
**P14.11-3** $H(s) = \frac{s+2}{(s+1+i)(s+1-i)}$ zeros: $s = -2$

poles: $s = -1+i,\ -1-i$

This circuit is stable since the poles are in the left-hand s-plane.




**P14.11-4**



$$\frac{V_i(s) - V_0(s)}{\frac{1}{10}} + \frac{V_i(s) - V_0(s)}{\frac{1}{2s}} + K\left[10\left(V_i(s) - V_0(s)\right)\right] = \frac{V_0(s)}{3s}$$

$$\Rightarrow \quad T(s) = \frac{V_0(s)}{V_i(s)} = \frac{s^2 + 5(1+K)s}{s^2 + 5(1+K)s + \frac{1}{6}}$$

This circuit is stable for $K > -1$.

This circuit is critically damped when $\left(5(1+K)\right)^2 - 4\left(\frac{1}{6}\right) = 0$

i.e., when $K = -1 + \sqrt{\frac{2}{75}}$ $\left(\text{but not for } K = -1 - \sqrt{\frac{2}{75}}\right)$

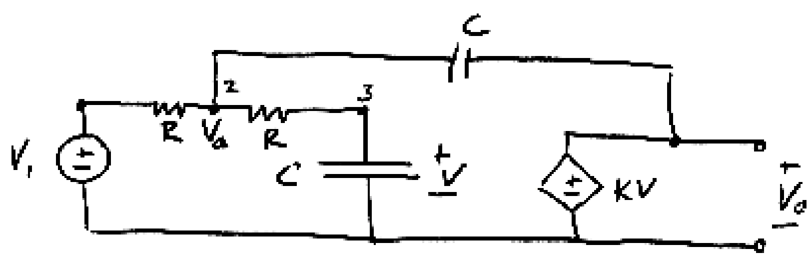
When K=2

$$T(s) = \frac{s^2 + 15s}{s^2 + 15s + \frac{1}{6}} \quad \Rightarrow \quad \omega_0 = \frac{1}{\sqrt{6}}, \quad Q = \frac{1}{15\sqrt{6}}$$

When $K = -1$, this circuit will oscillate at $\omega = \frac{1}{\sqrt{6}}$ rad/sec

Since $Q < \frac{1}{2}$, this circuit is overdamped.

**P14.11-5**




node 2: $\frac{V_a - V_1}{R} + \frac{V_a - V}{R} + (V_a - V_0)\, sC = 0$

node 3: $\frac{V - V_a}{R} + (V - 0)\, sC = 0$ , also $V_0 = KV$

$$\frac{V_0}{V_1} = \frac{K/R^2C^2}{s^2 + \frac{(3-K)s}{RC} + \frac{1}{R^2C^2}}$$

a) K for oscillator $\Rightarrow K = 3$

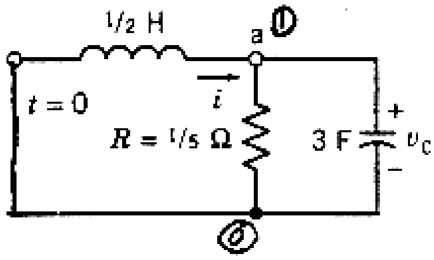
b) impulse response $K=1, R=1k\Omega, C = 0.5$ mF

$$\frac{V_0}{V_1} = \frac{1/0.25}{s^2 + \frac{2s}{0.5} + \frac{1}{0.25}} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

