Resistance and Capacitance Electrostatic Boundary value problems;

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https://www.youtube.com/watch?v=JEIkB_8v7qk

Last Chapters: we knew either V or charge distribution, to find E,D.

NOW: we only know values of V or Q at some places (boundaries).

Some applications

- *Resistance*
- *Capacitors*
- *Microstrip lines capacitance*

To find E, we will use:

- Poisson's equation: $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
- Laplace's equation: (if charge-free) $\nabla^2 V = 0$

They can be derived from Gauss's Law

$$\nabla \cdot D = \nabla \cdot \varepsilon E = \rho_v$$
$$E = -\nabla V$$

Depending on the geometry:

$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$ (It's a scalar)

We use appropriate coordinates:

Cartesian:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

cylindrical:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

spherical:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_v}{\varepsilon}$$

Resistance

- Defined as:

$$R = \frac{V}{I}\,[Ohms] \qquad R = \frac{\int E \cdot dl}{\oint \sigma E \cdot dS}$$

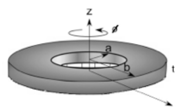
The problem of finding the resistance of a conductor of nonuniform cross section can be treated as a boundary-value problem. Using eq. (6.16), the resistance R (or conductance $G = 1/R$) of a given conducting material can be found by following these steps:

1. Choose a suitable coordinate system.
2. Assume V_o as the potential difference between conductor terminals.
3. Solve Laplace's equation $\nabla^2 V$ to obtain V. Then determine $\mathbf{E}$ from $\mathbf{E} = -\nabla V$ and I from $I = \int \sigma \mathbf{E} \cdot d\mathbf{S}$.
4. Finally, obtain R as V_o/I.

Resistance PE 6.8

A disc of thickness t has radius b and a central hole of radius a. Take σ = conductivity, find R

a. between hole and rim of the disk
b. Between the 2 fat sides of disk



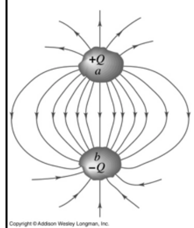
Answers:

$$R_a = \frac{\int E \cdot dl}{\oint \sigma E \cdot dS} = \frac{\ln(b/a)}{2\pi t \sigma}$$

$$R_b = \frac{t}{\sigma\pi\left(b^2 - a^2\right)}$$

$$\nabla = \frac{\partial}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

Capacitance



- Is defined as the ratio of the charge on one of the plates to the potential difference between the plates: $C = \frac{Q}{V}\left[Farads\right]$
- Assume Q and find V (*Gauss or Coulomb*)
- Assume V and find Q (*Laplace*)
- And substitute E in the equation.

$$C = \frac{Q}{\Delta V} = \frac{\varepsilon \oint_S \vec{E} \cdot d\vec{S}}{\int_l \vec{E} \cdot d\vec{l}} \quad [F]$$

The (-) sign in V can be ignored because we want the absolute value of V

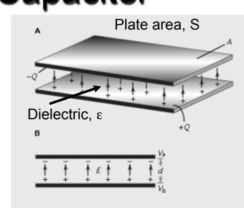
Two cases: Capacitance

1. Parallel plate
2. Coaxial

Parallel plate Capacitor

- Charge Q and $-Q$

$$\rho_s = \frac{Q}{S} \qquad \vec{E} = \frac{\rho_s}{\varepsilon}\hat{a}_x$$



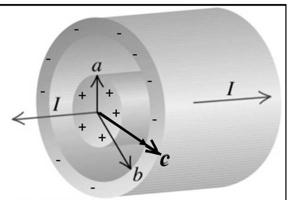
- *or*

$$Q = \varepsilon \oint \vec{E} \cdot d\vec{S} = \varepsilon E_x S$$

$$V = -\int_d^0 \vec{E} \cdot d\vec{l} = -\int_d^0 \frac{Q}{\varepsilon S} dx = \frac{Qd}{\varepsilon S}$$

$$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$$

Coaxial Capacitor



- Charge $+Q$ & $-Q$

$$Q = \varepsilon \oint \vec{E} \cdot d\vec{S} = \varepsilon E_\rho 2\pi\rho L$$

$$V = -\int \vec{E} \cdot d\vec{l} = -\int_b^a \frac{Q}{2\pi\varepsilon\rho L}\hat{\rho} \cdot d\rho\hat{\rho} = \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\left[\frac{b}{a}\right]}$$

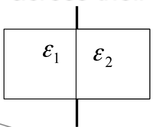
Capacitors connection

- Series $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$
- Parallel

$$C = C_1 + C_2$$

How to tell if C is in:

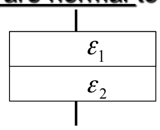
- Parallel: when they have same voltage across their plates. *E* is || to interface.



$$C = C_1 + C_2$$

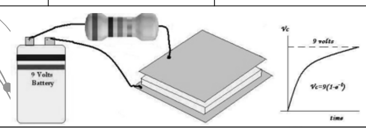
- Series: when E & D are normal to the dielectric interface.

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

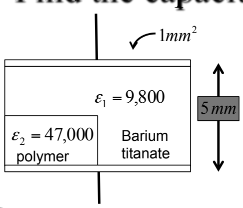


So In summary we obtained:

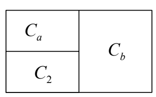
Capacitor	C	R (not derived)
Parallel Plate	$\frac{\varepsilon S}{d}$	$\frac{\sigma d}{S}$
Coaxial	$2\pi\varepsilon L / \ln\frac{b}{a}$	$\frac{\ln\frac{b}{a}}{2\pi\sigma L}$
Spherical (not derived)	$\frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}$	$\frac{\left[\frac{1}{a}-\frac{1}{b}\right]}{4\pi\sigma}$



Find the capacitance of



They are connected in parallel and series



$$C = \frac{C_a C_2}{C_a + C_2} + C_b$$

$$C_a = \frac{\varepsilon S}{d} = \frac{9{,}800\varepsilon_o(0.5/1000^2)}{(2.5mm)} = 17.3pF$$

$$C_2 = \frac{\varepsilon S}{d} = \frac{47000\varepsilon_o 0.5/1000^2}{(2.5mm)} = 8.3pF$$

$$C_b = \frac{\varepsilon S}{d} = \frac{9800\varepsilon_o(0.5)/1000^2}{(5mm)} = 8.7F$$

$$C = 14.3p\,F$$