Resistance and Capacitance Electrostatic Boundary value problems;

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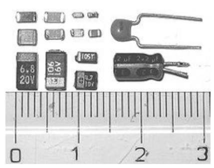
https://www.youtube.com/watch?v=JEIkB_8v7qk

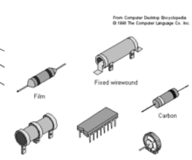
Last Chapters: we knew either V or charge distribution, to find E,D.

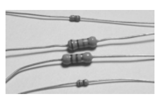
NOW: we only know values of V or Q at some places (boundaries).

Some applications

- *Resistance*
- *Capacitors*
- *Microstrip lines capacitance*







To find E, we will use:

- Poisson's equation: $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
- Laplace's equation: (if charge-free) $\nabla^2 V = 0$

They can be derived from Gauss's Law

Depending on the geometry:

$\nabla^2 V = 0$ $\quad \nabla^2 V = -\frac{\rho_v}{\varepsilon}$ (It's a scalar)

We use appropriate coordinates:

Cartesian:
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

cylindrical:
$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

spherical:
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_v}{\varepsilon}$$

1. Solve for Laplace or Poisson depending on value of ρ_v
2. Use Separation of Variables to solve for V
3. Apply Boundary values to find unique answer for V

Resistance

- Defined as:

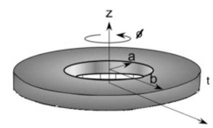
$$R = \frac{V}{I}\,[Ohms]$$

The problem of finding the resistance of a conductor of nonuniform cross section can be solved by :
1. Choose a suitable coordinate system
2. Assume V_o a the potential difference between conductor's terminals
3. Solve Laplace's eq. for V, then obtain E and I from $E = -\nabla V$ and $I = \oint \sigma E \cdot dS$
4. Finally obtain R as V_o/I

Resistance PE 6.8

A disc of thickness *t* has radius b and a central hole of radius a. Take σ = conductivity, find R

a. between hole and rim of the disk
b. Between the 2 fat sides of disk



P.E. 6.8 Part a) find Resistance of disk of radius *b* and central hole of radius *a*.

$$\nabla^2 V = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0$$

$$V = A\ln\rho + B$$

$$BC: \left\rangle \begin{matrix} V(\rho=a)=0 \\ V(\rho=b)=V_o \end{matrix}\right.$$

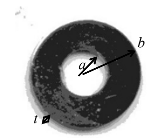
$$V = \frac{V_o}{\ln(b/a)}\ln\frac{\rho}{a}$$

$$E = -\nabla V = -\frac{dV}{d\rho}\hat{\rho} = \frac{-V_o}{\rho\ln(b/a)}\hat{\rho}$$

$$d\vec{S} = -\rho\, dz\, d\phi\, \hat{\rho}$$

$$I = \int_S \sigma\vec{E}\cdot d\vec{S} = \frac{2\pi t V_o \sigma}{\ln(b/a)}$$

$$R = \frac{V_o}{I} = \frac{V_o}{\int_S \sigma\vec{E}\cdot d\vec{S}}$$



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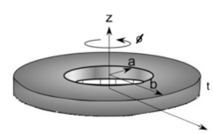


Answers:

$$R = \frac{\int E\cdot dl}{\oint \sigma E\cdot dS}$$

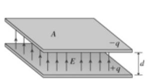
$$R_b = \frac{t}{\sigma\pi\left(b^2-a^2\right)}$$

Resistence

Resist the flow of electrons

0	Negro
1	Marrón
2	Rojo
3	Naranja
4	Amarillo
5	Verde
6	Azul
7	Violeta
8	Gris
9	Blanco

Capacitance



- Is defined as the ratio of the charge on one of the plates to the potential difference between the plates: $C = \frac{Q}{V}\left[Farads\right]$
- Assume Q and find V (*Gauss or Coulomb*)
- Assume V and find Q (*Laplace*)
- And substitute E in the equation.