$\therefore$ $\underline{v_0(t) = 4te^{-2t} \text{ V}}$

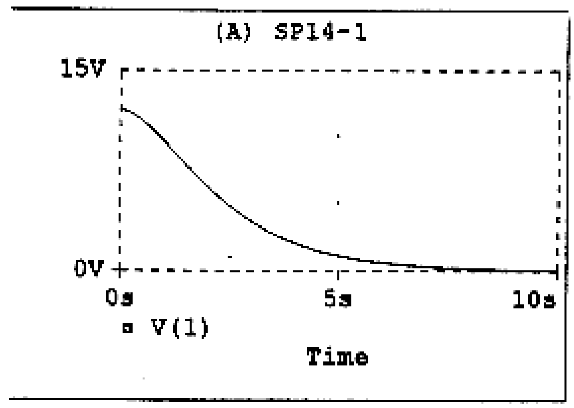
## PSpice Problems

**SP14-1**




Circuit @ t > 0

Probe output

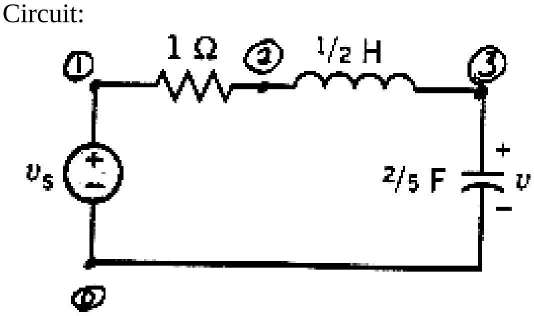



Inut file:

```
R1   1   0   0.2
L1   1   0   0.5  IC=-60
C1   1   0   3  IC=12

.tran  0.1   10   UIC
.plot  tran  V(1)
.probe
.end
```

**SP14-2**

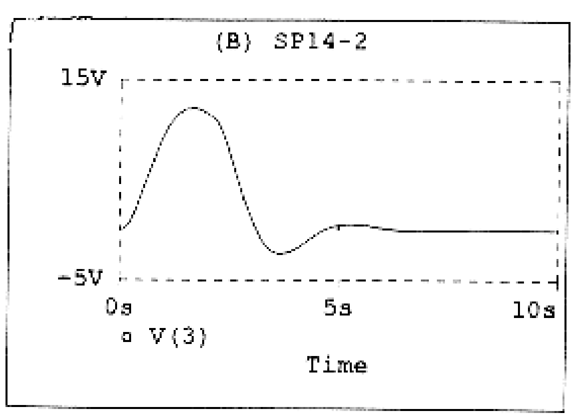
Circuit:



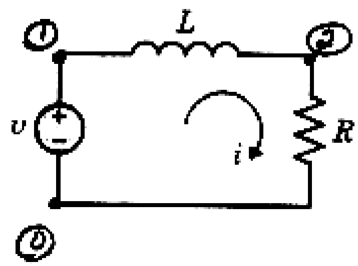

Inut file:

```
Vs   1   0   pulse(0  10  0  0  0  2
10)
R1   1   2   1
L1   2   3   0.5   IC=0
C1   3   0   0.4   IC=0

.tran  0.1   10   UIC
.plot  tran  V(3)
.probe
.end
```

Probe output



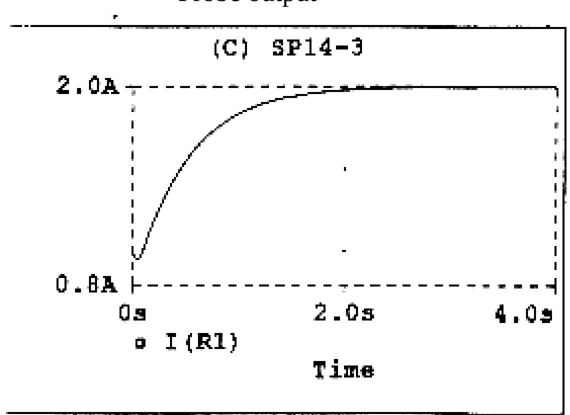
**SP14-3**



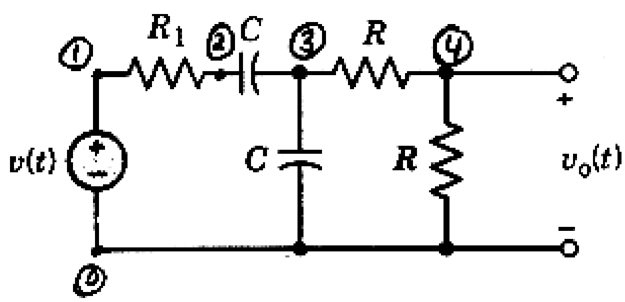

Inut file:

```
Vs    1   0   pulse(0    2)
L1    1   2   0.5   IC=1
R1    2   0   1

.tran    0.1   5  UIC
.probe
.end
```

