Electrostatic fields

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Some applications

- *Power transmission, X rays, lightning protection*
- *Solid-state Electronics: resistors, capacitors, FET*
- *Computer peripherals: touch pads, LCD, CRT*
- *Medicine: electrocardiograms, electroencephalograms, monitoring eye activity*
- Agriculture: seed sorting, moisture content monitoring, spinning cotton, …
- Art: spray painting
- …

We will study Electric charges:

- Coulomb's Law- $F = \frac{kQ_1Q_2}{R^2}$
  - Use when charge distribution is known
- Gauss' s Law – $\Psi = \oint_S \vec{D} \cdot d\vec{S} = Q_{enc}$
  - Use when charge distribution is symmetrical
- Electric Potential $V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$

(uses scalar, not vectors)
  - Use when potential V is known

Coulomb' s Law (1785)

- Force one charge exerts on another

$$F = \frac{kQ_1Q_2}{R^2}$$

Point charges

where k= 9 x 10^9
or $k = 1/4\pi\varepsilon_o$
ε_0=8.85 x 10^{-12}

*Superposition applies

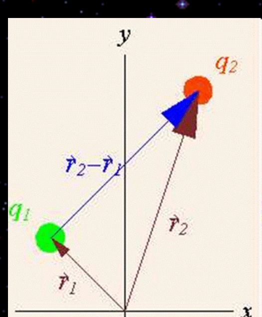
Force with direction

Force that Q_1 exerts on Q_2

$$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_o R^2}\hat{a}_{12}$$

Note: Observation point goes First!

$$\hat{a}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$



Example

Example: Point charges 5nC and -2nC are located at r_1=(2,0,4) and r_2=(-3,0,5), respectively.

a) Find the force on a 1nC point charge, Q_x, located at (1,-3,7)

Apply superposition:

$$\vec{F}_x = \frac{1}{4\pi\varepsilon_o}\left[\frac{Q_1Q_x(\vec{r}_x - \vec{r}_1)}{|\vec{r}_x - \vec{r}_1|^3} + \frac{Q_xQ_2(\vec{r}_x - \vec{r}_2)}{|\vec{r}_x - \vec{r}_2|^3}\right]$$

Electric field intensity

- Is the force per unit charge when placed in the E field

$$E = \frac{F}{Q}$$

$$E = \frac{Q}{4\pi\varepsilon_o R^2}\hat{a}_R$$

Example: Same point charges 5nC and -2nC are located at (2,0,4) and (-3,0,5), respectively.

b) Find the E field at r_x=(1,-3,7).

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\left[\frac{Q_1(r_x - r_1)}{|r_x - r_1|^3} + \frac{Q_2(r_x - r_2)}{|r_x - r_2|^3}\right] = (-1.0, -1.29, 1.4)\,V/m$$

If we have many charges

$$E = \frac{F}{Q} \qquad E = \frac{Q}{4\pi\varepsilon_o R^2}\hat{a}_R$$

Line charge density, ρ_L	C/m
Surface charge density ρ_S	C/m²
Volume charge density ρ_v	C/m³

$$Q = \int_L \rho_L dl \qquad Q = \int_S \rho_S dS \qquad Q = \int_v \rho_v dv$$

$$E_{\text{point charge}} = \frac{Q}{4\pi\varepsilon_o R^2}\hat{a}_R$$

The total E-field intensity is

$$E = \int \frac{(\rho_L dl)}{4\pi\varepsilon_o R^2}\hat{a}_R$$

$$E = \int \frac{(\rho_S dS)}{4\pi\varepsilon_o R^2}\hat{a}_R$$

$$E = \int \frac{(\rho_v dv)}{4\pi\varepsilon_o R^2}\hat{a}_R$$

More Charge distributions

Find E from

- Point charge (we just saw this one)
- Line charge
- Surface charge
- Volume charge

