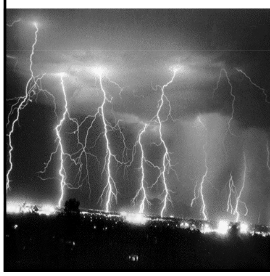
Dipolo Infinitesimal



Sandra Cruz-Pol, Ph. D.
INEL 5305
ECE UPRM
Mayagüez, PR

Outline

- Maxwell's equations
  - Wave equations for A and for Φ
- Power: Poynting Vector
- Dipole antenna

Maxwell Equations

Ampere: $$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}$$

Faraday: $$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Gauss: $$\nabla\cdot\vec{B} = 0$$
$$\nabla\cdot\vec{D} = \rho_v$$

Relaciones de Continuidad

$$\vec{D} = \varepsilon\vec{E}$$
$$\vec{B} = \mu\vec{H}$$
$$\nabla\cdot J = -\frac{\partial\rho}{\partial t}$$
$$J = \sigma E$$

Potencial Magnético Vectorial A

$$B = \nabla\times A = \mu H$$

- Si lo sustituimos en la Ley de Faraday:

$$\nabla\times E = -\frac{\partial}{\partial t}(\nabla\times A) = -\nabla\times\left(\frac{\partial}{\partial t}A\right)$$

$$\nabla\times\left(E + \frac{\partial A}{\partial t}\right) = 0$$

Identidad vectorial: $\nabla\times(\nabla\phi) = 0$

$$E + \frac{\partial A}{\partial t} = -\nabla\Phi$$

Φ is the Electric Scalar Potential

Usando Ley de Ampere:

$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J} \qquad \nabla\times\frac{B}{\mu} = \varepsilon\frac{\partial E}{\partial t} + \vec{J}$$

$$\nabla\times\nabla\times A = \nabla(\nabla\cdot A) - \nabla^2 A$$

$$\frac{1}{\mu}\left[\nabla(\nabla\cdot A) - \nabla^2 A\right] = \vec{J} + \varepsilon\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial A}{\partial t}\right)$$

$$\frac{1}{\mu}\left[\nabla(\nabla\cdot A) - \nabla^2 A\right] = \vec{J} + \varepsilon\frac{\partial\nabla\Phi}{\partial t} - \varepsilon\frac{\partial^2 A}{\partial t^2}$$

$$\nabla(\nabla\cdot A) + \mu\varepsilon\frac{\partial\nabla\Phi}{\partial t} = \nabla^2 A - \mu\varepsilon\frac{\partial^2 A}{\partial t^2} + \mu\vec{J}$$

Lorentz' condition

- Si escogemos $(\nabla \cdot A) = -\mu\varepsilon \dfrac{\partial \Phi}{\partial t}$

$$\nabla(\nabla \cdot A) + \mu\varepsilon \frac{\partial \nabla\Phi}{\partial t} = \nabla^2 A - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} + \mu\vec{J}$$

- Queda la Ecuacion de Onda

$$\nabla^2 A - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} = -\mu\vec{J}$$

donde J es la densidad de una fuente de corriente, si hay.

de la Ec. de Gauss Eléctrica

$$\nabla \cdot \vec{D} = \rho_v \qquad \nabla \cdot E = \frac{\rho_v}{\varepsilon}$$

$$\nabla \cdot \left(-\nabla\Phi - \frac{\partial A}{\partial t} \right) = \frac{\rho_v}{\varepsilon}$$

$$-\nabla^2\Phi - \frac{\partial(\nabla \cdot A)}{\partial t} = \frac{\rho_v}{\varepsilon}$$

$$-\nabla^2\Phi + \frac{\partial}{\partial t}\left(\mu\varepsilon \frac{\partial \Phi}{\partial t} \right) = \frac{\rho_v}{\varepsilon} \qquad \nabla^2\Phi - \mu\varepsilon \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