Probe output



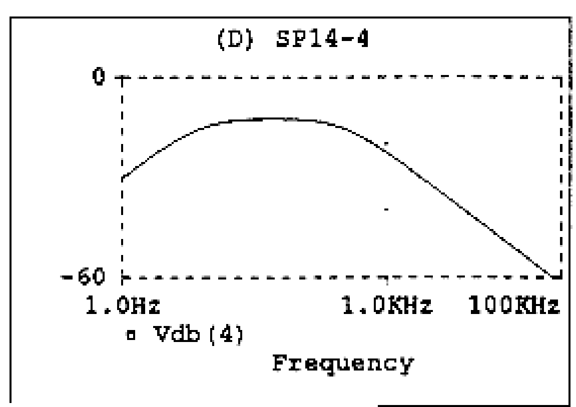

**SP14-4**



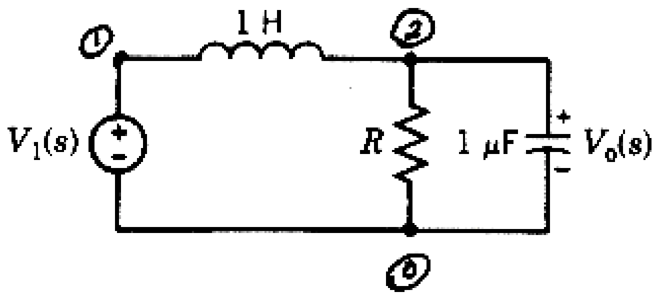

Probe output

Input file:

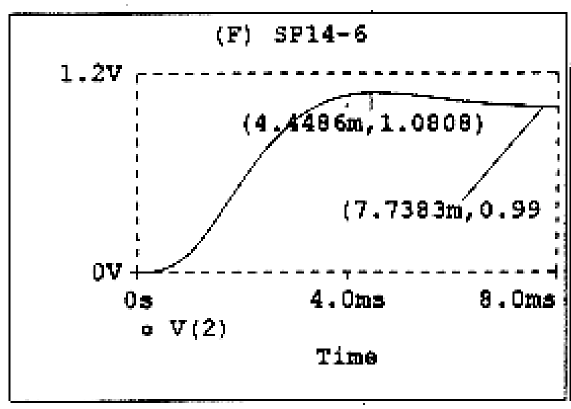
```
Vs    1   0    ac   1
R1    1   2    10
C1    2   3    100u
C2    3   0    100u
R2    3   4    50
R3    4   0    50

.ac   dec   100   1
      100k
.probe
.plot     ac   Vdb
          (4)
.end
```



**SP14-5**




Probe output




Input file:

```
Vs    1  0  pulse (0 1)
 L1   1  2  1
 R1   2  0  807.8
 C1   2  0  1u
 .tran 0.001   0.01
 .probe
 .end
```

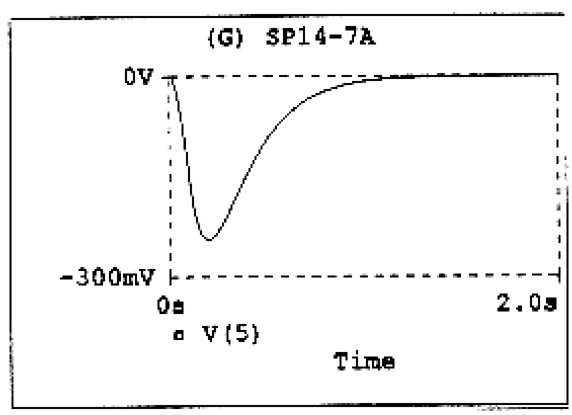
**SP14-6**

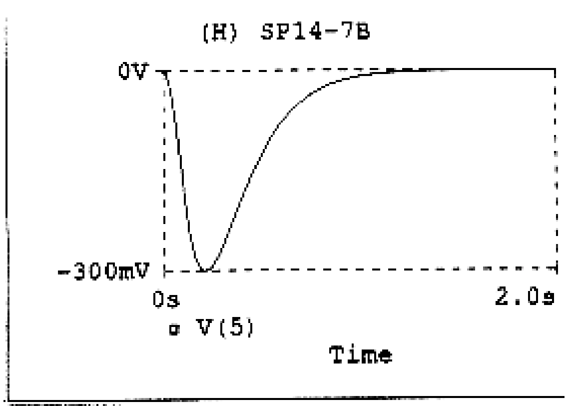
Input file:

