1. (20 points) Prove the following "if and only if" statement: 
"\(m|n \text{ and } n|m \text{ if and only if } m = n \text{ or } m = -n\)".

Answer:

(a) Proof of "If \(m|n \text{ and } n|m\) then \(m = n \text{ or } m = -n\)"

Since \(m|n\), there exists \(p\) integer such that \(n = pm\).
Since \(n|m\), there exists \(q\) integer such that \(m = qn\).
Therefore, \(n = pqm\), and thus, necessarily, \(pq = 1\).
Since \(p\) and \(q\) are integers, either \(p = q = 1\) or \(p = q = -1\).
In the first case \(m = n\). In the second, \(m = -n\).

(b) Proof of "If \(m = n \text{ or } m = -n\), then \(m|n \text{ and } n|m\)"

If \(m = n\), then \(n = 1 \cdot m\) and thus \(m|n\).
If \(m = -n\), then \(m = -1 \cdot n\) and thus \(n|m\).

2. (10 points) Verify the next statement using a truth table:

\[P \implies (Q \implies R) \iff (P \land Q) \implies R.\]

Answer:

<table>
<thead>
<tr>
<th>(P)</th>
<th>(Q)</th>
<th>(R)</th>
<th>(Q \implies R)</th>
<th>(P \implies (Q \implies R))</th>
<th>(P \land Q)</th>
<th>((P \land Q) \implies R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
3. (5 points) Find the **smallest** set $A$ such that

$\{\{a\}, \{a, b\}, \{\phi\}\} \subset power(A)$. 

**Answer:** Define $A = \{a, b, \phi\}$. Then,

$power(A) = \{\phi, \{a\}, \{b\}, \{\phi\}, \{a, b\}, \{a, \phi\}, \{b, \phi\}, \{a, b, \phi\}\}$

Therefore,

$\{\{a\}, \{a, b\}, \{\phi\}\} \subset power(A)$

and no proper subset of $A$ satisfies this property.

4. (10 points) Compute $|A \cap B \cap C|$, knowing that: the union of these sets has 280 elements, and $|A| = 100$, $|B| = 200$, $|C| = 150$, $|A \cap B| = 50$, $|A \cap C| = 80$, and $|B \cap C| = 90$.

**Answer:** Apply

$|A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |A \cap C| + |B \cap C| - (|A| + |B| + |C|)$

to get:

$|A \cap B \cap C| = 280 + 50 + 80 + 90 - (100 + 200 + 150)$

$= 500 - 450$

$= 50$.

5. (10 points) Given $L = \{\Lambda, ab\}$ and $M = \{ba, a\}$, construct: $LM^2$.

**Answer:** Compute first $M^2$:

$$
\begin{array}{c|ccc}
& ba & a \\
\hline
ba & baba & baa \\
a & aba & aa
\end{array}
$$

Thus, $M^2 = \{baba, baa, aba, aa\}$. Now $LM^2$ is computed as:

$$
\begin{array}{c|cccc}
& baba & baa & aba & aa \\
\hline
\Lambda & baba & baa & aba & aa \\
ab & abbaba & abba & ababa & abaa
\end{array}
$$

Thus, $LM^2 = \{baba, baa, aba, aa, abbaba, abba, ababa, abaa\}$
6. (15 points) Given the list \( L = \langle a, < c, d >, < f, g > \rangle \) find:
   (a) (7 points) \( \text{tail(tail}(L)) \),
   Answer:
   \[
   \text{tail}(L) = \langle < c, d >, < f, g > \rangle
   \]
   \[
   \text{tail(tail}(L)) = \langle < f, g > \rangle
   \]
   (b) (8 points) \( \text{cons(head}(L), (\text{cons}(x, \text{tail}(L)))) \).
   Answer:
   \[
   \text{tail}(L) = \langle < c, d >, < f, g > \rangle
   \]
   \[
   \text{cons}(x, \text{tail}(L)) = \langle x, < c, d >, < f, g > \rangle
   \]
   \[
   \text{head}(L) = a
   \]
   \[
   \text{cons}(\text{head}(L), (\text{cons}(x, \text{tail}(L)))) = \langle a, x, < c, d >, < f, g > \rangle.
   \]

7. (10 points) Find the language \( L \) that satisfies
   \[
   L \{ b, \Lambda, ab \} = \{ abb, ab, abab, bab, b, bb \}.
   \]
   Answer: Through the following table:
   \[
   \begin{array}{c|ccc}
   & b & \Lambda & ab \\
   \hline
   ab & abb & ab & abab \\
   b & bb & b & bab \\
   \end{array}
   \]
   we show that the language is \( L = \{ ab, b \} \).

8. (20 points) Provide a precise description for each of the languages:
   (a) (10 points) \( \{ 0, 1 \}^* - \{ 1 \}^* \),
   Answer: \( \{ 0, 1 \}^* - \{ 1 \}^* \) is the language of all strings of 0’s and 1’s with at least one 0.
   (b) (10 points) \( (L \cup M)^* \), where \( L \) and \( M \) are languages over \( \{ 0, 1 \} \).
   Answer:
   \[
   (L \cup M)^* = \{ s : s = s_0s_2\cdots s_k \land s_i \in L \cup M \text{ } 0 \leq i \leq k \land k \text{ natural} \}
   \[
   = \{ s : s = s_0s_2\cdots s_k \land s_i \in L \lor s_i \in M \text{ } 0 \leq i \leq k \land k \text{ natural} \}. 
   \]