Results Preview

Line charge $\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\hat{a}_\rho$

Sheet charge $\vec{E} = \frac{\rho_S}{2\varepsilon_o}\hat{a}_n$

Volume Charge $\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{a}_r$

We will derive these 3 cases

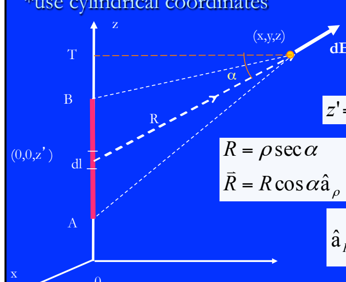
1. **Using Coulomb**
2. **Using Gauss**

Find E from LINE charge

- Line charge w/uniform charge density, ρ_L

$$Q = \int_A^B \rho_L dl \qquad dl = dz'$$

*use cylindrical coordinates



$$E = \int \frac{\rho_L dz'}{4\pi\varepsilon_o R^2}\hat{a}_R$$

$$z' = OT - \rho\tan\alpha$$

$$R = \rho\sec\alpha$$

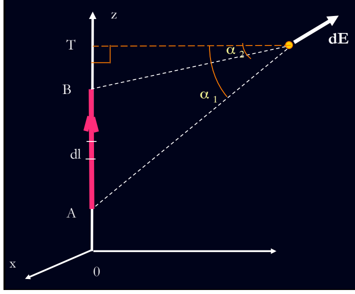
$$\bar{R} = R\cos\alpha\,\hat{a}_\rho + R\sin\alpha\,\hat{a}_z$$

$$\hat{a}_R = \frac{\bar{R}}{R} = \cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z$$

Defining angles α_1 and α_2

α_1 =imaginary perpendicular line with the **back**

α_2=imaginary perpendicular line with the front



LINE charge

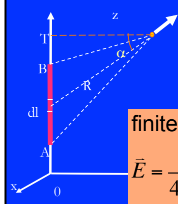
- Substituting in: $E = \int \frac{\rho_L dz'}{4\pi\varepsilon_o R^2}\hat{a}_R$

$z' = OT - \rho\tan\alpha$

$dz' = [0 - \rho\sec^2\alpha]d\alpha$

$R = \rho\sec\alpha$

$\hat{a}_R = \cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z$



$$\bar{E} = \int_{\alpha_1}^{\alpha_2} \frac{\rho_L[-\rho\sec^2\alpha]d\alpha}{4\pi\varepsilon_o\rho^2\sec^2\alpha}[\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z]$$

finite Line Charge:

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_o\rho}[-(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\hat{a}_z]$$

infinite Line Charge $(\alpha_{1,2} = \mp 90^o)$

$$E = \frac{\rho_L}{2\pi\varepsilon_o\rho}\hat{a}_\rho$$

More Charge distributions

- Point charge
- Line charge
- Surface charge
- Volume charge

Find E from Surface charge

- Sheet of charge w/ uniform density ρ_S

$dQ = \rho_S dS$

$dE = \frac{\rho_S dS}{4\pi\varepsilon_o R^2}\hat{a}_R$

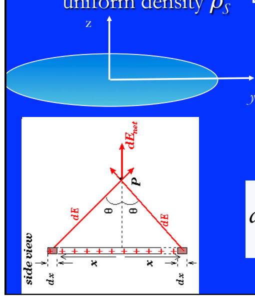
Element of area is:

$dS = \rho\, d\phi\, d\rho$

Observation point is at z-axis:

$R = \rho(-\hat{a}_\rho) + h\hat{a}_z$

$\hat{a}_R = \frac{\bar{R}}{R}$



$$dE = \frac{\rho_S \rho\, d\phi\, d\rho[-\rho\hat{a}_\rho + h\hat{a}_z]}{4\pi\varepsilon_o[\rho^2 + h^2]^{3/2}}$$

SURFACE charge

- Due to SYMMETRY the ρ component cancels out.

$$E_z = \frac{\rho_S}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{\infty} \frac{h\rho\, d\rho}{[\rho^2 + h^2]^{3/2}}$$

$$E_z = \frac{\rho_S}{4\pi\varepsilon_o}2\pi\left[\frac{-h}{\sqrt{\rho^2 + h^2}}\right]_0^\infty$$

infinite Surface Charge:

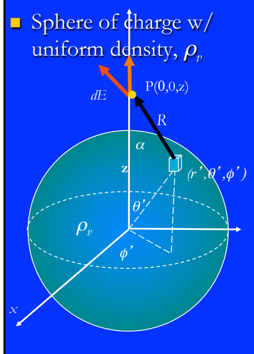
$$E = \frac{\rho_S}{2\varepsilon_o}\hat{a}_n$$