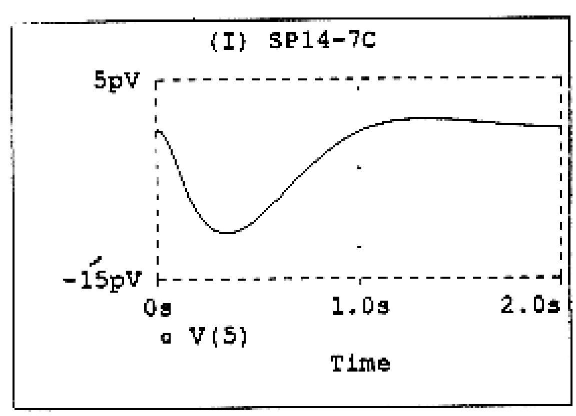
```
Vs   1    0   pulse (0   1)
R1   1    2   50k
C1   2    3   2u
R2   3    4   50k
R3   4    5   262.5k *
C2   2    5   2u
XOA1   4   0    5   OA

.subckt  OA   1   2   3
*nodes  listed in order – + o
E   3   0    1   2   –1G
.ends OA

.tran   0.1   2
.probe
.end
```

* This value varies:

a) 50 kΩ
b) 62.5 kΩ
c) 262.5 kΩ

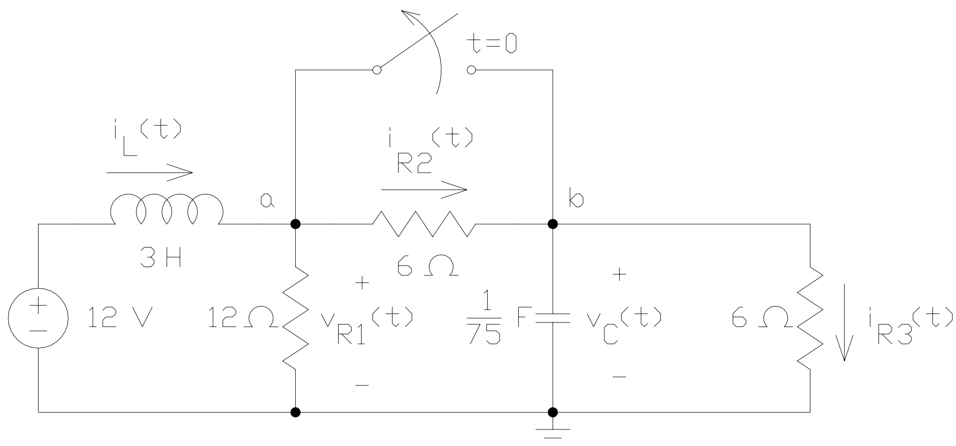


## Verification Problems

**VP 14-1**




$$v_L(t) = 3\frac{d}{dt}i_L(t) = -6e^{-2.1t} - 2e^{-15.9t}$$

$$i_C(t) = \frac{1}{75}\frac{d}{dt}v_C(t) = -0.092e^{-2.1t} - 0.575e^{-15.9t}$$

$$v_{R1}(t) = 12 - v_L(t) = 12 + 6e^{-2.1t} + 2e^{-15.9t}$$

$$i_{R2}(t) = \frac{12 - (v_L(t) + v_C(t))}{6} = 1 + 0.456e^{-2.1t} - 0.123e^{-15.9t}$$

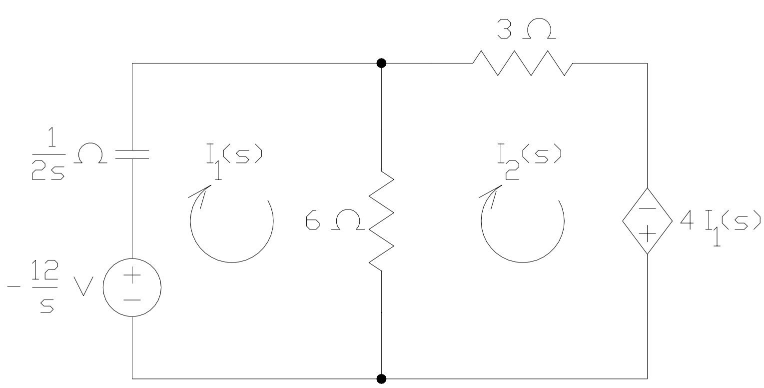
$$i_{R3}(t) = \frac{v_C(t)}{6} = 1 + 0.548e^{-2.1t} + 0.452e^{-15.9t}$$

Thus,

$$-12 + v_L(t) + v_{R1}(t) = 0 \text{ and } i_{R2}(t) = i_C(t) + i_{R3}(t)$$

as required. The analysis is correct.