Wave equation

- For sinusoidal fields (harmonics):

$$\nabla^2 A - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} = -\mu\vec{J}$$

$$\nabla^2 A - (j\omega)^2 \mu\varepsilon A = -\mu\vec{J}$$

$$\nabla^2 A + k^2 A = -\mu\vec{J}$$

where

$$k^2 = \omega^2 \mu\varepsilon$$

Outline

- Maxwell's equations
  - Wave equations for A and for Φ
- Power: Poynting Vector
- Dipole antenna

Poynting Vector $S = \frac{1}{2} E \times H^*$

$$S_{ave} = \frac{1}{2} \mathrm{Re}\{E \times H^*\}$$

$$E = \hat{x}E_x + \hat{y}E_y \qquad E_x = E_1 e^{j(\omega t - \beta z)}, \quad E_y = E_2 e^{j(\omega t - \beta z)}$$

$$H = \hat{x}H_x + \hat{y}H_y \qquad H_x = H_1 e^{j(\omega t - \beta z - \theta_\eta)}, \quad H_y = H_2 e^{j(\omega t - \beta z - \theta_\eta)}$$

$$S_{ave} = \frac{1}{2} \mathrm{Re}\{(\hat{x}E_x + \hat{y}E_y) \times (\hat{x}H_x + \hat{y}H_y)^*\}$$

Average S

$$S_{ave} = \frac{1}{2} \mathrm{Re}\{(\hat{x}E_x + \hat{y}E_y) \times (\hat{x}H^*_x + \hat{y}H^*_y)\}$$

$$S_{ave} = \frac{1}{2} \mathrm{Re}\{E_x H_y^* - E_y H_x^*\} \hat{z}$$

$$S_{ave} = \frac{1}{2} (E_1 H_2^* - E_2 H_1^*) e^{-2\alpha z} \cos\theta_\eta \quad \hat{z}$$

$$S_{ave} = \frac{1}{2} \left(\frac{|E_1|^2}{|Z_o|} + \frac{|E_2|^2}{|Z_o|} \right) e^{-2\alpha z} \cos\theta_\eta \quad \hat{z}$$

Para medios sin pérdidas:

$$S_{ave} = \frac{1}{2R_o} \left(|E_1|^2 + |E_2|^2 \right) \quad \hat{z}$$

Outline

- Maxwell's equations
  - Wave equations for A and for Φ
- Power: Poynting Vector
- Infinitesimal Dipole antenna

Find A from Dipole with current J

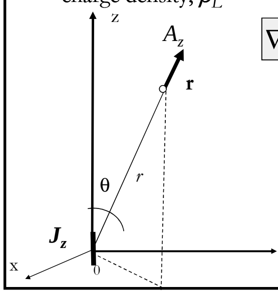
- Line charge w/uniform charge density, ρ_L

$$\nabla^2 A - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} = -\mu \vec{J}$$

Assume the simplest solution $A_z(r)$:

$$\nabla^2 A_z - k^2 A_z = -\mu \vec{J}_z(x, y, z)$$

$$= -\mu_o I_o \delta(x)\delta(y)$$



To find…. $\nabla^2 A = \nabla \cdot \nabla A$

$$\nabla A = \frac{\partial A}{\partial r}\hat{a}_r + \frac{\partial A}{\partial \theta}\hat{a}_\theta + \frac{1}{\sin\theta}\frac{\partial A}{\partial \phi}\hat{a}_\theta$$

Assume the simplest solution $A_z(r)$:

$$\nabla \cdot F = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(F_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\nabla^2 A_z(r) = \frac{1}{r}\frac{d^2}{dr^2}(rA_z)$$

Fuera de la Fuente (J=0)

$$\frac{1}{r}\frac{d^2}{dr^2}(rA_z) + k^2 A_z = 0$$

$$\frac{d^2}{dr^2}(rA_z) + k^2(rA_z) = 0$$

Which has general solution of:

$$rA_z = c_1 e^{jkr} + c_2 e^{-jkr}$$

Apply B.C.