More Charge distributions

- Point charge
- Line charge
- Surface charge
- Volume charge

Find E from Volume charge

- Sphere of charge w/ uniform density, ρ_v



$$dQ = \rho_v dv$$

$$dE = \frac{\rho_v dv}{4\pi\varepsilon_o R^2}\hat{a}_R$$

$$dv = r'^2 \sin\theta' d\theta' d\phi' dr'$$

Law of cosines:

(Eq. *) $R^2 = z^2 + r'^2 - 2zr'\cos\theta'$

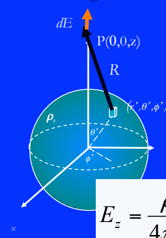
$r'^2 = z^2 + R^2 - 2zR\cos\alpha$

Differentiating (Eq. *) $\sin\theta' d\theta' = \frac{RdR}{zr'}$

Due to symmetry *only* $dE_z = dE\cos\alpha$ *survives.*

Find E from Volume charge

- Substituting…



$$dE_z = \frac{\rho_v dv}{4\pi\varepsilon_o R^2}\cos\alpha\,\hat{a}_z$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

$$dv = r'^2 \sin\theta' d\theta' d\phi' dr'$$

$$\sin\theta' d\theta' = \frac{RdR}{zr'}$$

$$E_z = \frac{\rho_v}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int^{a}\int^{z+r'} \ldots \frac{RdR}{} \ldots \frac{z^2+R^2-r'^2}{} \ldots \frac{1}{R^2}$$

$$E = \frac{1}{4\pi\varepsilon_o z^2}\left(\frac{4}{3}\pi a^3\rho_v\right)\hat{a}_z = \frac{Q}{4\pi\varepsilon_o r^2}\hat{a}_r$$

De donde salen los limites de R?

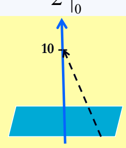
P.E. 4.5

- A square plate at plane z=0 and $x \le \pm 2, y \le \pm 2$ carries a charge $12|y|$ mC/m2 . Find the total charge on the plate and the electric field intensity at (0,0,10). $Q = \int_S \rho_S dS$

$$Q = \int_{x=-2}^{2} dx \int_{y=-2}^{2} 12|y| dy = \int_{x=-2}^{2} dx \int_{y=0}^{2} 12(2) y dy = 4\cdot 12(2)\left.\frac{y^2}{2}\right|_0^2 \; 192\,\text{mC}$$

$$\vec{E} = \int \frac{\rho_s}{4\pi\varepsilon_o r^2} dS\hat{a}_r = \int \frac{\rho_s dS}{4\pi\varepsilon_o}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{r}-\vec{r}' = (0,0,10)-(x',y',0) = (-x',-y',10)$$



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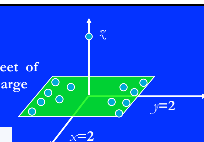


$$\vec{E} = \int_{-2}^{2}\int_{y=-2}^{2}\frac{12|y|dxdy}{4\pi\varepsilon_o}\frac{(-x,-y,10)}{(x^2+y^2+100)^{3/2}}$$

$$= 108\cdot 10^6\left[\int_{-2}^{2}|y|\int_{x=-2}^{2}\frac{-xdxdy\hat{a}_x}{(x^2+y^2+100)^{3/2}} + \int_{-2}^{2}\int_{x=-2}^{2}\frac{-y|y|dxdy\hat{a}_y}{(x^2+y^2+100)^{3/2}} + \int_{-2}^{2}|y|\int_{x=-2}^{2}\frac{10|y|dxdy\hat{a}_z}{(x^2+y^2+100)^{3/2}}\right]$$

Due to symmetry only E_z survives:

$$\vec{E} = 108\cdot 10^6\cdot 2\left[\int_{-2}^{2}\int_{0}^{2}\frac{10\,ydxdy\hat{a}_z}{(x^2+y^2+100)^{3/2}}\right]$$

$$= 16.5\hat{a}_z\; MV/m$$

Chapter Outline

- Coulomb's Law- $F = \frac{kQ_1Q_2}{R^2}$
  - Use when charge distribution is known
- Gauss's Law – $\Psi = \oint_S \vec{D}\cdot d\vec{S} = Q_{enc}$
  - Use when charge distribution is symmetrical
- Electric Potential (uses scalar, not vectors) $V(r) = -\int_{\infty}^{r}\vec{E}\cdot d\vec{l}$
  - Use when potential V is known

Electric Flux Density

D is independent of the medium in which the charge is placed.

$$\vec{D} = \varepsilon_o\vec{E} = \int\frac{\rho_v dv}{4\pi R^2}\hat{a}_R \quad [C/m^2]$$

Gauss's Law

$$\Psi = \oint_S \vec{D}\cdot d\vec{S} = Q_{enc}$$

Therefore:

$$\rho_v = \nabla \cdot \vec{D}$$

This is the 1st of the Maxwell's equations derived here.