**VP 14-2**



$$I_1(s)=\frac{18}{s-\frac{3}{4}} \text{ and } I_2(s)=\frac{20}{s-\frac{3}{4}}$$

KVL for left mesh: $$\frac{12}{s}+\frac{1}{2s}\left(\frac{18}{s-\frac{3}{4}}\right)+6\left(\frac{18}{s-\frac{3}{4}}-\frac{20}{s-\frac{3}{4}}\right)=0 \quad ✔$$

KVL for right mesh: $$-6\left(\frac{18}{s-\frac{3}{4}}-\frac{20}{s-\frac{3}{4}}\right)+3\left(\frac{20}{s-\frac{3}{4}}\right)-4\left(\frac{18}{s-\frac{3}{4}}\right)=0 \quad ✔$$

The analysis is correct.

**VP 14-3**

Initial value of $I_L(s)$: $$\lim_{s\to\infty} s\frac{s+2}{s^2+s+5}=1 \quad ✔$$

Final value of $I_L(s)$: $$\lim_{s\to 0} s\frac{s+2}{s^2+s+5}=0 \quad ✔$$

Initial value of $V_C(s)$: $$\lim_{s\to\infty} s\frac{-20(s+2)}{s(s^2+s+5)}=0 \quad ✘$$

Final value of $V_C(s)$: $$\lim_{s\to 0} s\frac{-20(s+2)}{s(s^2+s+5)}=-8 \quad ✘$$

Apparently the error occurred as $V_C(s)$ was calculated from $I_L(s)$. Indeed, it appears that $V_C(s)$ was calculated as $-\frac{20}{s}I_L(s)$ instead of $-\frac{20}{s}I_L(s)+\frac{8}{s}$. After correcting this error

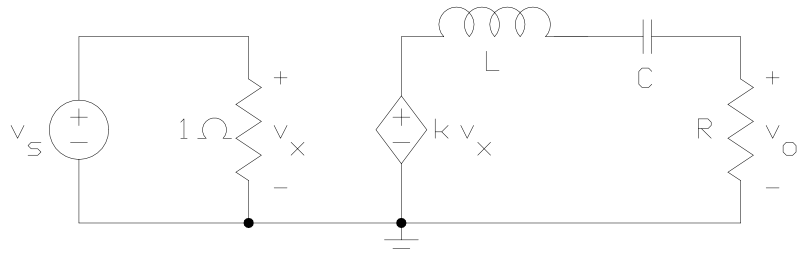
$$V_C(s)=-\frac{20}{s}\left(\frac{s+2}{s^2+s+5}\right)+\frac{8}{s}.$$

Initial value of $V_C(s)$: $$\lim_{s\to\infty} s\left(\frac{-20(s+2)}{s(s^2+s+5)}+\frac{8}{s}\right)=8 \quad ✔$$

Final value of $V_C(s)$: $$\lim_{s\to 0} s\left(\frac{-20(s+2)}{s(s^2+s+5)}+\frac{8}{s}\right)=0 \quad ✔$$

## Design Problems

**DP 14-1**




Equating the Laplace transform of the step response of the give circuit to the Laplace transform of the given step response:

$$V_o(s) = \frac{\frac{kR}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{5}{(s+4)^2}$$

Equating the poles:

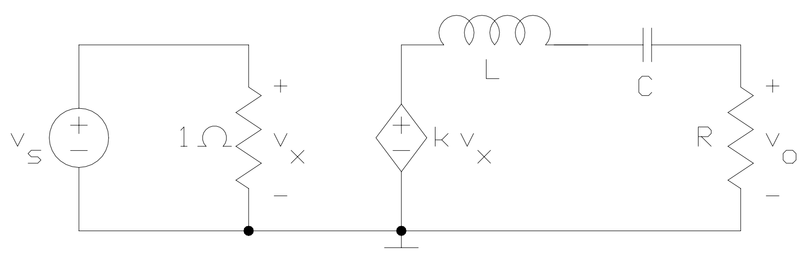
$$s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -4 \pm 0$$

Summarizing the results of these comparisons:

$$\frac{R}{2L} = 4,\ R = \frac{2}{\sqrt{LC}} \text{ and } \frac{kR}{L} = 5$$

Pick $L$ = 1 H, then $k$ = 0.625 V/V, $R$ = 8 Ω and $C$ = 0.0625 F.