The (-) sign in V can be ignored because we want the absolute value of V

Two cases: Capacitance

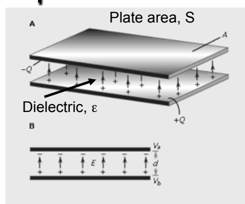
1. Parallel plate
2. Coaxial

Parallel plate Capacitor

- Charge Q and $-Q$

$$\rho_s = \frac{Q}{S}$$

$$\vec{E} = \frac{\rho_s}{\varepsilon}\hat{a}_x$$



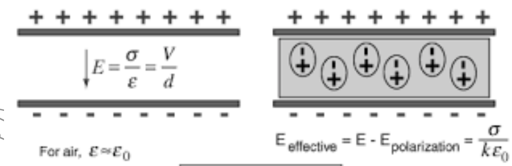
- or

$$Q = \varepsilon\oint \vec{E}\cdot d\vec{S} = \varepsilon E_x S$$

$$V = -\int_d^0 \vec{E}\cdot d\vec{l} = -\int_d^0 \frac{Q}{\varepsilon S}\,dx = \frac{Qd}{\varepsilon S}$$

Effect of ε_r on Capacitance

$$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$$



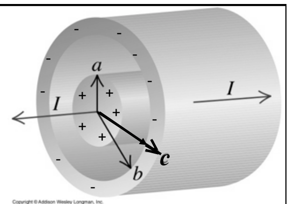
$$C = \frac{\varepsilon_0 A}{d}$$

The capacitance is increased by the factor k.

$$C = \frac{k\varepsilon_0 A}{d}$$

Coaxial Capacitor

- Charge $+Q$ & $-Q$



$$Q = \varepsilon\oint \vec{E}\cdot d\vec{S} = \varepsilon E_\rho 2\pi\rho L$$

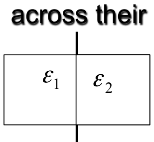
$$V = -\int \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{2\pi\varepsilon\rho L}\hat{\rho}\cdot d\rho\hat{\rho} = \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}$$

Capacitors connection

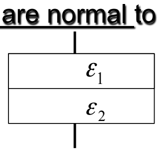
- Series
- Parallel

How to tell if C is in:

- Parallel: when they have same voltage across their plates. ***E* is ||** to interface.

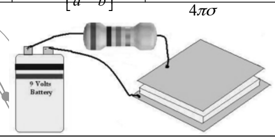


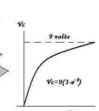
- Series: when E & D are normal to the dielectric interface.



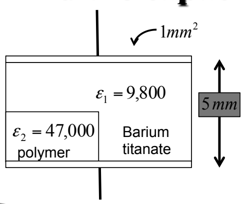
So In summary we obtained:

Capacitor	C	R (not derived)
Parallel Plate	$\frac{\varepsilon S}{d}$	$\frac{\sigma d}{S}$
Coaxial	$\frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$	$\frac{\ln\frac{b}{a}}{2\pi\sigma L}$
Spherical (not derived)	$\frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}$	$\frac{\left[\frac{1}{a}-\frac{1}{b}\right]}{4\pi\sigma}$

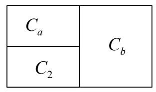




Find the capacitance of



They are connected in parallel and series



$$C = \frac{C_a C_2}{C_a + C_2} + C_b$$

$$C_a = \frac{\varepsilon S}{d} = \frac{9{,}800\varepsilon_o(0.5/1000^2)}{(2.5mm)} = 17.3pF$$

$$C_2 = \frac{\varepsilon S}{d} = \frac{47000\varepsilon_o 0.5/1000^2}{(2.5mm)} = 8.3pF$$

$$C_b = \frac{\varepsilon S}{d} = \frac{9800\varepsilon_o(0.5)/1000^2}{(5mm)} = 8.7F$$

$$\boxed{C = 14.3p\,F}$$