Gauss's Law

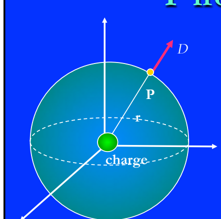
- The total electric flux Ψ, through any closed surface is equal to the total charge enclosed by that surface.

$$\vec{D} = \varepsilon_o \vec{E} = \int \frac{\rho_v dv}{4\pi R^2}\hat{a}_R$$

$$\Psi = Q_{enc} = \oint_S \vec{D}\cdot d\vec{S} = \int_v \rho_v dv$$

The key is to choose the Gauss surface to simplify the problem.
Follow the symmetry of the particular case.
Pick surface so that D is ⊥

Some examples: Finding *D* at point P from the charges:



Point Charge is at the origin.

$$Q = \oint_S \vec{D}\cdot d\vec{S}$$

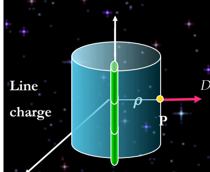
- Choose a spherical dS
- Note where *D* is perpendicular to this surface.

$$Q = D_r \int_S dS = D_r 4\pi r^2$$

$$D = \frac{Q}{4\pi r^2}\hat{a}_r$$

Some examples: Finding *D* at point P from the charges:

Infinite Line Charge



$$\rho_l dl = Q = \oint_S \vec{D}\cdot d\vec{S}$$

- Choose a cylindrical dS
- Note that integral =0 at top and bottom surfaces of cylinder

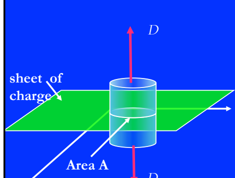
$$\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$$Q = D_\rho \oint_S dS = D_\rho 2\pi\rho l$$

Some examples: Find *D* at point P from the charges:

- Infinite Sheet of charge

$$\rho_s \int dS = Q = \oint_S \vec{D}\cdot d\vec{S}$$



- Choose a cylindrical box cutting the sheet

$$\rho_S A = Q = D_s\left[\int_{top} dS + \int_{bottom} dS\right]$$

Note that D is parallel to the sides of the box.

$$\rho_S A = D_s[A + A]$$

$$\vec{D} = \frac{\rho_S}{2}\hat{a}_z$$

P.E. 4.7

A point charge of 30nC is located at the origin, while plane y=3 carries charge 10nC/m^2.
Find *D* at (0, 4, 3)

$$\vec{D} = \vec{D}_Q + \vec{D}_\rho = \frac{Q}{4\pi r^2}\hat{a}_r + \frac{\rho_s}{2}\hat{a}_n$$

$$\vec{D} = \frac{30\cdot 10^{-9}}{4\pi\left(\sqrt{4^2+3^2}\right)^3}\left[(0,4,3)-(0,0,0)\right] + \frac{10n}{2}\hat{a}_y$$

$$\vec{D} = \frac{30\cdot 10^{-9}}{4\pi(5)^3}(0,4,3) + 5n\hat{a}_y$$

$$= 5.08\hat{a}_y + 0.057\hat{a}_z \text{ nC/m}^2$$

P.E. 4.8

If $\vec{D} = (2y^2 + z)\hat{a}_x + 4xy\hat{a}_y + x\hat{a}_z$ C/m2 . Find :

- volume charge density at (-1,0,3)

$$\rho_v(-1,0,3) = \nabla \cdot \vec{D} = 4x = -4\text{C/m}^3$$

- Flux thru the cube defined by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$

$$\Psi = Q_{enc} = \int_v \rho_v dv = \int_0^1\int_0^1\int_0^1 4x\,dx\,dy\,dz$$

- Total charge enclosed by the cube

$$Q = \Psi = 2C$$

Review

Point charge or volume Charge distribution

$$D = \frac{Q}{4\pi r^2}\hat{a}_r$$

Line charge distribution

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

Sheet charge distribution

$$\vec{D} = \frac{\rho_S}{2}\hat{a}_n$$

We will study Electric charges:

- Coulomb's Law (general cases)
- Gauss' s Law (symmetrical cases)
- Electric Potential (uses scalar, not vectors)

Electric Potential, V

- The work done to move a charge Q from A to B is

$$dW = -\vec{F}\cdot dl$$
$$= -Q\vec{E}\cdot dl$$

- *The (-) means the work is done by an external force.*
- *The total work= potential energy required in moving Q:*

$$W = -Q\int_A^B \vec{E}\cdot d\vec{l}$$

- *The energy per unit charge= potential difference between the 2 points:*

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}\cdot d\vec{l} \quad \left[\frac{\text{J}}{\text{C}}\right] = [\text{V}]$$

V is independent of the path taken.

The Potential at any point is the potential difference between that point and a chosen reference point at which the potential is zero. (choosing infinity):

$$V(r) = -\int_\infty^r \vec{E}\cdot d\vec{l} = -\int_\infty^r \frac{Q}{4\pi\varepsilon_o r'^2}\hat{a}_r \cdot dr'\hat{a}_r = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r'}\Big|_\infty^r = \frac{Q}{4\pi\varepsilon_o r}[\text{V}]$$

For many Point charges at r_k: (apply superposition)

$$V(r) = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{n}\frac{Q_k}{|\vec{r} - \vec{r}_k|}[\text{V}]$$

For Line Charges:

$$V(\hat{r}) = \frac{1}{4\pi\varepsilon_o}\int_L \frac{\rho_L(\hat{r}')dl'}{|\hat{r} - \hat{r}'|}$$

For Surface charges:

$$V(\hat{r}) = \frac{1}{4\pi\varepsilon_o}\int_S \frac{\rho_s(\hat{r}')dS'}{|\hat{r} - \hat{r}'|}$$

For Volume charges:

$$V(\hat{r}) = \frac{1}{4\pi\varepsilon_o}\int_v \frac{\rho_v(\hat{r}')dv'}{|\hat{r} - \hat{r}'|}$$

P.E. 4.10

A point charge of -4μC is located at (2,-1,3)
A point charge of 5μC is located at (0,4,-2)
A point charge of 3μC is located at the origin

Assume V(∞)=0 and Find the potential at (-1, 5, 2)

$$V(\vec{r}) = \sum_{k=1}^{3}\frac{Q_k}{4\pi\varepsilon_o|\vec{r} - \vec{r}_k|} + C$$

$$|\vec{r} - \vec{r}_1| = (-1,5,2) - (2,-1,3) = \sqrt{46}$$

$$|\vec{r} - \vec{r}_2| = (-1,5,2) - (0,4,-2) = \sqrt{18}$$

$$|\vec{r} - \vec{r}_3| = (-1,5,2) - (0,0,0) = \sqrt{30}$$

$$V(-1,5,2) = \frac{10^{-6}}{1/9\cdot 10^9}\left[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}}\right]$$ =10.23 kV

Example

A line charge of 5nC/m is located on line x=10, y=20

Assume V(0,0,0)=0 and Find the potential at A(3, 0, 5)

$$V(\hat{r}) = -\int \vec{E}\cdot d\bar{l} = -\int \frac{\rho_L}{2\pi\varepsilon_o \rho}\hat{a}_\rho \cdot d\rho\hat{a}_\rho$$

$$V(\hat{r}) = -\frac{\rho_L}{2\pi\varepsilon_o}\ln\rho + C$$

$$V_{origin} - V_A = -\frac{\rho_L}{2\pi\varepsilon_o}\left[\ln\rho_o - \ln\rho_A\right]$$

$$0 - V_A = -4.8 \qquad V_A = +4.8V$$

ρ_0= | (0,0,0)-(10,20,0) | =22.36 and ρ_A= | (3,0,5)-(10,20,0) | = 21.2

QUIZ #2: A point charge of 7nC is located at the origin

V(0,3,-5)=2V and Find C

$$V = \frac{Q}{4\pi\varepsilon_o r} + C$$

P.E. 4.11 A point charge of 5nC is located at the origin

V(0,6,-8)=2V and Find the potential at A(-3, 2, 6)