**DP 14-2**




Equating the Laplace transform of the step response of the give circuit to the Laplace transform of the given step response:

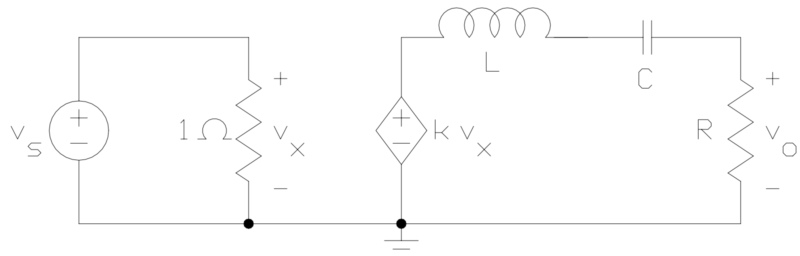
$$V_o(s) = \frac{\frac{kR}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{10}{(s+4)^2 + 4} = \frac{10}{s^2 + 8s + 20}$$

Equating coefficients:

$$\frac{R}{L} = 8,\ \frac{1}{LC} = 20, \text{ and } \frac{kR}{L} = 10$$

Pick $L$ = 1 H, then $k$ = 1.25 V/V, $R$ = 8 Ω and $C$ = 0.05 F.

**DP 14-3**




Equating the Laplace transform of the step response of the give circuit to the Laplace transform of the given step response:

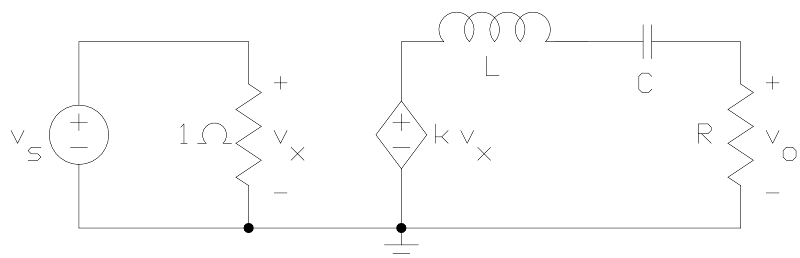
$$V_o(s) = \frac{\frac{kR}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{5}{(s+2)} - \frac{5}{(s+4)} = \frac{10}{s^2+6s+8}$$

Equating coefficients:

$$\frac{R}{L} = 6,\ \frac{1}{LC} = 8, \text{ and } \frac{kR}{L} = 10$$

Pick $L$ = 1 H, then $k$ = 1.667 V/V, $R$ = 6 Ω and $C$ = 0.125 F.

**DP 14-4**




Comparing the Laplace transform of the step response of the give circuit to the Laplace transform of the given step response:

$$V_o(s) = \frac{\frac{kR}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \neq \frac{5}{(s+2)} + \frac{5}{(s+4)} = \frac{10s+30}{s^2+6s+8}$$

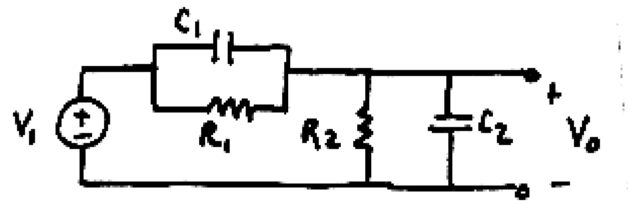
These two functions can not be made equal by any choice of $k$, $R$, $C$ and $L$ because the numerators have different forms.

**DP14-5**

a) Use voltage divider

$$\frac{V_o(s)}{V_1(s)} = \frac{\dfrac{R_2}{sC_2R_2+1}}{\dfrac{R_1}{sC_1R_1+1}+\dfrac{R_2}{sC_2R_2+1}}$$

$$= \frac{C_1}{C_1+C_2}\left[\frac{s+1/C_1R_1}{s+\dfrac{R_1+R_2}{R_1R_2(C_1+C_2)}}\right]$$




b) To make natural response zero, eliminate the pole in $\frac{V_o}{V_1}$ by causing it to cancel with the zero.