- If radiated wave travels outwards from the source:

$$rA_z(r) = c_2 e^{-jkr} \qquad A_z(r) = \frac{c_2 e^{-jkr}}{r}$$

- To find C2, let's examine what happens near the source. (in that case k tends to 0)

$$A_z(r) = \frac{c_2}{r}$$

- So the wave equation reduces to

$$\nabla^2 A_z = \nabla \cdot \nabla A_z = -\mu_o I_o \delta(x)\delta(y)$$

Now we integrate the volume around the dipole:

$$\iiint \nabla \cdot \nabla A_z dv = -\mu_o I_o \iiint \delta(x)\delta(y) dxdydz$$

- And using the Divergence Theorem

$$\oint \nabla A_z dS = -\mu_o I_o \nabla z$$

$$dS = r^2 \sin\theta d\theta d\phi$$

$$\oint \frac{\partial A_z}{\partial r} r^2 \sin\theta d\theta d\phi = 4\pi r^2 \frac{\partial A_z}{\partial r} = -\mu_o I_o \nabla z$$

Comparing both, we get:

$$\frac{dA_z}{dr} = -\frac{\mu_o I_o \nabla z}{4\pi r^2}$$

$$\frac{dA_z}{dr} = -\frac{c_2}{r^2}$$

$$c_2 = \frac{\mu_o I_o \nabla z}{4\pi}$$

$$A_z(r) = \frac{\mu_o I_o \nabla z}{4\pi r} e^{-jkr}$$

Now from A we can find E & H

$$H = \frac{1}{\mu}\nabla\times A = \frac{1}{\mu}\nabla\times A_z\hat{a}_z$$

- Using the Victoria IDENTITY: $\nabla\times fG = \nabla f\times G + f(\nabla\times G)$

$$\nabla\times A_z\hat{a}_z = \nabla A_z\times\hat{a}_z + A_z(\nabla\times\hat{a}_z)$$

- And

$$H = \frac{1}{\mu}(\nabla A_z\times\hat{a}_z) = \frac{1}{\mu}\left[\frac{\partial}{\partial r}A_z(r)\hat{a}_r\times\hat{a}_z\right]$$

$$\hat{a}_r\times\hat{a}_z = -\hat{a}_\phi\sin\theta$$

- Substitute:

$$H = -\frac{1}{\mu}\left[\frac{\partial A_z}{\partial r}\hat{a}_\phi\sin\theta\right] = H_\phi\hat{\phi}$$

The magnetic filed intensity from the dipole is:

$$H = -\frac{1}{\mu}\left[\frac{\partial A_z}{\partial r}\hat{a}_\phi\sin\theta\right] = H_\phi\hat{\phi}$$

$$A_z(r) = \frac{\mu_o I_o \nabla z}{4\pi r} e^{-jkr}$$

$$H_\phi = \frac{I_o l}{4\pi}\left[\frac{jk}{r} + \frac{1}{r^2}\right]e^{-jkr}\sin\theta$$

Now the E field:

$$\nabla\times\vec{H} = \varepsilon\frac{\partial E}{\partial t}$$

$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}$$

$$j\omega\varepsilon E = \nabla\times\vec{H}$$

$$\nabla\times A = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right) + \frac{\hat{\theta}}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)$$

$$\nabla\times H = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(H_\phi\sin\theta)\right) - \frac{\hat{\theta}}{r}\left(\frac{\partial}{\partial r}(rH_\phi)\right)$$
$$= j\omega\varepsilon(E_r\hat{r} + E_\theta\hat{\theta})$$

The electric field from infinitesimal dipole:

$$j\omega\varepsilon E_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi\sin\theta) =$$
$$= \frac{1}{r\sin\theta}\frac{I_o l}{4\pi}\left[\frac{jk}{r} - \frac{1}{r^2}\right]e^{-jkr}2\sin\theta\cos\theta$$

$$E_r = \frac{I_o l}{2\pi}\left[\frac{\sqrt{\mu/\varepsilon}}{r^2} - \frac{1}{j\omega\varepsilon r^3}\right]e^{-jkr}\cos\theta$$

$$j\omega\varepsilon E_\theta = \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) =$$
$$= \frac{jI_o l}{4\pi\omega\varepsilon r}\left[k^2 - \frac{jk}{r} - \frac{1}{r^2}\right]e^{-jkr}\sin\theta$$

$$E_\theta = \frac{I_o l}{4\pi}\left[\frac{j\mu\omega}{r} + \frac{\sqrt{\mu/\varepsilon}}{r^2} - \frac{1}{j\omega\varepsilon r^3}\right]e^{-jkr}\sin\theta$$