Find the potential at B(1,5,7), the potential difference V_{AB}

$$V = \frac{Q}{4\pi\varepsilon_o r} + C \qquad r = |(0,0,0)-(0,6,-8)| = 10 \qquad 2 = \frac{5n}{4\pi\varepsilon_o(10)} + C \qquad \therefore C = -2.5$$

$$V_A = \frac{5n}{4\pi\varepsilon_o|(-3,2,6)-(0,0,0)|} + C = 3.93V$$

$$V_B = \frac{5n}{4\pi\varepsilon_o|(1,5,7)-(0,0,0)|} - 2.5 = 2.696V$$

$$V_{AB} = V_B - V_A = -1.233V$$

Relation between E and V

V is independent of the path taken.

$$V_{AB} = -V_{BA}$$

$$V_{AB} + V_{BA} = \oint \vec{E}\cdot d\bar{l} = 0$$

*Esto aplica sólo a campos estáticos

Significa que no hay trabajo NETO en mover una carga en un paso cerrado donde haya un campo estático E.

$$\oint \vec{E}\cdot d\bar{l} = \int_S \left(\nabla\times\vec{E}\right)\cdot d\bar{S} = 0$$

Static E satisfies:

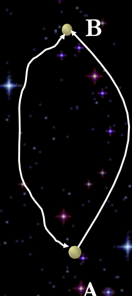
$$\nabla\times\vec{E} = 0$$

Condition for Conservative field = independent of path of integration

$$dV = -\vec{E}\cdot d\bar{l} = -E_x dx - E_y dy - E_z dz$$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$E = -\nabla V$$



Example

Given the potential $V = \frac{10}{r^2}\sin\theta\cos\phi$

Find D at $\left(2,\frac{\pi}{2},0\right)$.

$$\vec{D} = \varepsilon_o\vec{E} = \varepsilon_o(-\nabla V)$$

In spherical coordinates:

$$\vec{E} = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$$

$$\vec{E} = -\left[\frac{20}{r^3}\sin\theta\cos\phi\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\hat{a}_\theta + \frac{10}{r^3}\sin\theta\frac{\partial V}{\partial\phi}\hat{a}_\phi\right]$$

$$\vec{D} = \varepsilon_o\vec{E}\Big|_{(2,\pi/2,0)} = \varepsilon_o\left(\frac{20}{8}\hat{a}_r + -0\hat{a}_\theta + 0\hat{a}_\phi\right)$$

$$\vec{D} = 22.1\hat{a}_r\ C/m^2$$

P.E. 4.12 Given that $E=(3x^2+y)a_x + x\,a_y$ kV/m, find the work done in moving a -2μC charge from (0,5,0) to (2,-1,0) by taking the straight-line path.

a) (0,5,0)→(2,5,0) →(2,-1,0)

$$\frac{-W}{Q} = \int E \cdot dl = \int \left[(3x^2 + y)dx + xdy \right]$$

$$\frac{-W}{Q} = \int_0^2 (3x^2 + y)dx \int_5^{-1} xdy$$

$W = (-Q)(18-12)$ $\quad W = 6(-2\mu)12mJ$

b) $y = 5-3x$

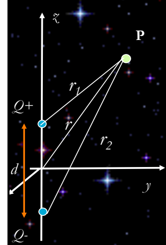
$$dy = -3dx$$

$$\frac{-W}{Q} = \int E \cdot dl = \int \left[(3x^2 + 5 - 3x)dx + x(-3dx) \right] =$$

$$\frac{-W}{Q} = \int_0^2 (3x^2 - 6x + 5)dx \qquad -\frac{W}{Q} = 8 - 12 + 10 = 6 \qquad W = 12mJ$$

Electric Dipole

- Is formed when 2 point charges of equal but opposite sign are separated by a small distance.



$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_o}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

For far away observation points ($r >> d$):

$$V = \frac{Q}{4\pi\varepsilon_o}\frac{d\cos\theta}{r^2}$$

Energy Density in Electrostatic fields

- It can be shown that the total electric work done is:

$$W_E = \frac{1}{2}\int_v \vec{D}\cdot\vec{E}dv = \frac{\varepsilon_o}{2}\int_v E^2 dv$$