$$\Rightarrow \ -\frac{1}{C_1R_1} = -\frac{R_1+R_2}{R_1R_2(C_1+C_2)} \text{ leads to } \underline{\frac{C_2}{C_1} = \frac{R_1}{R_2}}$$

c) If $v_1(t) = u(t)$, $V_1(s) = \frac{1}{s}$

$$V_o(s) = \frac{C_1}{C_1+C_2}\left[\frac{s+\frac{1}{R_1C_1}}{s\left\{s+\dfrac{R_1+R_2}{R_1R_2\,(C_1+C_2)}\right\}}\right] = \frac{K_1}{s}+\frac{K_2}{s+\dfrac{R_1+R_2}{R_1R_2(C_1+C_2)}}$$

where $K_1 = \frac{R_2}{R_1+R_2}$ and $K_2 = \frac{C_1}{C_1+C_2}-\frac{R_2}{R_1+R_2}$

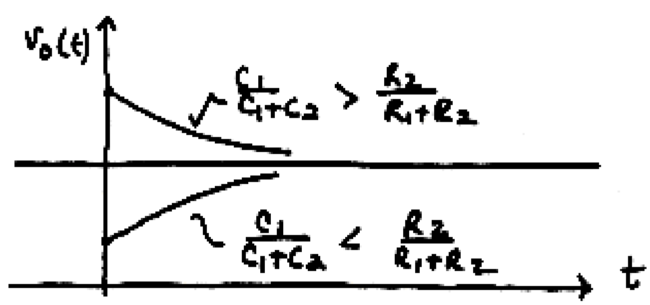
So $v_o(t) = \frac{R_2}{R_1+R_2}+\left[\frac{C_1}{C_1+C_2}-\frac{R_2}{R_1+R_2}\right]e^{-t/\tau} \qquad t > 0$

where $\tau = \frac{R_1R_2(C_1+C_2)}{R_1+R_2}$

1) If $\frac{C_1}{C_1+C_2} = \frac{R_2}{R_1+R_2} \Rightarrow v_o(t) = \frac{R_2}{R_1+R_2} = \frac{C_1}{C_1+C_2}$

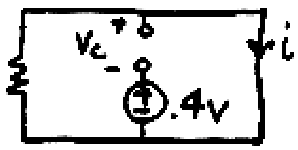
2) and 3) $v_o(t = 0^+) = \frac{C_1}{C_1+C_2}$ and $v_o(t\to\infty) = \frac{R_2}{R_1+R_2}$

$\frac{C_1}{C_2+C_1} = \frac{R_2}{R_1+R_2}\rightarrow$




**DP14-6**

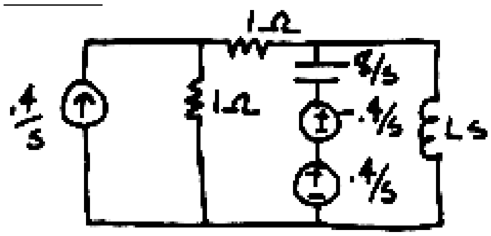
<u>For t < 0</u>



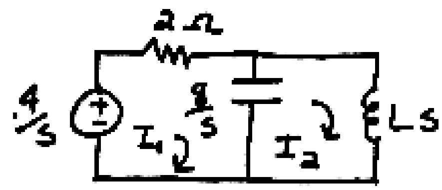

$v_c(0) = -0.4\text{V}$

$i(0) = 0$

<u>For t > 0</u>




⇒




mesh equations $\left(2+8/s\right) I_1 - 8/s\, I_2 = 0.4/s$

$-8/s\, I_1 + \left(Ls + 8/s\right) I_2 = 0$

Solving for $I_2$: $I_2 = \dfrac{1.6}{s\,(Ls^2+4Ls+8)}$

Thus $s^2+4s+8/L = 0$ need char. eqn. with complex roots with significant damping

If $L = 1\text{H} \Rightarrow s^2+4s+8 = 0$

So $I(s) = \dfrac{1.6}{s\,(s^2+4s+8)} = \dfrac{.2}{s} + \dfrac{-.2(s+4)}{s^2+4s+8} = \dfrac{.2}{s} + \dfrac{-.2(s+2)}{(s+2)^2+4} - \dfrac{.8}{(s+2)^2+4}$

So $\underline{i(t) = .2 - .2e^{-2t}\cos 2^t - .4e^{-2t}\sin 2t}$ (A), $t \ge 0$