General @Far field $r > 2D^2/\lambda$

General	@Far field
$H_\phi = \frac{I_o l}{4\pi}\left[\frac{jk}{r} + \frac{1}{r^2}\right]e^{-jkr}\sin\theta$	$H_\phi = \frac{I_o l}{4\pi}\left[\frac{jk}{r}\right]e^{-jkr}\sin\theta$
$E_r = \frac{I_o l}{2\pi}\left[\frac{\sqrt{\mu/\varepsilon}}{r^2} - \frac{1}{j\omega\varepsilon r^3}\right]e^{-jkr}\cos\theta$	$E_r = 0$
$E_\theta = \frac{I_o l}{4\pi}\left[\frac{j\mu\omega}{r} + \frac{\sqrt{\mu/\varepsilon}}{r^2} - \frac{1}{j\omega\varepsilon r^3}\right]e^{-jkr}\sin\theta$	$E_\theta = \frac{I_o l}{4\pi}\left[\frac{j\mu\omega}{r}\right]e^{-jkr}\sin\theta$

Note that the ratio of E/H is the intrinsic impedance of the medium.

Power :Hertzian Dipole

$$S = \frac{1}{2}\left(E \times H^*\right)$$

$$\frac{1}{2}\left(E_\theta H_\phi^* \hat{r} - E_r H_\phi^* \hat{\theta}\right)$$

$$P = \oiint \vec{S} \cdot d\vec{A} = \oiint \vec{S} \cdot dA\hat{r} = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_0^{\pi} \left(E_\theta H_\phi^*\right) r^2 \sin\theta d\theta d\phi$$

$$S_r = \left(\frac{1}{2}\right)\frac{I_o l}{4\pi}\left[\frac{j\mu\omega}{r} + \frac{\sqrt{\mu/\varepsilon}}{r^2} - \frac{1}{j\omega\varepsilon r^3}\right] e^{-jkr} \sin\theta \frac{I_o^* l}{4\pi}\left[\frac{-jk}{r} + \frac{1}{r^2}\right] e^{+jkr} \sin\theta$$

$$S_r = \frac{\left|I_o l\right|^2}{32\pi^2}\sin^2\theta\left[\frac{j\mu\omega}{r} + \frac{\sqrt{\mu/\varepsilon}}{r^2} - \frac{1}{j\omega\varepsilon r^3}\right]\left[\frac{-jk}{r} + \frac{1}{r^2}\right]$$

$$S_r = \frac{\left|I_o l\right|^2}{8\lambda^2 r^2}\eta \sin^2\theta\left[1 - \frac{1}{(kr)^3}\right]$$

$$P = \int d\phi \int \frac{\left|I_o l\right|^2}{8\lambda^2 r^2}\eta \sin^2\theta\left[1 - \frac{1}{(kr)^3}\right] r^2 \sin\theta d\theta$$

$$P = \frac{\pi\eta}{3}\left|\frac{I_o l}{\lambda}\right|^2\left[1 - \frac{j}{(kr)^3}\right]$$

Resistencia de Radiacion

- La potencia tiene parte real y parte reactiva

$$P = \frac{\pi\eta}{3}\left|\frac{I_o l}{\lambda}\right|^2\left[1 - \frac{j}{(kr)^3}\right] = P_{rad} + P_{reactiva}$$

- La potencia irradiada es:

$$P_{rad} = \frac{\pi\eta}{3}\left|\frac{I_o l}{\lambda}\right|^2 = \frac{1}{2}\left|I_o\right|^2 R_{rad}$$

- Comparando con la P total, se halla la Impedancia

$$Z = \frac{2\pi\eta}{3}\left|\frac{l}{\lambda}\right|^2\left[1 - \frac{j}{(kr)^3}\right]\Omega$$

- COmparando cno la P_{rad}, halla la R_{rad}.

$$R_{rad} = 80\pi^2\left|\frac{l}{\lambda}\right|^2 \Omega$$